The *Fourier Phase Sphere*: A Method for Computer-Assisted Atonal Composition

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Abstract: In post-tonal music since the early 20th century, harmonic syntax evolved to almost as many theories and compositional systems as the number of composers. In this context, it is important to assist them in their creative process as intended by the tool proposed in this paper. To this end, we extended recent methods by Yust on adopting the phase information from the discrete Fourier transform of pitch-class sets resulting in the creation of a spherical space of harmonic representation that ultimately helps the composer establishing a harmonic syntax for his pieces.

Keywords: Discrete Fourier Transform. Computer-assisted Composition. Atonal Music.

1. Introduction

In music, multiple levels of syntactic organization govern the creation of harmonic structure (ROHRMEIER, 2018). For example, in the context of tonal music, the different harmonic functions (i.e., tonic, dominant, subdominant, etc.) that one chord can assume in distinct phrases are well known. The syntactic organization of musical structure is less understood in post-tonal music since the early 20th century. The foundational compositional principles lack a generalized systematization. We can find many idiosyncratic proposals to the extent that has never been historically experienced within the canonic Western music tradition. In the 20th century, we witnessed the emergence of as many compositional systems as composers or individual works.

In this context, we explore a referential set of tools and methods to assist contemporary composers in their creative decision-making process, namely in the emergent syntactic behaviour of non-tonal harmony, inspired by the work of Ferguson and Parncutt (2004). Our approach is computationally grounded on the recent

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application of the discrete Fourier transform (DFT) of pitch-class (pc) sets within music theory (YUST, 2020; QUINN, 2006-7; BERNARDES *et al.*, 2016). Beyond the many properties of theoretical musical value that the DFT of pc sets has shown to elicit, we expand its application as a computer-assisted composition method. In short, we propose a harmonic description space, named *Fourier phase sphere*. It aims to unpack emergent lexical and syntactic behaviours from a pool of user-defined pc sets. To this end, we adopt the phase information from the DFT of pc sets to compute a spherical geometrical space, where distance between elements are relevant to the design of compositional principles. The *Fourier phase sphere* should ultimately enable: (1) the definition of a geometric space from a collection of pc sets defined *a priori* by the composer; (2) the identification of harmonic clusters in the resulting spatial information, i.e., group different pc sets according to their position in space, and (3) its usefulness in the composer's creative process.

2. The DFT of pc sets in the construction of geometric spaces of harmonic representation

Yust (2015) has shown the potential of geometric Fourier spaces of harmonic representation for musical analysis and its syntactic interpretation. He states that such geometrical spaces provide access to a set of metaphors commonly used for the interpretation and study of music, and in particular, harmony. However, they may be put at the service of musical composition as a systematization of compositional principles and design behaviours. The idea of harmonic spaces assisting the composer, on which much of the motivation of this study is based, gained particular momentum with the computer-assisted composition practice and the recent advances on the application of the DFT of pc sets (LEWIN, 1959, 2001; QUINN, 2006-7; AMIOT, 2018).

2.1 The DFT of pc sets

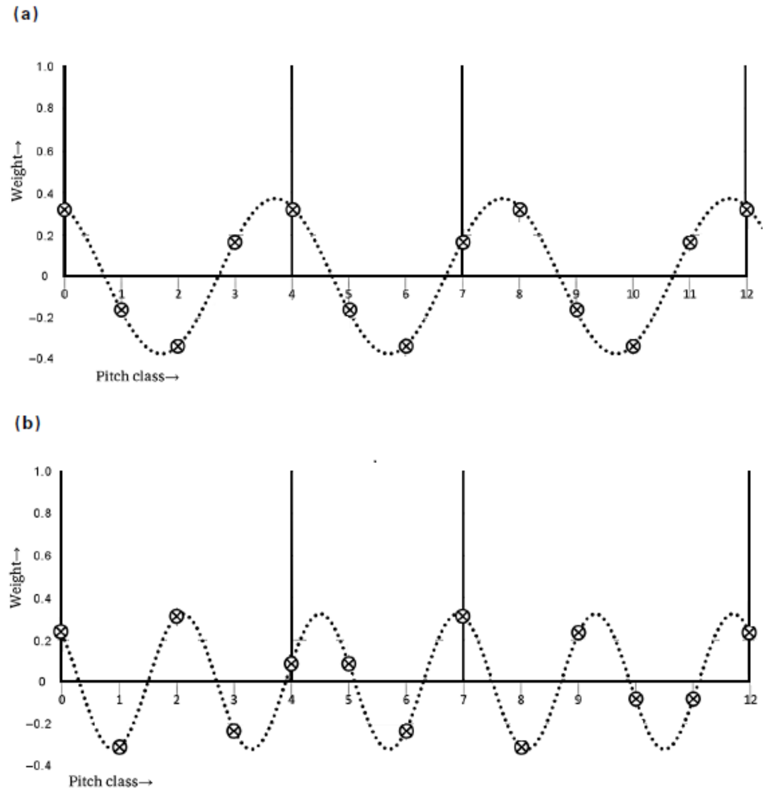

Figure 1: Two Fourier components for the C Major triad. In this example it is possible to see components $\hat{f}_3$ – in point (a) – and $\hat{f}_5$ – in point (b) – with three and five cycles per octave, respectively. Adapted from Yust (2015).

The DFT is traditionally known among musicians, researchers and experts as the mathematical process used to extract frequency information from audio signals. When applied to pc sets (LEWIN, 1959; 2001) and a 12-dimensional pc set distribution (0-11) where binary activation denotes the presence of a pc, 12-complex values (i.e., DFT coefficients) are computed. Excluding redundant information resulting from the symmetrical properties of this transform, the $1 \geq k \geq 6$ DFT coefficients remain. Complex values resulting from the DFT analysis can be further defined as magnitude and phase information (QUINN, 2006-7).

In Figure 1, we show the pitch classes 0, 4 and 7 of the C major chord as bold vertical lines and their DFT decomposition into two of the six nonsymmetrical coefficients. The notation $\hat{f}_0$, $\hat{f}_1$, ..., $\widehat{f}_{11}$ represent the Fourier coefficient whose index indicates the number of cycles per octave of their sinusoid. Additionally, the coefficients under analysis in Figure 1 have three and five peaks per octave, i.e., sinusoid a) has period = 3 and sinusoid b) period = 5.

2.2 Phase spaces

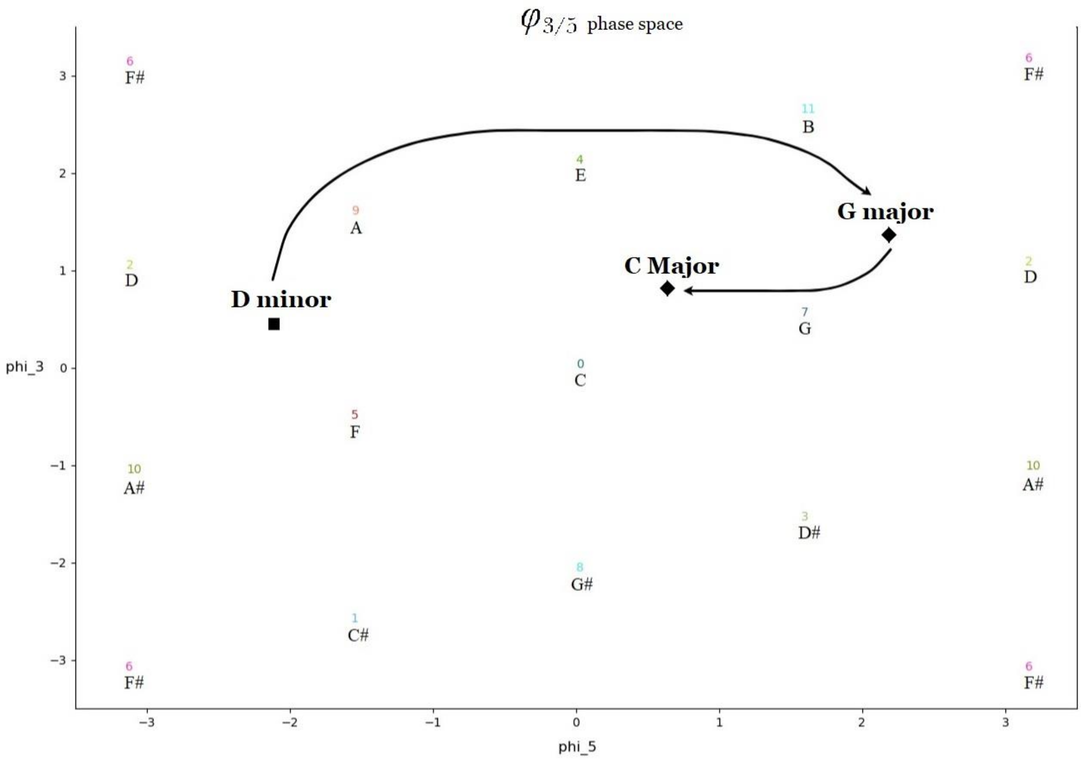

Figure 2: The phase space proposed by Yust (2015), the positioning of three different chords and prototypical trajectories between them. Adapted from Yust (2015).

The phase of a Fourier coefficient indicates the sinusoid offset from a reference point (along the horizontal axis in Figure 1). For the Fourier coefficient $\hat{f}_n$, its phase is represented by φ_n. Moreover, phases are cyclic and measured between $-\pi$ and π radians or between -180º and 180º. The two extremes are coincident in the phase space (AMIOT, 2012-3). On the other hand, phases of pc sets are dependent on transposition or inversion. Generally, they allow us to identify the positioning of a given pc set in the space as the combination of its component pc (YUST, 2015). The phases of the Fourier coefficients are at the basis of Yust's (2015) geometric space of harmonic representation.

Phase distances between pc sets are computed by their angle and have been explored in the literature as two-dimensional spaces whose axis represent the phase of two DFT coefficients. Yust (2015) adopts the $\hat{f}_3$ and $\hat{f}_5$ coefficients in the study of the tonal music of Schubert, thus resulting in a $\varphi_{3/5}$ phase space.

In Figure 2, we show the distance of multiple major and minor chords, as well as prototypical harmonic trajectories and the location of all pitch classes in the $\varphi_{3/5}$ phase space. This last feature is particularly relevant because the positioning of a pc set equates the linear interpolation of its component pitch classes (YUST, 2015). For example, C major chord results from the linear interpolation of the coordinates of the 0, 4, and 7 pitch class (i.e., the C, E, and G notes).

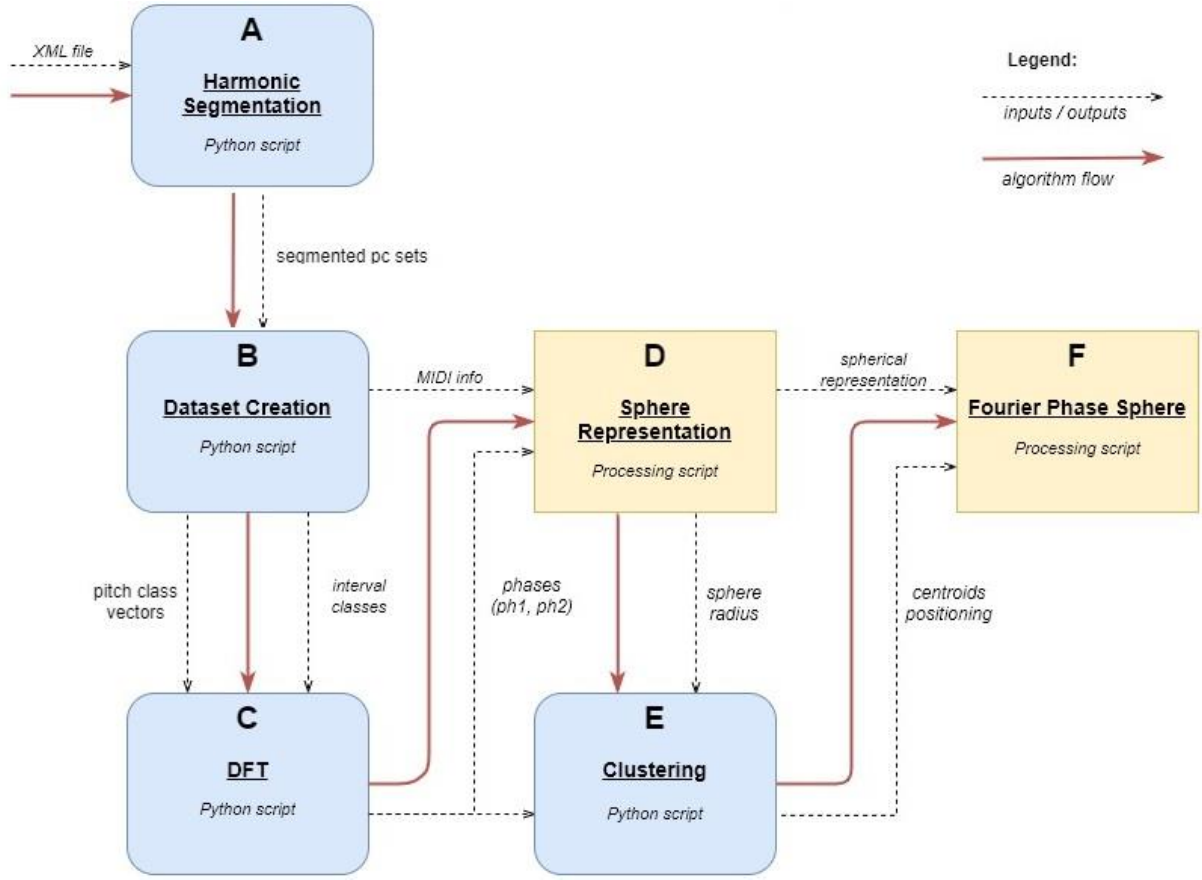

Figure 3: Modular architecture of the Fourier phase sphere. The system is composed by six distinct modules that allow the creation of a spherical DFT phase space. Adapted from Pereira (2020).

Additionally, it should be noted that the two-dimensional representation of Figure 2 does not capture well the cyclic nature of phase information. For example, the positioning of pitch class 6 (F♯) in the upper-left corner of Figure 2 is superimposed on the placement of the same pitch class in the upper-right corner. This idea is fundamental because the development of a computational tool that allows the automatic clustering of different pc sets according to their proximity in space, requires a different and more adapted representation.

3. The *Fourier phase sphere*

In this section, we discuss the implementation of a computer-assisted composition tool named *Fourier phase sphere* that takes advantage of a spherical space of harmonic representation for the DFT phase information. In Figure 3 we show the architecture of the computational system behind the space. Points in this space represent pc sets pre-defined by the composer computed as their DFT phases. To this end, we start by (1) encoding pc sets on an XML digital format; then we (2) compute the DFT of each pc set and their phase information in python libraries *music21* and *scipy*; which we use to (3) draw a spherical space of harmonic representation where each point represents a pc set in *Processing*[17] software; and, finally, (4) group these points into distinct clusters. Please refer to Pereira (2020) for a comprehensive description of the software components and their code.

3.1 Spherical space

The alternative 3-dimensional *Fourier phase sphere* representation proposed in our work, as opposed to the existing 2-dimensional representations (YUST, 2016), relies on the fact that phase information is cyclic and therefore it is measured in radians (varies in the range $[-\pi, \pi]$) (AMIOT, 2012-3). We represent the phase values from two coefficients of the DFT pc sets into three Cartesian coordinates. The mapping between the two-phase values and the three-dimensional sphere adopts the geographical

[17] The website of this software can be found at: https://processing.org/

concepts of latitude and longitude (PEREIRA, 2020). Latitude indicates how far north or south a point is and varies between -90º and 90º and longitude how far east or west a point is and varies between -180º and 180º. The concepts of latitude and longitude allow the identification of any point on the surface of the globe and, similarly, the values of two phases of DFT can plot any pc set on the surface of a sphere (PEREIRA, 2020). Figure 4 shows the resulting visual configuration of the *Fourier phase sphere*.

The *Fourier phase sphere* ultimately guarantees a topographically simpler solution than the torus suggested by Amiot (2012-3) but allows a better representation of the cyclical character in phase information of the DFT of pc sets than that discussed in Section 2. In particularly, it allows the adoption of the representation as a metrical space and group multiple pc sets into different harmonic clusters.

3.2 Clustering

Once the *Fourier phase sphere* is created for harmonic representation, we can group pc sets into several clusters (PEREIRA, 2020). The optimal number of clusters is estimated by an algorithm for a given configuration of multiple pc sets in the space as well as the positioning of each cluster centroid, followed by the clustering itself, i.e., the association between each pc set and its corresponding cluster.

In greater detail, we adopt the *k-means* algorithm to cluster the pc sets into a finite number of n clusters, represented by a centroid location, according to the spatial configuration of the pc sets. The optimal number of clusters is then computed using the elbow method (THORNDIKE, 1953). In Figure 4, we show the *Fourier phase sphere* where seven pc sets are represented. The optimal number of clusters is $n = 3$ defined by different colours in the representation. Each pc set is then assigned to the nearest centroid (defined as capital letter A, B and C).

3.3 Properties of the Fourier phase sphere – distances and circularity

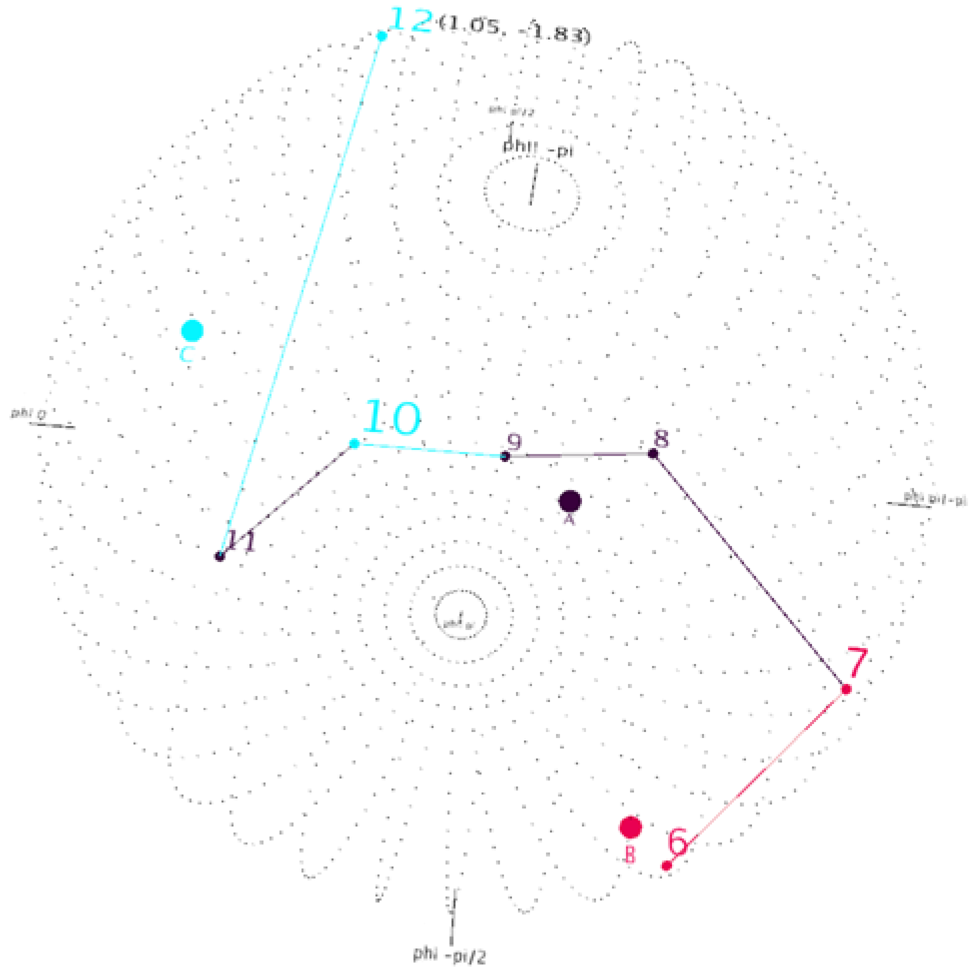

Figure 4: The Fourier phase sphere design and the positioning of seven distinct pc sets plotted in its surface. Purple, pink, and blue colours represent clusters A, B and C respectively (PEREIRA, 2020).

It is important to discuss the concept of distance (Euclidean distance) within a phase space and its musical interpretation. The Euclidean distance between pc sets in the spherical phase space is based on the idea of shared pitch classes, i.e., the closer two pc sets are, the more pc they share, since their positioning can be seen as the linear interpolation from the positioning of its component pitch classes (YUST, 2015). The clustering algorithm above, relies on this property to group pc sets that share a greater number of notes. Therefore, it is expected that pc sets belonging to distinct clusters tend to have fewer common pc.

On the other hand, a circular path in the *Fourier phase sphere* (see Figure 5), has been observed as an emergent behaviour across clusters in the space (PEREIRA, 2020). This path not only relates to the idea of functional syntax in tonal music, but also to the exploration of the total 12 pitch classes in post-tonal music. This musical interpretation can lead to the speculation of the underlying principles behind a generalized syntactical model for music composition and has been adopted in the definition of our computer-assisted composition strategies.

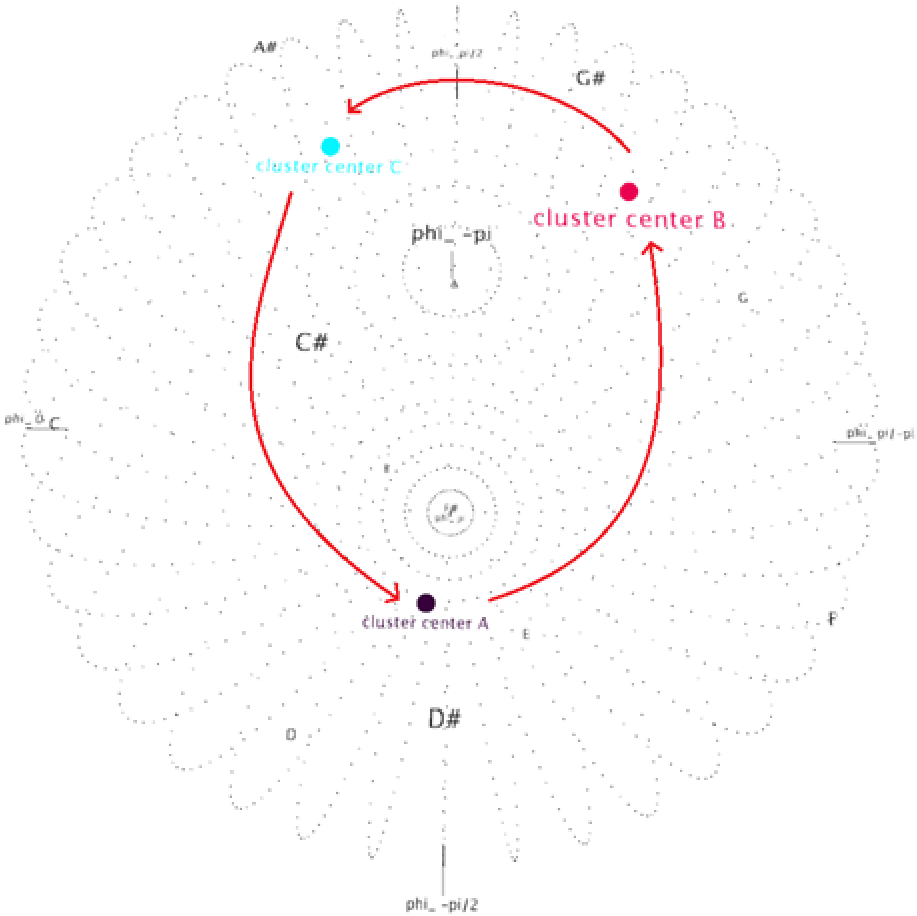

Figure 5: The placement of three cluster centroids and an example of a circular path [BCA] in the Fourier phase sphere used in the composition process of a sting quintet (PEREIRA, 2020).

4. Compositional method

It is undoubtedly important to understand whether the *Fourier phase sphere* described above can play a role in the composer's creative process, namely by establishing a temporal harmonic organization, i.e., to define sequences of pc sets. In greater detail, the main objective of the phase sphere is to assist the composer in defining a harmonic syntax that derives from a predefined harmonic lexicon from standard trajectories within the sphere. It should be noted the proposed tool does not aim at style imitation. Instead, it targets the writing of works with idiosyncratic compositional traits and musical language as the study case presented next, a work for string quintet (two violins, two violas and one cello). The phase sphere suggesting only sequences of pc sets between different clusters as a means for defining a harmonic syntax, whose effective choice, ultimately, is always at the composer's discretion, is the guarantee of his complete freedom in the creative process using the *Fourier phase sphere*.

4.1 Pc set selection

The property detailed in Section 3 on the circularity of the phase sphere exploring all twelve different pitch classes, was adopted as the trajectory for the harmonic path in the space. The composer, and first author, defined the lexicon, a collection of pc sets, and plotted it in the sphere so that it was separated into different clusters. Finally, with this information, it is up to the composer to make the compositional choices that he most pleases, always with the idea of circular harmonic sequences between clusters like those discussed in Figure 5.

The lexicon includes thirty-one pc sets of tri, tetra, penta, and heptachords as well as some dense pc sets with more than seven notes. This material was digitized as an XML format and algorithmically processed using the method presented in Section 3 prior to the representation in the *Fourier phase sphere*. One of the key points of this process is the selection of the Fourier coefficient phases to be mapped onto the sphere. To this end, it was determined the most prominent interval classes present in the harmonic data and, for this reason, the interval class 1 (referring to intervals of minor

2^{nd} or major 7^{th}) and 5 (for intervals of perfect 4^{th} or perfect 5^{th}) were more predominant, resulting, therefore, in a phase space $\varphi_{1/5}$.

Table 1: The correspondence between pc sets and their respective cluster.

Cluster	Number of the pc set
A	1, 4, 7, 14, 16, 20, 25, 27, 30, 31
B	2, 3, 5, 6, 8, 12, 13, 15, 17, 18, 24, 29
C	9, 10, 11, 19, 21, 22, 23, 26, 28

4.2 Definition of the harmonic clusters

Following the computation of the phase value for the $\hat{f}_1$ and $\hat{f}_5$ coefficients for all thirty-one pc sets chosen, the algorithm determined the definition of $n = 3$ cluters. The results of such grouping can be seen in Table 1.

Please note that these results are not representative of a harmonic sequence, but only of a set of harmonic information individually thought – a harmonic lexicon. Harmonic sequences are inferred through the *Fourier phase sphere*. From the data shown in Table 1, there were ten pc sets in cluster A, twelve in cluster B and nine in cluster C. Additionally, as shown in Figure 5, the positioning of the three cluster centroids in $\varphi_{1/5}$ requires that, for circular movements inside the sphere, sequences between clusters like [ABC], [BCA] or [CAB] can be chosen. These three options were the basis of the composition process that is best described in the next subsection to ensure harmonic sequences that use all pitch classes.

4.3 The creative process

For the composition process, we opted for the primordial use of sequences [BCA] in the definition of the harmony of the piece. It is recalled that this type of circular sequences within the *Fourier phase sphere* guarantees an element of harmonic syntax for the work because such sequences tend to systematically use almost or even the twelve pitch classes. For this, knowing the respective harmonic cluster of each pc set previously selected, the sequences of type [BCA] most appealing to the composer were chosen, still safeguarding that if they did not compositionally appeal him, other pc sets could be chosen. Figure 6 shows an example of a [BCA] sequence at the beginning of the work. In the last part of bar 2 is the pc set belonging to set class [016] consisting of the notes C, C♯ and F♯, which is numbered in the harmonic material with the number 2 and belongs to cluster B. This pc set is followed by two heptachords, numbered with numbers 19 and 20, and which belong respectively to clusters C and A, thereby conferring, at the beginning of the piece, the establishment of the sequence [BCA] for its harmonic organization. Throughout the writing process, the idea of creating harmonic sequences (mostly of the type [BCA]) was reinforced.

It was notorious that the *Fourier phase sphere* allowed a more direct and objective approach to the creative process because its use requires, from the outset, the creation of a harmonic lexicon by the composer and also because it infers a harmonic syntax through geometric elements in space. This fact is not negligible since the beginning of the creative process is, very often, the moment when the composer experiences any kind of creative block.

The proposed system was adopted with relative freedom, suggesting only harmonic progressions following a specific trajectory inside the phase sphere. The space and compositional method equally made possible the exploration of different musical textures. That happened because the tool does not interfere in compositional ideas such as articulation, timbre, or register, and, in that sense, it was possible to explore in the quintet distinct textures. On the other hand, the possible harmonic progressions suggested by the trajectories in the *Fourier phase sphere* were not always ideal for given points in the work. At times, when the composer sought to write contrasting sections to the existing material, the harmonic suggestions were not always possessing the desired level of contrast, thus forcing the search for new pc sets. Of course, this aspect can be rectified if, in the act of defining the harmonic lexicon to be used, the composer outlines different pc sets for each of the sections he has in mind and idealizes.

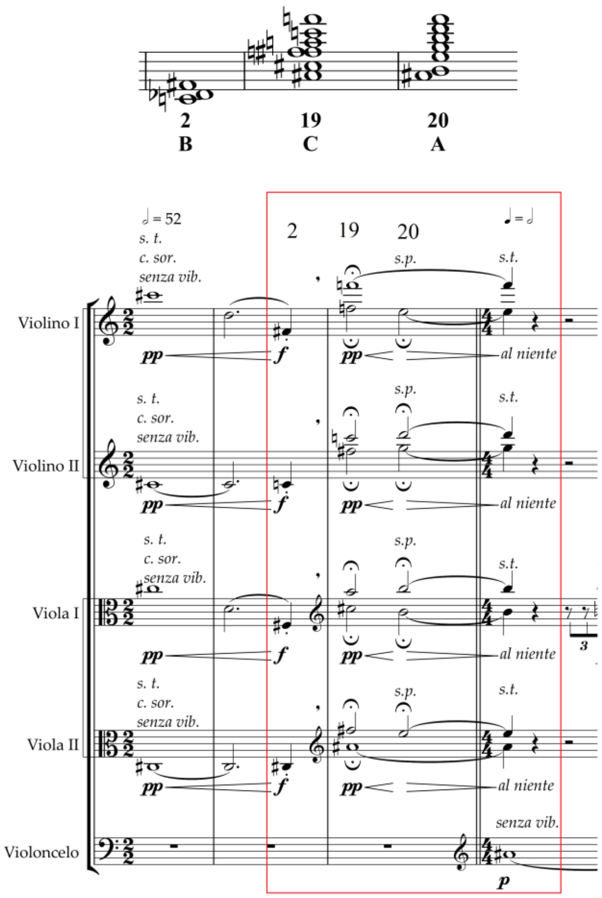

Figure 6: An example of the harmonic sequence written with pc sets 2, 19 and 20 that represents a [BCA] circular movement within the Fourier phase sphere (PEREIRA, 2020).

5. Discussion and final considerations

In Section 4, the adoption of the *Fourier phase sphere* as a tool for assisting the composer's creative process has been detailed across two main directives: 1) a harmonic lexicon grouped into different categories and 2) a harmonic syntax as circular paths across the phase space. This method can, to a certain extent, give greater harmonic coherence to the works by structuring a harmonic lexicon and syntax. This is based on the idea of trajectories within the phase space without, thereby, removing the freedom of the composer in his writing, and continuing to place his musical ideas and intuition at the centre. This approach facilitated the writing process, mainly by limiting the harmonic options available thus fostering greater fluidity to the creative act.

On the other hand, it was also noted the potential in identifying distinct clusters for multiple pc sets within the phase space. This process, considering the interpretation of the distance between points discussed previously, creates clusters whose pc sets generally share several pitch classes. This computation is only possible due to the development of a new geometric space of harmonic representation that represents pc sets as points on the surface of a sphere which confers greater representativeness to the circular aspects of the DFT phase information. With our approach, we extend the phase space proposed by Yust (2015), towards a meaningful metrical space for pitch classes.

The contributions of our work focus mainly on two different areas. On the one hand, from a technical point of view, spherical representation is a new geometric harmonic space that also allows the clustering of a myriad of harmonic material in a completely automatic way, and, from an artistic point of view, the tool assists the composer in defining a lexicon and harmonic syntax for his works.

Finally, there is certainly a long way to explore the usage of the *Fourier phase sphere* as an assistant to composition. First, it would be relevant to understand how other composers would use it in their creative process and, thus, one could reflect on what improvements are needed to better respond to their needs. One of these paths may be to increase the interactivity level of the application to allow the composer, in real-time, to select, for example, two pc sets, one for the beginning and the other to finish his harmonic sequence, and, automatically, the system proposes a variable number of harmonic progressions in which he can not only navigate and hear but also consult its interval content or its score.

References

AMIOT, Emmanuel. 2012-3. The Torii of Phases. *Mathematics and Computation in Music* - Lecture Notes in Computer Science, Berlin, v.7937, pp. 1-18.

AMIOT, Emmanuel. 2018. *Music through Fourier Space*: Discrete Fourier Transform in Music Theory. Springer.

BERNARDES, Gilberto; COCHARRO, Diogo; CAETANO, Marcelo; GUEDES, Carlos; DAVIES, Matthew. 2016. A Multi-Level Tonal Interval Space for Modelling Pitch Relatedness and Musical Consonance. *Journal of New Music Research*, v.45 n.4, pp. 281-294.

FERGUSON, Sean; PARNCUTT, Richard. 2004. Composing in the Flesh: Perceptually Informed Harmonic Syntax. *Proceedings of Sound and Music Computing*.

LEWIN, David. 1959. Re: Intervallic Relations between Two Collections of Notes. *Journal of Music Theory*, v.3 n.2, p. 298-301.

LEWIN, David. 2001. Special Cases of the Interval Function between Pitch-Class Sets X and Y. *Journal of Music Theory*, v.45 n.1, p. 1-29.

PEREIRA, Samuel. 2020. *A Sintaxe Harmónica Atonal no Espaço de Fourier*: Para Uma Representação Esférica das Fases. Dissertação (Mestrado em Multimédia), Universidade do Porto, Porto.

QUINN, Ian. 2006-7. General Equal-Tempered Harmony (in two parts). *Perspectives of New Music*, v.44 n.2; v.45 n.1, pp. 114-159; pp. 4-63.

ROHRMEIER, Martin; PEARCE, Marcus. 2018. Musical Syntax I: Theoretical Perspectives. In: *Springer Handbook of Systematic Musicology*, pp. 473-486.

THORNDIKE, Robert. 1953. Who Belong in the Family? *Psychometrika*, v.18 n.4, pp. 267–276.

YUST, Jason. 2015. Schuberts Harmonic Language and Fourier Phase Space. *Journal of Music Theory*, v.59 n.1, pp. 121–181.

YUST, Jason. 2016. Special Collections – Renewing Set Theory. *Journal of Music Theory*, v.60 n.2, pp. 213-262.

YUST, Jason. 2020. Generalized Tonnetze and Zeitnetze, and the Topology of Music Concepts. *Journal of Mathematics and Music*, pp. 1-35.