Hauer Tropes as Clockface Diagrams: Dialogs among Forte, Carter and Ŝedivý

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Abstract: This paper puts into dialogue the concept of tropes, idealized by Joseph Matthias Hauer - and explored in detail by Dominik Ŝedivý (2011), - with the hexachordal combinatoriality presented by Allen Forte (1973), and later by Elliott Carter (2002). Starting with an inductive introduction to permutation classes, the main purpose is to present an integrated way to look at the Hauer tropes through its relation with Forte/Carter numbers, modulo 12 clockface diagrams and some reverse engeneering of 12-tone rows.
Keywords: Hauer tropes. Hexachordal combinatoriality. Second Viennese School.

Tropos de Hauer como Diagramas *clockface*: diálogos entre Forte, Carter e Ŝedivý

Resumo: Este artigo coloca em diálogo o conceito de tropos, idealizado por Joseph Matthias Hauer – e explorado em detalhes por Dominik Ŝedivý (2011), - com a combinatorialidade hexacordal apresentada por Allen Forte (1973), e posteriormente por Elliott Carter (2002). Começando com uma introdução indutiva às classes de permutação, o objetivo principal é apresentar uma maneira integrada de entendimento dos tropos de Hauer por meio de sua relação com os números de Forte/Carter, os diagramas circulares em modulo 12 e alguma engenharia reversa de séries de 12 sons.
Palavras-chave: Tropos de Hauer. Combinatorialidade Hexacordal. Segunda Escola Vienense.

1. Introduction

It´s amazing how Joseph Matthias Hauer (1883-1959) had a great intuition about a system to organize all the possible 12 tone rows into 44 tropes or "permutation classes" many years before the musical set theory established the standards for the enumeration of set classes. According to Gustafson:

> What Hauer did, in essence, was to incorporate all the possibilities of tonerow construction (479,00I,600) into forty-four constellations or tropes, each trope, then, representing a multiplicity of possibilities. Trope number one, for example, includes all rows or series in which (in any order) occur in the first half of the row and occur in the second half. (GUSTAFSON, 1979, p. 23)

In 1973, Allen Forte did the summary of all harmonic structures available in the 12-tone tempered system (FORTE, 1973, p. 179-181). Alongside with Forte, Elliott Carter, in his *Harmony Book* (2002), also carried out an equivalent exhaustive study of the combinations between sets and their complements. Carter used his own naming convention of the sets, but choose to make a consensus between his particular nomenclature and that used by Forte (CARTER, 2002, pp. 23-26).

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In the path to a better understanding of the harmonic materials derived from pitch class set theory, the Hauer system of tropes provides a comprehensive yet summarized insigth about how the classes of permutations can be used as melodic and harmonic sources in the 12-tone universe. According to Ŝedivý:

> The 44 tropes are a system that allows one to gain a thorough overview of all existing twelve-tone rows (479,00I,600) and to classify them with regard to common properties. Generally, a twelve-note set is divided into two complementary groups of six tones. These two groups are then examined with regard to their intrinsic and mutual interval relation. Since it is the relations of intervals that counts, the particular order of tones within each hexachord is as irrelevant as the order of the two complementary hexachords within a whole trope. Any existent six-note set can be associated with a particular half of a trope. (ŜEDIVÝ, 2011, p. 83)

Before exploring the Hauer system, let´s make an introdutory inductive exploration of an experimental "mini system" of "seeds".

2. An inductive "mini system" of "seeds" as introduction

For the sake of simplicity, let´s work with a universe of just 6 elements, since the total permutations of 6 elements is 6! = 720, a small number compared to the permutations of 12 elements: 12! = 479001600. In the next section there is a sequence of steps to make the reasoning as clear as possible. Let´s work out the permutation classes in our 6-element universe with these steps in mind:

1. Let´s take the set (0 1 2 3 4 5) as our entire universe of discrete elements;
2. From this universe, we want to build rows with 6-elements without repetition and exhaustive, i.e., each permutation contains all 6 elements, likewise the concept of aggregate, used in 12-tone music theory;
3. Every 6-element nonrepeating ordered row is divided into 2 mutually complementary 3-element sets, like (0 1 2) (3 4 5) or (2 5 1) (3 4 0);
4. Based in the premises above, since we are dealing with 6-element sets without repetition, choosing just the first 3-element set implies necessarily that the remaining (complementary) 3-element will comprise the whole 6-element unordered set;
5. If all the 720 6-element permutations are comprised of mutually exclusive 3-element unordered sets and it´s enough to define just the first 3-element to get the total unordered set, then a smaller number of 3-element permutations (just 120, in total) potencially contains all the possible 6-elements permutations if we just project the complementary elements. Basically, the same expressed in the previous item;

6. Below, we have a list with all the 120 3-element combinations without repeating elements from our 6-element universe (0, 1, 2, 3, 4, 5):

012 021 102 120 201 210 **013** 031 103 130 301 310 **014** 041 104 140 401 410
015 051 105 150 501 510 **023** 032 230 203 302 320 **024** 042 204 240 402 420
025 052 205 250 502 520 **034** 043 304 340 403 430 **035** 053 305 350 503 530
045 405 450 504 540 054 **125** 152 215 251 512 521 **123** 132 213 231 312 321
124 142 214 241 412 421 **135** 153 315 351 513 531 **134** 143 314 341 413 431
145 154 415 451 514 541 **234** 243 324 342 423 432 **235** 253 325 352 523 532
245 254 425 452 524 542 **345** 354 435 453 534 543

Example 1: Three-element combinations without repeating elementsof the set (0,1,2,3,4,5).

7. If we take the above combinations in ascendent order and exclude the repetitions, then the 3-element sets are reduced to the list below, comprised of 10 mutually exclusive 3-element sets:

(012,345),(013,245),(014,235),(015,234),(023,145)
(024,135),(025,134),(034,125),(035,124),(045,123)

Example 2: Combinations of (0,1,2,3,4,5), ascendent order, without repetitions and as mutually exclusive 3-element sets**.**

The 10 mutually exclusive 3-element unordered sets listed above are the basic “seeds” for all the 720 ordered permutations of 6-elements;

8. “Seeds” are the group permutation classes. For example, from the unordered “seed” (0 1 2), we get the following ordered sets: (0 1 2) (0 2 1) (1 0 2) (1 2 0) (2 0 1) and (2 1 0). These seeds are in bold in the example 6.
9. To find the “seeds”, the permutations classes, the order is not important. But, in terms of musical creation, the content is very important. For the sake of clarity, let´s first associate the numbers with some words comprising some specific properties that can be subsumed to classes.

0	dog	Animals
1	cat	
2	bird	
3	bus	Vehicles
4	car	
5	bike	

Example 3: Association of numbers to words and words to classes.

Assuming the non algebric properties expressed by the associated words, the numbers (0, 1 2) can be generally subsumed to the class of "animals" and the numbers (3 4 5) can be subsumed to the class of "vehicles".

10. According to this class association, and according with the idea of a limited number of "seeds" from wich we derive all the combinations, there are limited possibilities to arrange the 10 "seeds" of unordered combinations:

(0 1 2) (3 4 5)	3 Animals / 3 Vehicles
(0 1 3) (2 4 5)	2 animals 1 vehicle / 2 vehicles 1 animal
(0 1 4) (2 3 5)	2 animals 1 vehicle / 2 vehicles 1 animal
(0 1 5) (2 3 4)	2 animals 1 vehicle / 2 vehicles 1 animal
(0 2 3) (1 4 5)	2 animals 1 vehicle / 2 vehicles 1 animal
(0 2 4) (1 3 5)	2 animals 1 vehicle / 2 vehicles 1 animal
(0 2 5) (1 3 4)	2 animals 1 vehicle / 2 vehicles 1 animal
(0 3 4) (1 2 5)	2 vehicles 1 animal / 2 animals 1 vehicle
(0 3 5) (1 2 4)	2 vehicles 1 animal / 2 animals 1 vehicle
(0 4 5) (1 2 3)	2 vehicles 1 animal / 2 animals 1 vehicle

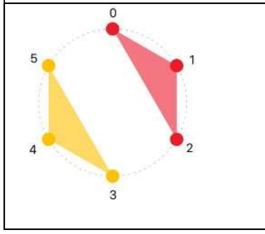
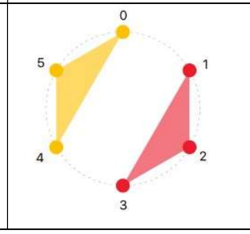
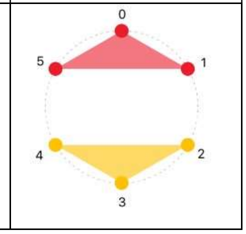
Example 4: Limited possibilities of association between "seeds" and classes.

11. If we disregard the order of elements and also disregard the order of the 3element mutually exclusive sets, we end with just 2 classes of permutations based on the properties of the words;

3 animals	3 vehicles
2 animals, 1 vehicle *or reversed:* 2 vehicles, 1 animal	2 vehicles, 1 animal *or reversed:* 2 animals, 1 vehicle

Example 5: Permutation classes expressed as propoerties of words.

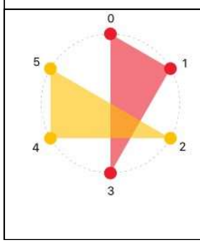
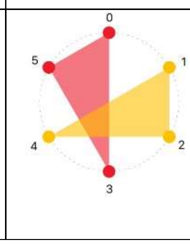
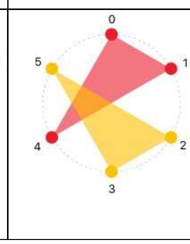
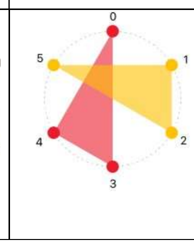
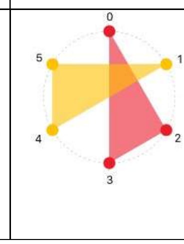
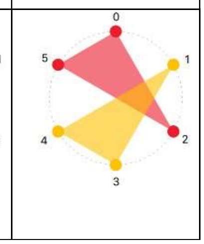
12. Instead of the "extra-algebraic" properties, let´s introduce notions applied to musical set theory, as the transposition (rotation) and inversion ("flipping", mirroring) equivalence in a modular space, using necklace/clock representation, similar to pitch class theory. Now we can create mathematical classes of properties, similar to "animals" and "vehicles", to subsume the elements of our universe of 6 elements. For example, the seeds (0 1 2) (3 4 5), (1 2 3) (0 4 5) and (0 1 5) (2 3 4) are equivalent, since the result are the same geometric figure:

(0 1 2) (3 4 5)	(1 2 3) (0 4 5)	(0 1 5) (2 3 4)
		

Example 6: Equivalent geometry shapes from different permutations[1].

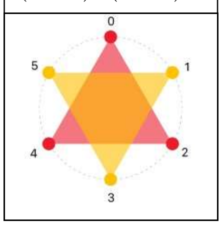
All the above examples and shapes could be obtained from operations (rotation and inversion) applyed to the same "seed" (0 1 2) (3 4 5).

The same occur withe the "seeds" (0 1 3) (2 4 5), (0 3 5) (1 2 4), (0 1 4) (2 3 5), (0 3 4) (1 2 5), (0 2 3) (1 4 5), (0 2 5) (1 3 4), wich produces the same geometric shape, in different rotations and mirrorings, like shown below:

(0 1 3) (2 4 5)	(0 3 5) (1 2 4)	(0 1 4) (2 3 5)	(0 3 4) (1 2 5)	(0 2 3) (1 4 5)	(0 2 5) (1 3 4)
					

Example 7: Different rotations and mirroring from the (0 1 3) (2 4 5) seed.

Similarly, like the previous example, all this shapes can be derived by rotation or mirroring only from the seed (0 1 3) (2 4 5). Lastly, the seed (0 2 4) (1 3 5) produces the third shape, shown below.

(0 2 4) (1 3 5)


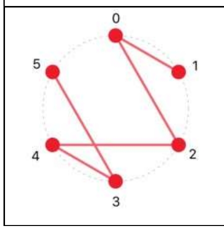
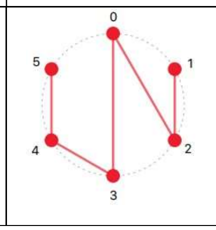
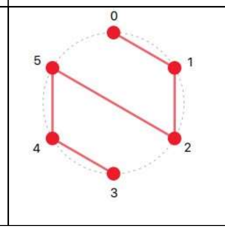
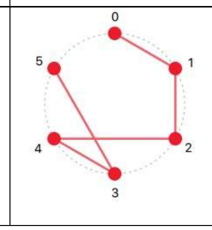
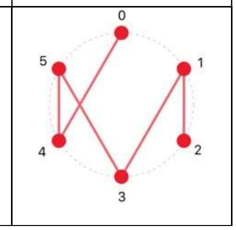
Example 8: Shapes resulting from the (0 2 4) (1 3 5) seed.

13. From the standpoint of summarizing, we can even say that all that initial 10 seeds can be reduced to only the following three: (0 2 4) (1 3 5), (0 1 3) (2 4 5) and (0 1 2) (3 4 5);
14. For the sake of a minimum expenditure of means, as we are dealing with complementary 6-element sets, it´s enough to say that all the "seeds" (unordered sets) can be reduced to just (0 1 2), (0 1 3) and (0 2 4). The reason

[1] All the clockface diagrams were made in OPUSMODUS software. https://opusmodus.com/

is that when choosing one of the 3-element unordered set, we are also choosing its complementary set, since we are dealing with exhaustive 6-element sets. Choosing (0 1 2) means that the complete seed will be (0 1 2) (3 4 5), and so on;

15. When finding the "parent" seeds, we stated that order is irrelevant, but each seed generate a number of ordered sets, or "childs". For example, the seed (0 1 2) (3 4 5), or simply (0 1 2) and its complementary set, can bring these (and many others) ordered sets:

(1 0 2 4 3 5)	(1 2 0, 3 4 5)	(0 1 2, 5 4 3)	(0 1 2, 5 3 4)	**(2 1 3 5 4 0)**
				

Example 9: Ordered sets from the same (0 1 2) (3 4 5) "parent" "seed".

16. The ordered sets derived from this very same parent seed (0 1 2, 3 4 5) also creates some "order classes" of its own, like the examples in bold above, wich comprises a rotation of the same shape.
17. "Order classes" can be conceptualized independently, forming properties that subsume some sets like: numbers in consecutive order, every other number, every-n numbers, descendent odd numbers, or any other conceivable order. Despite that, actually they are consequence of the rotation or mirroring of any ordered set. Classes of order pertain to specific parent seeds, since those classes are the actual permutations of the seeds that share properties, like rotation or mirror equivalence.

3. Hauer tropes and Carter hexachord families

From this perspective, we can understand the Hauer tropes as the exact same concept of "seeds" explained before but, instead of the 6-element universe with 3-element "mini system of seeds", we now have a 12-element universe with 6-element "seeds", or tropes. Let´s remind that, likewise our "mini system" example in the beggining, each trope is an unordered "seed" for many ordered rows. In his Harmony Book (p. 40), Carter made an ordering that organizes the hexachords into four families, designated here by the letters [A], [B], [C], [D].

> Carter grouped 6-note chords into four categories: nos. 1-7 are non-invertible (with the exception of number 3) and self-complementary by transposition; nos. 8-20 are invertible and self-complementary by inversion; numbers 21-34 are non-invertible , in which the compliment is another (not invertible) 6-note chord that shares the same total interval content; and, numbers 35-50

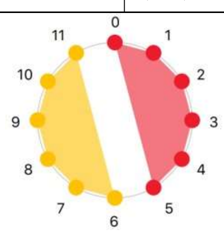
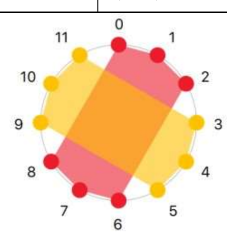
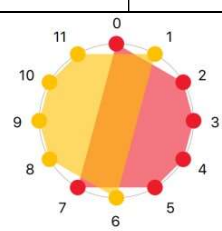
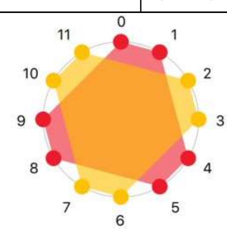
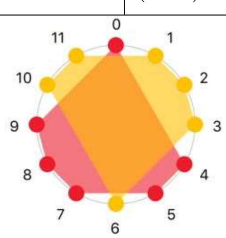
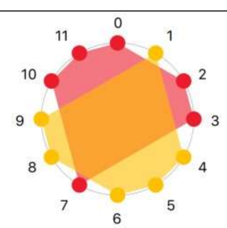
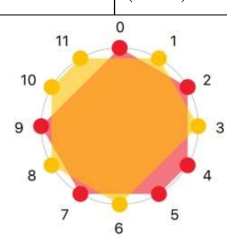
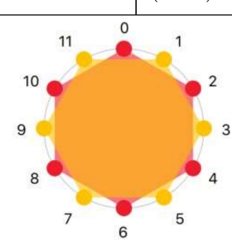
> are invertible, in which the compliment another (invertible) 6-note chord with the same total interval content. Carter conveniently arranges chords nos. 35-50 as complementary pairs the show which chords share the same interval content. (CARTER, 2002, p.40)

These categories are of great value to understand the remarks Ŝedivý make about each of the tropes in his book.

The next figures are a combination of the 44 Hauer tropes alongside with its correspondent Forte/Carter numbers and clockface diagrams, divided in the four categories provided by Carter. These diagrams can give a better insight about the nature and classification of Hauer tropes in relation to its properties. A deeper discussion of each category can be done in another oportunity, but I hope this can clarify the main properties of the tropes, regarding to its hexachordal content.

Group [A] – Self-complementary by Transposition (rotation) and noninvertable Tropes 1, 4, 10, 17, 28, 29, 41, 44. Hauer tropes in **Bold-Face** Algarisms – Forte Numbers in parenthesis.

Forte	6-35	6-20	6-14 [2]	6-1	6-8	6-32	6-7
Carter	1	2	3	4	5	6	7

1	(6-1)	**4**	(6-7)	**10**	(6-8)	**17**	(6-20)
							
28-29	(6-14)			**41**	(6-32)	**44**	(6-35)
							

The first group is comprised by the hexachords that are the self-complementary by transposition (or rotation, in spatial terms) and non-invertable, represented by tropes 1, 4, 10, 17, 28, 29, 41 and 44.

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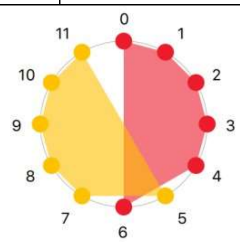
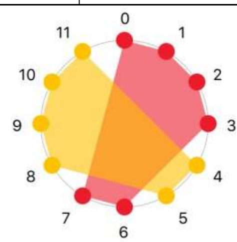
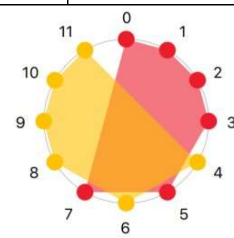
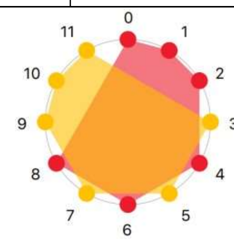
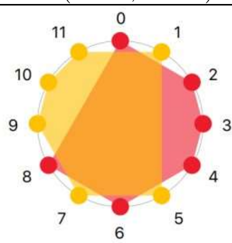
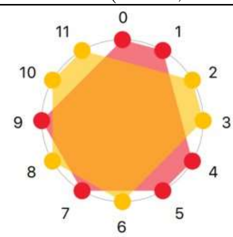
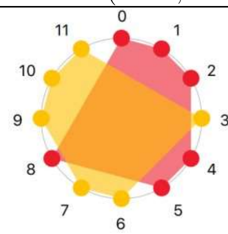
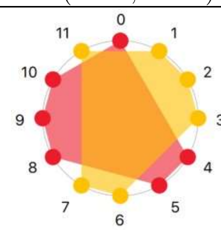
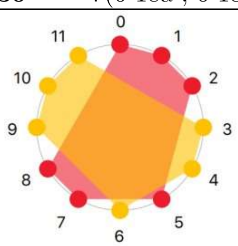
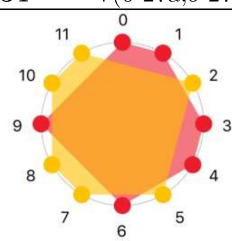
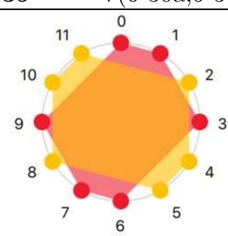
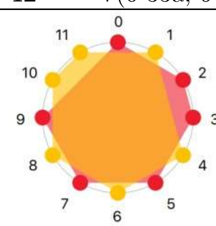
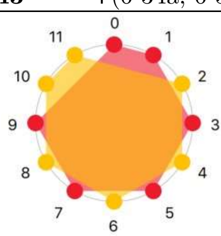
[2] All the chords in this category don´t have inverted forms in the Forte table, except the 6-14, wich have both original (6-14a) and inverted form (6-14b) in Forte classification. However, Carter groups this 614 chords among the non-invertible chords because it is self-complementary by transposition. For purposes of organizing his catalog, Carter prioritizes the property of self-complementary over invertibility.

Group [B] – Self-Complementary by Inversion (mirror) and invertable
Tropes 2 3 9 11 12 13 26 27 30 34 39 42 43

Forte	6-31	6-34	6-22	6-16	6-21	6-15	6-27	6-30	6-5	6-18	6-33	6-2	6-9
Carter	8	9	10	11	12	13	14	15	16	17	18	19	20

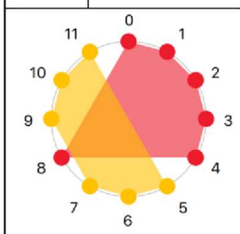
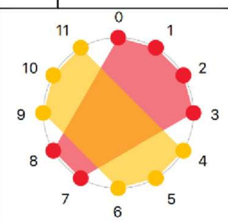
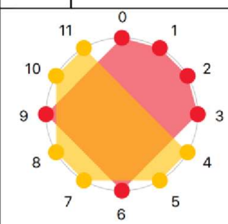
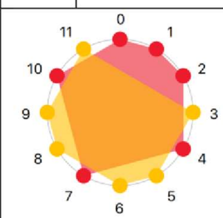
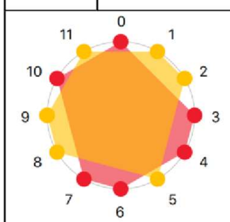
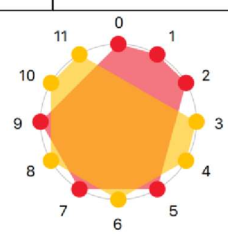
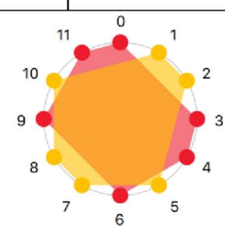
2 (6-2a, 6-2b)	**3** (6-5a, 6-5b)	**9** (6-9a, 6-9b)	**11** (6-22a,6-22b)
			
12 (6-21a, 6-21b)	**13** (6-31a,6-31b)	**26** (6-15a, 6-15b)	**27** (6-16a, 6-16b)
			
30 (6-18a , 6-18b)	**34** (6-27a,6-27b)	**39** (6-30a,6-30b)	**42** (6-33a, 6-33b)
			
43 (6-34a, 6-34b)			
			

In this category, the hexachord is complemented by its own inversion form.

The first and second hexachords have the same Forte number and interval vector.

Group [C] – Non-invertible Z-related - Complementary by Z-Relation
Tropes 7 8 14 32 35 36 40

Forte	6-z28	6-z37	6-z48	6-z23	6-z13	6-z50	6-z6
Forte (Compl)	6-z49	6-z4	6-z26	6-z45	6-z42	6-z29	6-z38
Carter	21	23	25	27	29	31	33
Carter (Compl)	22	24	26	28	30	32	34

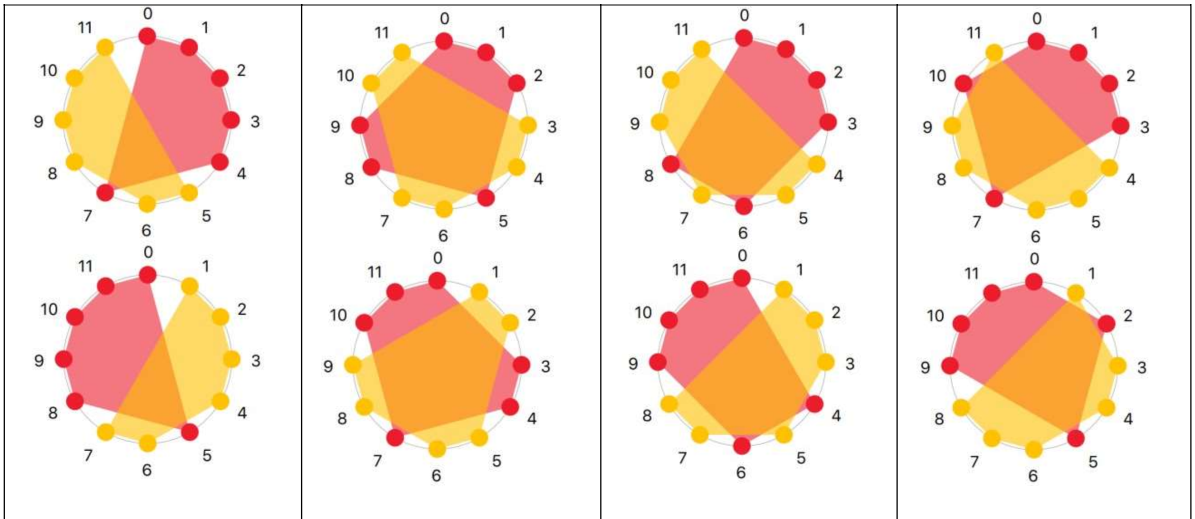
7	(6-z37, 6-z4)	**8**	(6 z-38, 6-z6)	**14**	(6 z-42, 6-z13)	**32**	(6 z-45, 6-z23)
							
35	(6 z-49, 6-z28)	**36**	(6 z-48, 6-z26)	**40**	(6 z-29, 6-z50)		
							

These hexachords are non-invertible, in which the compliment is another (not invertible) 6-note chord that shares the same total interval content (Carter, 2002, p. 40).

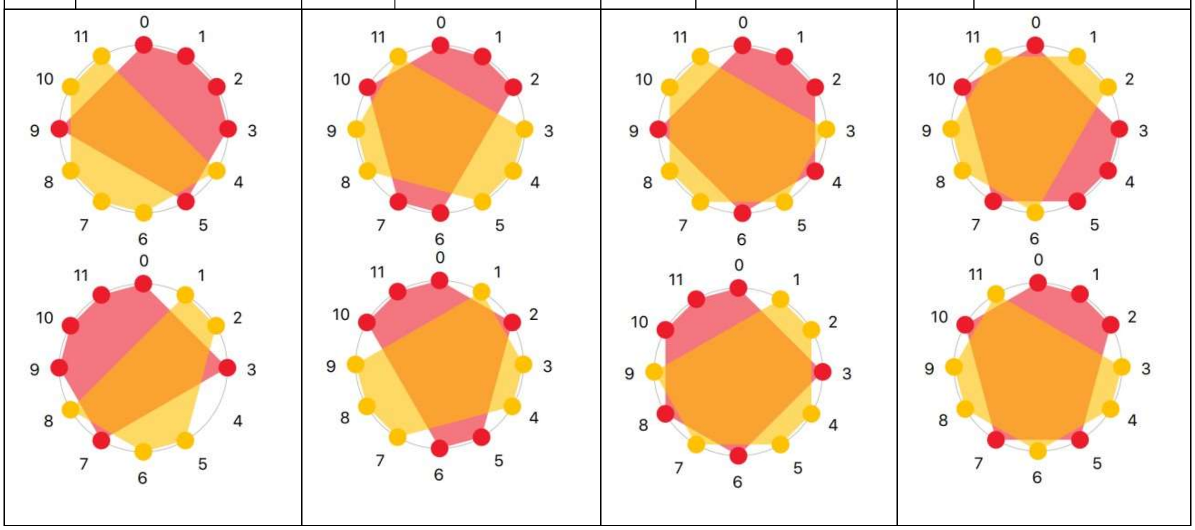
Group [D] Z-related invertable. Complementary by Z-Relation (crossing original and inverted form)
Tropes 6 15 18 20 21 25 33 37

Forte	6-z17	6-z19	6-z24	6-z39	6-z25	6-z41	6-z11	6-z3
Forte (Compl)	6-z43	6-z44	6-z16	6-z10	6-z47	6-z12	6-z40	6-z36
Carter	35	37	39	41	43	45	47	49
Carter (Compl)	36	38	40	42	44	46	48	50

5	(6-z36a,6-z3b)	**15**	(6-z44b, 6-z19)	**18**	(6-z41, 6-z12b)	**20**	(6-z40b, 6-z11)
6	(6-z36b, 6-z3)	**16**		**22**		**23**	



19 21	(6-z39a,6-z10a)	**24 25**	(6-z17, 6-z43b)	**31 33**	(6-z46b, 6-z24)	**37 38**	(6-z47,6-z25b)



The hexachords of the group [D] are invertible, such that their complement is another invertible hexachord with the same interval content. The hexachords in the same column below form an aggregate.

Forte	6-z17	6-z19	6-z24	6-z39	6-z25	6-z41	6-z11	6-z3
Forte (Compl)	6-z43	6-z44	6-z16	6-z10	6-z47	6-z12	6-z40	6-z36
Carter	35	37	39	41	43	45	47	49
Carter (Compl)	36	38	40	42	44	46	48	50

Example 10: Hexachords of the group D.

However, it is important to point out that the complementarity occurs in a crossed way: for example, the original form of the 6-z17, which we distinguish here by the letter "a" after the set, has as complement the inverted form (distinguished here by the letter "b", after the set) of the 6-z43. In this crossing way, the 6-z43b is complementary to 6-z17a. We can unfold the previous table as follows:

Forte	6-z17**a**	6-z19**a**	6-z24**a**	**6-z39a** [3]	6-z25**b**	6-z41**a**	6-z11**a**	6-z3**a**
Forte (Compl)	6-z43**b**	6-z44**b**	6-z46**b**	**6-z10a**	6-z47**a**	6-z12**b**	6-z40**b**	6-z36**b**
Forte	6-z17**b**	6-z19**b**	6-z24**b**	**6-z39b**	6-z25**a**	6-z41**b**	6-z11**b**	6-z3**b**
Forte (Compl)	6-z43**a**	6-z44**a**	6-z46**a**	**6-z10b**	6-z47**b**	6-z12**a**	6-z40**a**	6-z36**a**

Example 11: Crossing complementarity of some Z-related sets.

However, there is an exception to this cross relation: the set 6-z39a (**bold**, in the above table) is complementary to 6-z10a (original form with original form) and 6z39b is complementary to 6-z10b (inverted form with inverted form).

Trope Mindset

Beyond the set theory idea of listing complementary hexachords, the system of tropes is also a way to describe the interval content of the hexachord in such a way that can be easier to grasp its sonority and possible uses for composition. According to Ŝedivý:

> (...) a trope is neither a hexatonic scale nor a chord. Likewise, it is neither a pitch-class set, nor an interval-class set. A trope is a framework of contextual interval relations. Although it can be very well used for both, its primary purpose is composition, not analysis. (ŜEDIVÝ, 2011, p. 83)

For this reason, the hexachords provenient from the tropes are broken in several ways to make its properties more evident. The trope system structure and notation devised by Hauer helps the awareness of this content, also explored deeply in Ŝedivý´s book. [4] As Ŝedivý points out, from the information provided by trope organization, a composer may decide in a freer manner the way the materials are going to be arranged with respect to the desired manipulation:

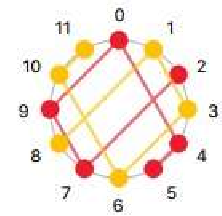
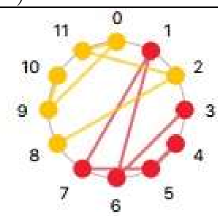
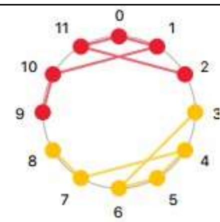
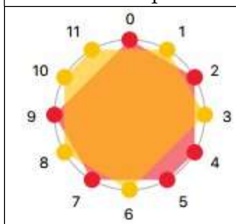
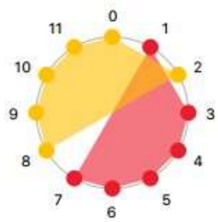
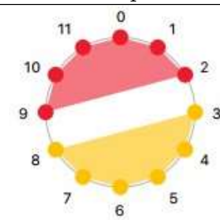
> For example, a composer wants to create two identical hexachords out of trope 36. The transposition interval is chosen to be a major second. With this information it becomes clear that the transposed hexachords can only be from trope 17, because this is the only trope that allows a transposition by whole tone. (ŜEDIVÝ, 2011, p. 99)

If we take a look on the geometric representation of trope 17, it´s easy to see that its second hexachord´s shape results from a rotation of two increments or, in musical terms, a transposition by major second of the first hexachord, while the

[3] Exception in the crossing relation.

[4] See Ŝedivý 2011, Appendix.

hexachords of trope 36 are not in a relation of transposition nor inversion (Carter group C, Noninvertible Z-related hexachords). So, this awareness is crucial to the deliberate manipulation of the material by the composer. In the next figure there are some examples of reverse engineering of some 12-tone rows.

Berg: Lyric Suite (5 4 0 9 7 2) (8 1 3 6 10 11)	Schoenberg: Op. 25 (4 5 7 1 6 3) (8 2 11 0 9 10)	Webern: Opus 30 (9 10 1 0 11 2) (3 6 5 4 7 8)
		
Parent Trope: 41	Parent Trope: 2	Parent Trope: 1
		

Example 12: Reverse engineering of three 12-tone rows from the Second Viennese School.

Conclusions

Nowadays, after the advances in the discussion of musical set theory, it´s possible to articulate the legacy of Forte and Carter with the pioneering Hauer (thanks to the great enlightment provided by Ŝedivý) and I hope that this paper can serve as a gentle introduction to this subject, providing some dialog among the literature. The Carter families and its clockface representations articulates the connection with the respective Hauer tropes and the concept of 12-tone hexachordal combinatoriality. In this way, it's much easier to understand the nature of each 12-tone row and to which Carter Family and Hauer trope their hexachords pertain. Doing the reverse engeneering of the tone rows by this kind of classification can help us to understand some compositional choices for some specific 12-tone rows and other choices related to some deliberate transformation applyed to these tone rows, based on nature and properties of the combinatoriality and content of the hexachords.

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