Non-Ptolemaic Nature of Musical Space

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Abstract: *This study constructs a discrete voice-leading orbifold for 12-TET, providing quantitative tools parallel to continuous manifold model developed by Tymoczko. The Ptolemaic inequality proves pitch-class and voice-leading spaces globally non-Ptolemaic, confirming intrinsic musical curvature incompatible with Euclidean embedding. Tonal-to-atonal shift emerges as geometric exploration from local Euclidean regions (diatonicism) to global non-Euclidean harmony (chromaticism).*

Keywords: *Voice-leading. Cayley graph. Ptolemaic inequality. Schoenberg embeddability.*

1. INTRODUCTION AND HISTORICAL CONTEXT

The Circle of Fifths (CoF), a cornerstone of Western music theory since the Baroque era, organizes the twelve pitch classes of equal temperament into a cycle of perfect fifths. This structure admits rigorous analysis via topology, complex analysis, and metric geometry. Historical developments include Hugo Riemann's dualist system relating triads through transformations [3], revived by David Lewin in neo-Riemannian theory [1, 2], and Richard Cohn's connections to voice-leading parsimony [11, 12]. Dmitri Tymoczko's *Geometry of Music* [13, 5] models chords as points in the orbifold voice-leading space T^n/S_n, the quotient of the n-torus T^n (pitch classes modulo 12) by the symmetric group S_n (permutations and octave equivalence). This framework unifies harmony and voice-leading, with singularities at symmetric chords (repeated notes, rotational symmetry, or evenly spaced divisions). Callender, Quinn, and Tymoczko extended this to generalized voice-leading spaces [4, 25]. Cyclic groups and Cayley graphs provide geometric representations of chords, scales, and the CoF [17, 18, 19, 21, 22, 23, 24]. Computational tools like persistent homology have been applied [7, 8]. Over 150 years, efforts to formalize chord space topologies aim to enhance voice-leading theory [10]. While similarity metrics exist [9, 25], topological models excel in analysis but face limitations for composition due to their abstract focus on pitch classes. Our work builds upon previous approaches specifically investigating the *Ptolemaic property* in musical spaces. To our knowledge, Ptolemaicity has never been studied in this context. We use this metric diagnostic to quantitatively demonstrate the non-Euclidean nature of Western 12-TET harmony, providing rigorous proof of its intrinsic discrete curvature through the failure of Schoenberg embeddability [14]. Furthermore, our analysis takes place within the discrete environment represented by the discrete equivalent of Tymoczko's continuous orbifold. In practice, Western music uses the 12-tone equal temperament system. This means we work in the discrete pitch-class space $\mathbb{Z}_{12}$ rather than the continuous space $\mathbb{R}/\mathbb{Z}$. Tymoczko's work addresses this by noting that a related result covers discrete pitch-class spaces, but the primary framework remains continuous. Shifting the analysis from continuous differential geometry to metric geometry based on graphs, combinatorics, and discrete metric spaces, perfectly suited to the combinatorial nature of Western harmony.

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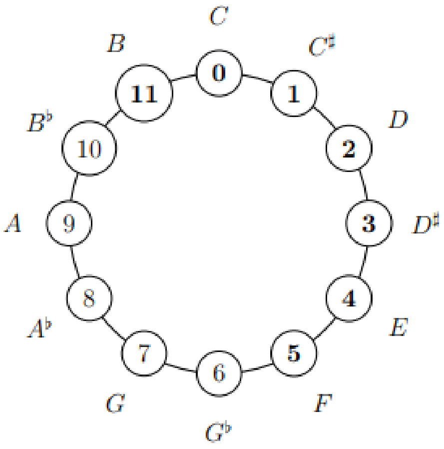


Figure 1: *The Tone-Clock Diagram: A geometric representation of the chromatic circle* $C_{12}(1, 11)$.

2. Construction of the chord space

The foundation of Western musical theory lies in the 12-tone equal temperament (12-TET), a system that discretizes the continuous frequency spectrum into 12 distinct pitch classes. Mathematically, this is modeled by the finite cyclic group $\mathbb{Z}_{12} = \{0, 1, \ldots, 11\}$ under addition modulo 12. The relationship between these notes is logarithmic; specifically, the frequency ratio between adjacent semitones is defined as $2^{1/12}$. This cyclic nature allows us to apply *Graph Theory* to visualize and analyze musical structures, where notes are vertices and intervals are edges. A primary tool for this modeling is the Cayley Graph, denoted as $Cay(\mathbb{Z}_n, S)$, where S is the generating set. The choice of S determines the musical context:

- **Chromatic Voice Leading:** Defined by $S = \{\pm 1\}$. This generates a directed n-cycle graph where edges represent movement by a single semitone.
- **Harmonic Progressions:** Modeled by $S = \{3, 4, 5\}$, representing the structure of major and minor triads. This results in a denser graph that allows for "leaps" between chord members.
- **Bidirectional Movement:** To account for both clockwise and counter-clockwise movement (e.g., ascending or descending intervals), we use $S = \{1, 11\}$, effectively creating the undirected cycle graph C_{12}.

By utilizing the Cayley structure of $\mathbb{Z}_{12}$, musicologists can visualize complex modulation paths and harmonic relationships. The *tone-clock*, also known as musical clock, is a circular disposition of the twelve pitch-classes and the elements of $\mathbb{Z}_{12}$ representing them bijectively. Tone-clock is a geometrical representation of pitch class space on a circle. The most common tone-clock in Western music are the *Chromatic Circle* and the *Circle of Fifths / Fourths*. The Pitch Class space $\mathbb{Z}_{12}$ denote the set of 12 pitch classes (the *chromatic scale*) along the circle, the so-called *chromatic circle*, as showed in Fig. 1, where each element of $\mathbb{Z}_{12}$ is put in one-to-one correspondence with the symbols: $0 = \mathrm{C}, 1 = \mathrm{C}\sharp, 2 = \mathrm{D}, \ldots, 11 = \mathrm{B}$. Each vertex represents the *pitch (equivalence) class* of all pitches that share the same name regardless of octave. The *diatonic scale* is a heptatonic (seven-note) scale composed of five whole steps (tones) and two half steps (semitones) per octave. Moreover, in Fig. 3 the Circle of Fifths (CoF) represents a different ordering of the elements of $\mathbb{Z}_{12}$ based on the interval of a perfect fifth (7 semitones) where, moving clockwise on the circle of fifths, there is an ascending perfect fifth between each key and, equivalently, moving counterclockwise there is a descending perfect fifth. Major keys are usually on the outer circle, and their relative minor keys are on the inner circle, sharing the same key signature.

0	1	2	3	4	5	6	7	8	9	10	11
C	$C\sharp/D\flat$	D	$D\sharp/E\flat$	E	F	$F\sharp/G\flat$	G	$G\sharp/A\flat$	A	$A\sharp/B\flat$	B

Figure 2: *Chromatic scale: one-to-one correspondence*

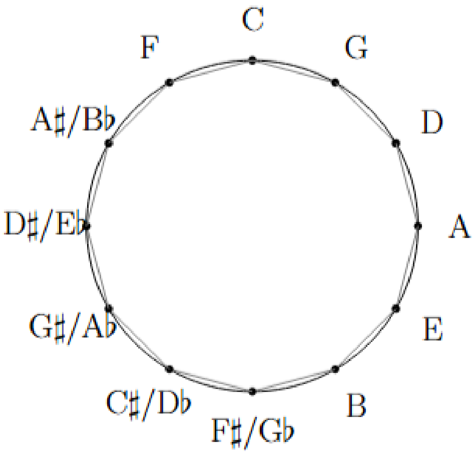


Figure 3: *Circle of Fifth starting from C (all major keys/chords visualized)*

Definition 1. *The* ordered chord space *of n-voices is defined as the Cartesian product:*

$$C^n = \mathbb{Z}_{12}^n = \underbrace{\mathbb{Z}_{12} \times \mathbb{Z}_{12} \times \cdots \times \mathbb{Z}_{12},}_{n \text{ times}}$$

with cardinality $|C^n| = 12^n$. Elements of C^n are ordered n-tuples (a multiset) $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ where each $x_i \in \mathbb{Z}_{12} = \{0, 1, 2, \ldots, 11\}$. Those elements are named as the chords *of pitch classes.*

Each element of C^n represents an *ordered chord* with n voices, where the i-th component x_i specifies which pitch class the i-th voice sings.

Definition 2. *The finite symmetric group S_n (on n elements) of all bijective functions $\{1, ..., n\} \to \{1, ..., n\}$ acts on C^n by permuting the voice coordinates:*

$$\sigma \cdot (x_1, x_2, \ldots, x_n) = (x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$$

for any $\sigma \in S_n$.

Notice that the group S_n acts on the n *voices* (the n components of the chord), not on the 12 pitch classes in $\mathbb{Z}_{12}$. The parameter n represents the number of voices in the chords. If we used S_{12} instead, we would be permuting the 12 pitch classes, which is not the musical structure we want.

Definition 3. *We define the* unordered chord space *as the quotient space:*

$$\mathcal{C}_n = C^n / S_n = \mathbb{Z}_{12}^n / S_n.$$

Elements of $\mathcal{C}_n$ are equivalence classes under the action of S_n:

$$[\mathbf{x}] = \{\sigma \cdot \mathbf{x} : \sigma \in S_n\} = \{(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}) : \sigma \in S_n\}.$$

The discrete orbifold $\mathcal{C}_n = \mathbb{Z}_{12}^n / S_n$ represents *unordered chords* (multisets of n pitch classes), identifying notes by name across octaves and voices by permutation (*C-E-G* $\sim$ *E-G-C*). Cardinality satisfies $|\mathcal{C}_n| \leq 12^n / n!$, with equality for distinct pitches. As will be demonstrated in the following sections, this discrete $\mathcal{C}_n$ features orbifold singularities (sharp edges/mirrors) arising from permutation identification—structures which are absent in the simple C_{12} cycle graph.

3. METRICS ON THE CHORD SPACE

Definition 4. *For $k, \ell \in \mathbb{Z}_{12}$, the* chromatic distance *is defined on $\mathbb{Z}_{12}$ as:*

$$d_C(k, \ell) = \min\{|k - \ell|, 12 - |k - \ell|\}.$$

The chromatic distance d_C measures the minimum semitones between pitch classes in $\mathbb{Z}_{12}$, analogous to the Lee distance (size 12) in coding theory and L_1 (Manhattan) distance. It is the shortest path distance in the Cayley graph $Cay(\mathbb{Z}_{12}, S)$ with $S = \{1, 11\}$ (or $\{1, -1\}$), forming a discrete elliptic metric space where all geodesics intersect, violating Euclid's parallel postulate [16]. This structure proves 12-TET musical space is non-Euclidean: the octave identification $0 = 12$ curves the line into a circle, manifesting global curvature. It underpins *parsimonious voice-leading*, where composers minimize semitone motion (e.g., C to B: 1 semitone vs. 11). The graph $Cay(\mathbb{Z}_{12}, \{1, 11\})$ is a discrete manifold with locally Euclidean geometry (Ollivier-Ricci curvature $\kappa(x, y) = 0$ for $x \neq y$) but globally elliptic topology, allowing linear analysis locally and group-theoretic models globally. The space $(\mathbb{Z}_{12}, d_C)$ is isometric to $Cay(\mathbb{Z}_{12}, S)$ iff $S = \{1, 11\}$ yields the chromatic distance; other generators (e.g., $S = \{3\}$) alter the metric and geometry. This extends to voice-leading in $\mathbb{Z}_{12}^n$, measuring total semitone motion between n-note chords.

Definition 5. *For n-chords* $\mathbf{x} = (x_1, \ldots, x_n)$ *and* $\mathbf{y} = (y_1, \ldots, y_n)$ *in* C^n*, we define the* voice-leading metric *as:*

$$\rho_n(\mathbf{x}, \mathbf{y}) = \min_{\sigma \in S_n} \left[\frac{1}{n} \sum_{i=1}^{n} d_C(x_i, y_{\sigma(i)})^2 \right]^{1/2}$$

where S_n *is the symmetric group of permutations on n elements. The metric* ρ_n *is a variant of the* Wasserstein distance *(or optimal transport metric) computed over a discrete and cyclic domain. It measures the minimum work required to transform one chord into another. Since* $d_C(x, y) \leq 6$ *for all* $x, y \in Z_{12}$*, the space* (C^n, ρ_n) *has bounded diameter:*

$$diam(C^n) = \sup_{\mathbf{x}, \mathbf{y} \in C^n} \rho_n(\mathbf{x}, \mathbf{y}) \leq 6.$$

Voice leading is the practice of smoothly transitioning from one chord to another using a minimum amount of movement in the process (minimun energy), preferably a whole tone or a half step. The minimization over S_n finds the voice-leading permutation that minimizes the total voice-leading distance (in the L^2 sense) between two chords. This captures the musical concept of *efficient voice leading*, where each voice moves to the nearest available pitch class in successive chord-tone.

Remark 1. *The distance* ρ_n *is invariant under the action of* S_n *on both arguments:*

$$\rho_n(\sigma \cdot \mathbf{x}, \tau \cdot \mathbf{y}) = \rho_n(\mathbf{x}, \mathbf{y}) \quad \text{for all } \sigma, \tau \in S_n$$

This follows immediately from the fact that ρ_n *minimizes over all permutations. Therefore,* ρ_n *is constant on equivalence classes and descends to a well-defined metric on the quotient space* $\mathcal{C}_n = C^n / S_n$*. This makes* $(\mathcal{C}_n, \rho_n)$ *a discrete or combinatorial version of* orbifold *defined in the Tymoczko's approach. It is the quotient of a discrete finite set by a finite group action. Unlike smooth Riemannian orbifolds,* $\mathcal{C}_n$ *does not possess a* C^k *differentiable structure.*

4. WHAT IS LOST/GAINED IN DISCRETIZATION

The relationship between discrete and continuous structures is captured by the following commutative diagram:

$$\begin{tikzcd}
\mathbb{Z}_{12}^n \arrow[r, hook] \arrow[d] & T^n \arrow[d] \\
\mathcal{C}_n \arrow[r, hook] & T^n / S_n
\end{tikzcd} \tag{1}$$

where the vertical arrows are quotient maps by S_n and the horizontal arrows are embeddings. The diagram commutes: quotienting then embedding is the same as embedding then quotienting. We have: $\mathcal{C}_n$ is a finite discrete approximation to the continuous orbifold T^n / S_n in the following sense:

1. every point in $\mathcal{C}_n$ corresponds to a point in T^n / S_n.
2. As we increase the number of divisions (replacing 12 with a larger $m > 12$), the space becomes a finer approximation to T^n / S_n.
3. Voice leadings between chords in $(\mathbb{Z}_{12})^n$ correspond to discrete paths in the continuous space.

In T^n/S_n, voices can move by arbitrarily small amounts. In C_n, the minimum non-zero movement is a semitone. Unlike Tymoczko's continuous orbifold the discrete formulation can be preferred, in a musical sense, if one want to highlight phenomena that otherwise would not be evident:

1. harmonic structure rests on discrete chromatic scales despite microtonal performance.
2. Preserves exact modular arithmetic mod 12 as used in Western music.
3. Finite size space ($\leq 12^n/n!$) enables exhaustive computation analysis of musical data using discrete metric spaces like $\mathcal{C}_n$.

The discrete $\mathbb{Z}_{12}^n$ (finite, combinatorial) contrasts sharply with continuous T^n (uncountable, smooth), requiring discrete mathematics over differential geometry. Notice that the voice-leading metric ρ_n is not a simple d_C extension: permutation minimization creates fundamentally new high-dimensional orbifold structure on $\mathbb{Z}_{12}^n/S_n$, beyond mere chromatic distance. Most of the geometric properties of d_C are inherited by ρ_n. For example, metric axioms, compactness, completeness, non-Hilbert, non-convexity. While the properties collected in Table 1 differ fundamentally between $(\mathbb{Z}_{12}, d_C)$ and (C_n, ρ_n).

Property	$(\mathbb{Z}_{12}, d_C)$	(C_n, ρ_n)
Dimension	0 (discrete points)	n (locally)
Number of elements	12	12^n
Diameter	6	$\sqrt{n} \cdot 6$
Graph structure	Simple cycle C_{12}	Complex graph, notcyclic
Symmetry group	$\mathbb{Z}_{12}$	$\mathbb{Z}_{12}^n \rtimes S_n$
Orbifold structure	None	Yes
Singularities	None	Yes
Curvature	Discrete positive	Mixed (positive at singularities)

Table 1: *Non-inherited properties between* $(\mathbb{Z}_{12}, d_C)$ *and* $(\mathbb{Z}_{12}^n, \rho_n)$.

5. The Ptolemaic Inequality

A metric space (X, d) is said to be *Ptolemaic* if for any four points $x, y, z, w \in X$, the following inequality holds:

$$d(x,z) \cdot d(y,w) \leq d(x,y) \cdot d(z,w) + d(y,z) \cdot d(x,w). \tag{2}$$

This is known as the *Ptolemy inequality*. Equality in (2) holds when the four points lie on a circle (or a line, as a degenerate circle) in the plane, in the order x, z, y, w or x, w, y, z around the circle. This inequality holds if and only if the space is an inner product space as proved by Schoenberg [15] in 1952. In fact, the inequality holds in every R^n, $n \geq 1$. In Tymoczko's continuous model, the Ptolemaic property is implicitly assumed because his quotient spaces are derived from Euclidean geometry, which is inherently Ptolemaic. Therefore, while Tymoczko tacitly assumes a Ptolemaic structure to map the harmonic movements of Western music (voice leading), the present result provides the necessary metric justification for $\mathbb{Z}_{12}$ to function rigorously as a Ptolemaic environment. In essence, this work formalizes the discrete foundation upon which continuous models of musical geometry rely. Tymoczko's orbifold T^n/S_n exhibits mixed Ptolemaic behavior [13, 5]:

1. **Globally non-Ptolemaic**: It cannot be isometrically embedded into Euclidean space due to orbifold singularities and toroidal topology.
2. **Locally Ptolemaic**: Away from singularities (repeated notes, rotational symmetry), regions behave Euclidean-like.

We present a rigorous treatment of Ptolemaic metric spaces and demonstrate that cycle graphs C_n for $n \geq 5$ are non-Ptolemaic using the Cayley-Menger determinant and geometric embedding theory. As an application, we prove that the musical pitch-class space $(\mathbb{Z}_{12}, d_C)$ with the chord metric admits no isometric embedding into Hilbert space. In the following, we collect some usefull results.

Theorem 1 (Cayley-Menger Embedding Criterion, [20])**.** *Let $D = (d_{ij})$ be the distance matrix for n points $\{p_1, \dots, p_n\}$ in a metric space. The points can be isometrically embedded in $\mathbb{R}^D$ if and only if the Cayley-Menger matrix* $\mathrm{CM}(D)$ *satisfies:*

1. $\mathrm{rank}(\mathrm{CM}(D)) \leq D + 2$,
2. *all principal minors of size $(D+2) \times (D+2)$ vanish,*
3. *all principal minors of size $k \times k$ ($k \leq D+2$) satisfy $(-1)^{k+1} \det(\mathrm{CM(D)}_k) \geq 0$.*

Here, $\mathrm{CM}(D)$ *is the $(n+1) \times (n+1)$* Cayley-Menger matrix *defined as:*

$$\mathrm{CM}(D) = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & d_{12}^2 & \cdots & d_{1n}^2 \\ 1 & d_{21}^2 & 0 & \cdots & d_{2n}^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & d_{n1}^2 & d_{n2}^2 & \cdots & 0 \end{pmatrix}. \tag{3}$$

Equivalently, in block form:

$$\mathrm{CM}(D) = \begin{pmatrix} 0 & \mathbf{1}^T \\ \mathbf{1} & D^{(2)} \end{pmatrix},$$

where $\mathbf{1} = (1, 1, \dots, 1)^T \in \mathbb{R}^n$ is the all-ones vector and $D^{(2)} = (d_{ij}^2)$ is the matrix of squared distances.

Theorem 2 (Schoenberg, 1938 [14])**.** *A metric space (X, d) is Ptolemaic if and only if it can be isometrically embedded into a Hilbert space.*

Theorem 3. *Cycle graphs C_n equipped with the shortest-path metric d_P are non-Ptolemaic metric spaces with $n \geq 5$.*

Proof. We prove this by demonstrating that C_5 violates the Ptolemy inequality, and then extending the result to all C_n with $n \geq 5$. Consider the quadruple (v_0, v_1, v_2, v_3) in $C_5 = \{v_0, v_1, v_2, v_3, v_4\}$. The distances are:

$$\begin{aligned} d(v_0, v_1) &= 1, \quad d(v_2, v_3) = 1, \\ d(v_0, v_2) &= 2, \quad d(v_1, v_3) = 2, \\ d(v_0, v_3) &= 2, \quad d(v_1, v_2) = 1. \end{aligned}$$

Computing the left-hand side of the Ptolemy's inequality:

$$d(v_0, v_1) \cdot d(v_2, v_3) = 1 \cdot 1 = 1.$$

Computing the right-hand side:

$$d(v_0, v_2) \cdot d(v_1, v_3) + d(v_0, v_3) \cdot d(v_1, v_2) = 2 \cdot 2 + 2 \cdot 1 = 6.$$

Thus the inequality $1 \leq 6$ is satisfied. However, consider the quadruple (v_0, v_1, v_3, v_4):

$$\begin{aligned} d(v_0, v_1) &= 1, \quad d(v_3, v_4) = 1, \\ d(v_0, v_3) &= 2, \quad d(v_1, v_4) = 2, \\ d(v_0, v_4) &= 1, \quad d(v_1, v_3) = 2. \end{aligned}$$

Left-hand side:

$$d(v_0, v_1) \cdot d(v_3, v_4) = 1 \cdot 1 = 1.$$

Right-hand side:

$$d(v_0, v_3) \cdot d(v_1, v_4) + d(v_0, v_4) \cdot d(v_1, v_3) = 2 \cdot 2 + 1 \cdot 2 = 6,$$

and non-Ptolemacity directly follows. It is possible to prove this result also by an alternative approach via non-embeddability. We show that C_5 cannot be isometrically embedded in $\mathbb{R}^3$ (or any $\mathbb{R}^k$), which by Schoenberg's Theorem 2 implies non-Ptolemaicity. Consider embedding the 5 vertices of C_5 in $\mathbb{R}^3$. The distance matrix requires:

- Each vertex is distance 1 from its two neighbors,
- each vertex is distance 2 from the two vertices at positions ±2 around the cycle.

Attempting to place v_0 at the origin and v_1 at $(1,0,0)$:

- v_4 must be at distance 1 from v_0 and distance 2 from v_1. This gives v_4 on a circle.
- v_2 must be at distance 1 from v_1 and distance 2 from v_0. This gives v_2 on another circle.
- v_3 must be at distance 1 from both v_2 and v_4, and distance 2 from both v_0 and v_1.

The constraints $d(v_0,v_2)=2$, $d(v_1,v_2)=1$, $d(v_0,v_4)=1$, $d(v_1,v_4)=2$ determine the positions of v_2 and v_4 up to reflections. However, requiring v_3 to simultaneously satisfy $d(v_2,v_3)=1$ and $d(v_4,v_3)=1$ creates an overdetermined system with no solution in $\mathbb{R}^3$. More rigorously, one can compute the Cayley-Menger determinant for suitable subsets of 4 points and verify that $\det(\mathrm{CM}(D)) \neq 0$, contradicting embeddability in $\mathbb{R}^3$. Now, in order to extend the result for any $n > 5$, we point out that the cycle C_n contains C_5 as an isometric subspace (take any 5 consecutive vertices). Since non-Ptolemaicity is preserved under taking subspaces, C_n is non-Ptolemaic for all $n \geq 5$. ■

Note that C_3 and C_4 *are* Ptolemaic: C_3 is isometric to an equilateral triangle in $\mathbb{R}^2$ and C_4 is isometric to a square in $\mathbb{R}^2$. The threshold $n = 5$ is sharp.

Theorem 4. *The metric space $(\mathbb{Z}_{12}, d_C)$ is not Ptolemaic and admits no isometric embedding into any Hilbert space.*

Proof. Observe that $(\mathbb{Z}_{12}, d_C)$ is isometric to the cycle graph C_{12}: there is a bijection $\phi : \mathbb{Z}_{12} \to V(C_{12})$ given by $\phi(k) = v_k$, where $V(C_{12}) = \{v_0, \ldots, v_{11}\}$ is the vertex set of C_{12}, and the chord metric d_C coincides with the shortest-path metric on C_{12}. By Theorem 3, C_{12} is non-Ptolemaic since $12 \geq 5$. By Schoenberg's Theorem 2, non-Ptolemaicity is equivalent to non-embeddability in Hilbert space. ■

The *wrap-around* structure of $\mathbb{Z}_{12}$ with octave equivalence $0 \equiv 12$ creates a cyclic topology that cannot be flattened into Euclidean space while preserving all pairwise distances. This has some musical implications:

- Pitch-class spaces cannot be faithfully represented in 2D or 3D Euclidean plots without distortion.
- Distance-based algorithms (clustering, nearest-neighbor search) must account for the non-Euclidean geometry.
- Extensions like Tonnetz or geometric group theory approaches are needed to capture harmonic relationships.

Theorem 5. *For $n \geq 3$, the unordered chord space $(\mathcal{C}_n, \rho_n)$ is non-Ptolemaic.*

Proof. We prove the result by demonstrating that $(\mathcal{C}_3, \rho_3)$ cannot be isometrically embedded into Euclidean space via the Cayley-Menger determinant criterion. Consider a set of five major triads in $\mathcal{C}_3$ forming a chromatic cycle:

$$\begin{aligned}
T_0 &= \{0,4,7\} && \text{(C major)},\\
T_1 &= \{1,5,8\} && \text{(C}\sharp\text{ major)},\\
T_2 &= \{3,7,10\} && \text{(E}\flat\text{ major)},\\
T_3 &= \{6,10,1\} && \text{(F}\sharp\text{/G}\flat\text{ major)},\\
T_4 &= \{9,1,4\} && \text{(A major)}.
\end{aligned}$$

To rigorously analyze the geometry of the chord space $\mathcal{C}_3$, we utilize the voice-leading metric ρ_3. For any two triads $T_a, T_b \in \mathcal{C}_3$, the distance is defined as:

$$\rho_3(T_a, T_b) = \min_{\sigma \in S_3} \sqrt{\frac{1}{3}\sum_{i=1}^{3} d_i^2} \tag{4}$$

where d_i represents the *individual chromatic displacement* of the i-th voice under a specific permutation σ of the symmetric group S_3. Specifically, $d_i = \mathrm{dist}(x_i, y_{\sigma(i)})$, where dist is the shortest path on the pitch-class circle. For T_0 and T_2, a simple parallel translation by 3 semitones $T_2 = T_0 + 3$ suggests $\rho_3(T_0, T_2) \leq 3$.

Specifically, using the permutation $\sigma = \text{id}$ (parallel translation), that is $(0 \to 3, 4 \to 7, 7 \to 10)$ for which $d_i = 3$ for all $i = 1,2,3$ we calculate:

$$\rho_3(T_0, T_2)_{parallel} = \sqrt{\frac{1}{3}(3^2 + 3^2 + 3^2)} = 3.$$

This represents a non-optimized result. However, the voice-leading metric ρ_3 requires exhaustive minimization over *all* permutations $\sigma \in S_3$. By utilizing the common pitch class 7 shared by T_0 and T_2 in 12-TET, consider the optimal distance $7 \in T_0 \to 7 \in T_2$ with $d_1 = 0$ and, then, matching the remaining pairs $(0 \to 10, 4 \to 3)$, with $d_2 = 2$ and $d_3 = 1$ respectively, to find a more efficient path yielding to a smaller distance value:

$$\rho_3(T_0, T_2)_{optimal} = \sqrt{\frac{1}{3}(0^2 + 2^2 + 1^2)} = \sqrt{5/3} \approx 1.29.$$

Thus, the optimized distance found via exhaustive search over all permutations is smaller. Whether using the parallel distance of 3 or the optimized distance of ≈ 1.29, the global metric structure remains non-Euclidean. Exhaustive computation of all pairwise distances $\rho_3(T_a, T_b)$ provides the following squared distances $\rho_3(T_a, T_b)^2$ (via exhaustive search over $3! = 6$ permutations per pair). For T_0 and T_3 we have previously shown that $\rho_3(T_0, T_2)^2 = 5/3$. For T_0 and T_3, the optimal matching is $(0 \to 1, 4 \to 6, 7 \to 10)$. The individual chromatic distances are $(1, 2, 3)$. The squared distance is therefore $(1^2 + 2^2 + 3^2)/3 = 14/3$. For T_0 and T_4, the optimal matching is $(0 \to 1, 4 \to 4, 7 \to 9)$. These mappings yield distances of $(1, 0, 2)$. The squared distance is $(1^2 + 0^2 + 2^2)/3 = 5/3$. While a parallel translation matching $(0 \to 9, 4 \to 1, 7 \to 4)$ results in a distance of 3, the exhaustive search over all permutations confirms that the common-tone matching $(4 \to 4)$ is more efficient, yielding the smaller $\rho_3(T_0, T_4)^2 = 5/3$. Due to the transpositional symmetry of the chosen major triads, the pairwise distances for the set $\{T_0, T_1, T_2, T_3, T_4\}$ are structured as follows:

$$\begin{array}{lll} \rho_3(T_0, T_2)^2 = 5/3, & \rho_3(T_0, T_3)^2 = 14/3, & \rho_3(T_0, T_4)^2 = 5/3 \\ \rho_3(T_1, T_3)^2 = 5/3, & \rho_3(T_1, T_4)^2 = 14/3, & \rho_3(T_2, T_4)^2 = 5/3. \end{array}$$

We construct the related 6×6 Cayley-Menger matrix CM for $\{T_0, T_1, T_2, T_3, T_4\}$:

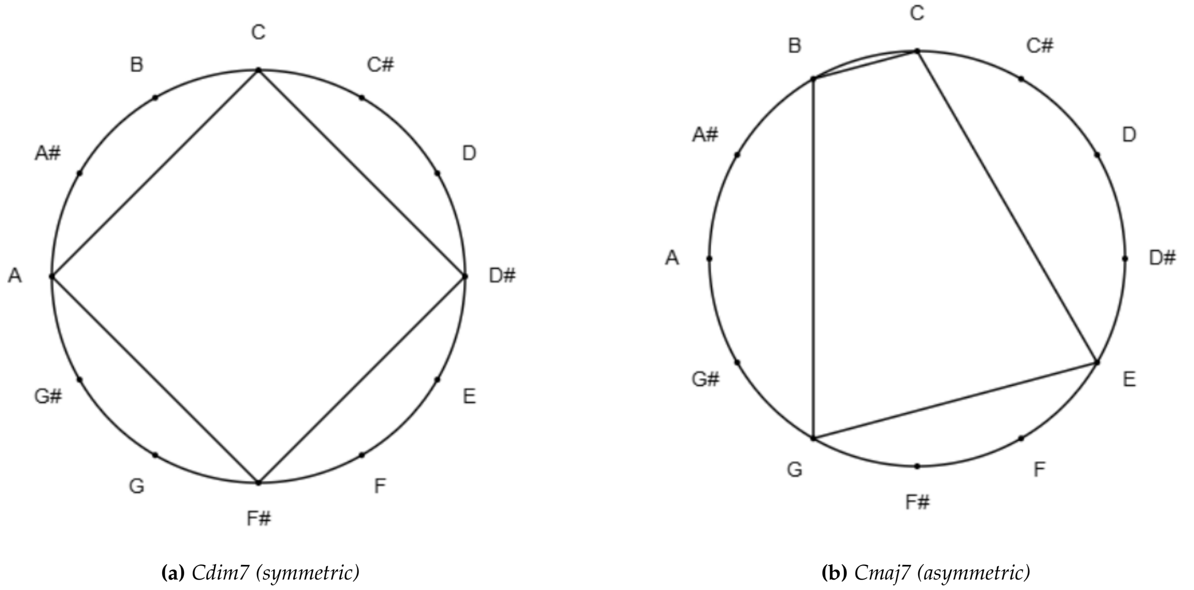
$$\text{CM} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 5/3 & 14/3 & 1 \\ 1 & 1 & 0 & 1 & 5/3 & 14/3 \\ 1 & 5/3 & 1 & 0 & 1 & 5/3 \\ 1 & 14/3 & 5/3 & 1 & 0 & 1 \\ 1 & 1 & 14/3 & 5/3 & 1 & 0 \end{pmatrix}.$$

We point out that, without the exhaustive search for d_i, the Cayley-Menger test is invalid. A necessary condition for these five points to be isometrically embeddable in $\mathbb{R}^3$ (or any Euclidean space) is that $\det(\text{CM}) = 0$. Using the NumPy library in Python, the function `np.linalg.det()` was applied to the Cayley-Menger matrix constructed from the optimized distances, yielding $\det(\text{CM}) \approx -19.35 \neq 0$. This violates the Schoenberg embeddability criterion. For $n > 3$, $(\mathcal{C}_3, \rho_3)$ embeds isometrically into $(\mathcal{C}_n, \rho_n)$ by padding chords with $n - 3$ constant pitch classes (e.g., zeros), which preserves the voice-leading distances. Thus, the non-Ptolemaic property holds for all $n \geq 3$.

■

This non-Ptolemaicity formally proves tempered harmony's intrinsic curvature: musical space is a discrete elliptic manifold, not embeddable in flat Euclidean geometry. Although both $(\mathbb{Z}_{12}, d_C)$ and $(\mathcal{C}_n, \rho_n)$ are non-Ptolemaic, $(\mathcal{C}_n, \rho_n)$ is not automatically inherited this property from its base space. Non-Ptolemaicity of $(\mathcal{C}_n, \rho_n)$ must be proved independently via direct verification on chord quadruples, or by embedding analysis of the discrete orbifold. In the sequel, we prove that the space of unordered n-note chords $(\mathcal{C}_n, \rho_n)$ equipped with the voice-leading distance is non-Ptolemaic. While the base pitch-class space $(\mathbb{Z}_{12}, d_C)$ is already non-Ptolemaic, this property does not automatically transfer to quotient spaces. We provide an independent proof via explicit computation of the Ptolemy inequality on chord quadruples and demonstrate that the orbifold singularities induced by permutation symmetry create additional geometric obstructions to Euclidean embedding. Despite being globally non-Ptolemaic, $(\mathbb{Z}_{12}, d_C)$ and $(\mathcal{C}_n, \rho_n)$ can satisfy the Ptolemaic inequality for certain quadruples of points.

Definition 6. *A metric space (X, d) is* locally Ptolemaic *if there exist non-trivial quadruples (x, y, z, w) satisfying the Ptolemaic inequality, even though the space is not globally Ptolemaic.*



(a) *Cdim7 (symmetric)* **(b)** *Cmaj7 (asymmetric)*

Figure 4: *Symmetrical vs. asymmetric chord geometry on chromatic circle*

Consider, for example, the following four pitch classes: $x = 0$ (C), $y = 2$ (D), $z = 5$ (F), $w = 7$ (G). Then, the inequality is satisfied. This particular quadruple is *locally* Ptolemaic. We point out that specific quaternas of points can satisfy the inequality, or even the equality, even within a non-Ptolemaic space. In a musical viewpoint, see Fig. 4, the Ptolemaic inequality can be verified locally when the chords or pitch sets are confined within a small neighborhood (typically less than half the chromatic circle). In particular, ptolemaic *equality* holds precisely for symmetric "co-circular" chromatic clusters. We analyze chord classes satisfying the Ptolemaic inequality via strict inequality or equality:

- **Strict Inequality Case:** This occurs for asymmetric chords, such as *Cmaj*7. In these cases, the inequality is strict because the points do not form a perfectly symmetrical configuration (like a square) on the circle, lacking the specific co-circularity required for equality.
- **Equality Case:** Highly symmetric "perfectly even" chords dividing the octave equally. $Cdim7 = \{0, 3, 6, 9\}$ (square, minor thirds) and $C_{aug} = \{0, 4, 8\}$ (equilateral triangle, major thirds) form regular polygons on the chromatic circle. Maximal symmetry ensures Ptolemaic equality, with C_{aug} dividing C_{12} into three equal arcs of 4 semitones.

Cayley graphs equip finitely generated groups with a natural path metric, central to geometric group theory. For a Cayley graph, we can define the *path distance* between any two vertices as the length of the shortest edge path.

Definition 7. *Let $G = \mathrm{Cay}(V, S)$ be a Cayley graph of a group V with respect to a finite symmetric generating set $S \subset V \setminus \{e\}$. The* path metric *$d_P(x, y)$ is the length of the shortest path between vertices $x, y \in V$ in G. Formally,*

$$d_P(x, y) = \min\{k \in \mathbb{N}_0 \mid \exists s_1, \ldots, s_k \in S \text{ s.t. } x \cdot s_1 \cdots s_k = y\}.$$

Theorem 6. *Let $G = \mathrm{Cay}(\mathbb{Z}_{12}, S)$ with $S = \{\pm a\}$ where $\gcd(12, a) = 1$, equipped with the path metric d_P. Then the chord metric space $(\mathbb{Z}_{12}, d_C)$ is isometric to (V, d_P), i.e., $d_C \cong d_P$.*

Proof. Define $\phi : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$, $\phi(x) = ax \bmod 12$. and show it preserves distances. Since $\gcd(12, a) = 1$, a is an invertible element in the ring $\mathbb{Z}_{12}$. Thus, ϕ is a bijection (specifically, a group automorphism of $\mathbb{Z}_{12}$). In any cycle graph C_n with generators $S = \{\pm s\}$, the path metric $d_P(u, v)$ is the minimum number of generator applications needed to reach v from u. For $S = \{\pm a\}$, this is the smallest k such that:

$$v - u \equiv \pm k \cdot a \bmod 12.$$

Since a is invertible, we can multiply by a^{-1}:

$$a^{-1}(v-u) \equiv \pm k \bmod 12.$$

This implies $k = d_C(a^{-1}u, a^{-1}v)$. We check the distance between $\phi(x)$ and $\phi(y)$ in the path metric d_P:

$$d_P(\phi(x), \phi(y)) = d_P(ax, ay).$$

The distance $d_P(ax, ay)$ is the smallest k such that $ay - ax \equiv k \cdot (\pm a) \bmod 12$. Factoring out a:

$$a(y-x) \equiv \pm k \cdot a \bmod 12.$$

Multiplying by a^{-1} (which exists since $\gcd(12, a) = 1$):

$$y - x \equiv \pm k \bmod 12.$$

The smallest such k is exactly $d_C(x, y)$. Therefore:

$$d_P(\phi(x), \phi(y)) = d_C(x, y).$$

Since ϕ is a bijection that preserves distances, the space $(\mathbb{Z}_{12}, d_C)$ is isometric to the Cayley graph $(\mathbb{Z}_{12}, d_P)$. ■

The *Tonnetz* (tone-network) can be viewed as living on the 2-torus T^2, with certain paths representing major and minor thirds and perfect fifths. More commonly, the Tonnetz is presented as a discrete graph formed by connecting pitches with intervals of perfect fifths (7 semitones), major thirds (4), and minor thirds (3). It is a Cayley graph on Z_{12}, but with a larger set of generators, specifically

Definition 8. *The discrete Tonnetz is the Cayley graph* $Cay((\mathbb{Z}_{12})^n, S)$ *where*

$$S = \bigcup_{k=1}^{n} \{(\pm 3, 0, \ldots, 0), (\pm 4, 0, \ldots, 0), (\pm 7, 0, \ldots, 0)\}_k$$

with generators acting on the k-th coordinate and representing minor third, major third, and perfect fifth intervals.

This structure has $v_n = 12^n$ vertices, $e_n = 3n \cdot 12^n$ edges and $f_n = 12^n(3n-1)$ faces. The Tonnetz provides a concrete illustration of the general non-Ptolemaicity theorem for cyclic Cayley graphs. With $S = \{3, 4, 7\}$, shortest path metric d_P fails Ptolemaicity due to toroidal "wrap-around" effect: $\mathbb{Z}_{12}$ identification creating torus topology in the Tonnetz lattice. For distant quadruples, wrap-around paths create metric compression:

$$d_P(x,z) d_P(y,w) > d_P(x,y) d_P(z,w) + d_P(y,z) d_P(x,w).$$

This modular shortcut violates Euclidean flatness, destroying Ptolemaic property and Hilbert embeddability. For example, consider points $x = 0$ (C), $y = 3$ (E♭), $z = 6$ (F♯), $w = 9$ (A). As distances approach the tritone diameter (6 semitones), modular paths compress: $d_P(x,z) d_P(y,w)$ exceeds the Ptolemaic right-hand side, violating Euclidean flatness required for Hilbert embedding. This confirms the Tonnetz inherits $\mathbb{Z}_{12}$'s intrinsic cyclic curvature. Notice that any visualization as 2D lattice is a local approximation, not globally metric-preserving.

6. On the Schoenberg's Embedding Theorem

According to Schoenberg's Isometric Embedding Theorem [14], a finite metric space (X, d) with $|X| = m + 1$ points is isometrically embeddable in $\mathbb{R}^n$ if and only if the associated Gramian matrix G is positive semidefinite (PSD). The Gramian matrix G of size $m \times m$ is constructed by fixing an origin $x_0 \in X$ and calculating for $i, j \in \{1, ..., m+1\}$:

$$g_{ij} = \frac{1}{2}\left(d(x_i, x_0)^2 + d(x_j, x_0)^2 - d(x_i, x_j)^2\right). \tag{5}$$

The minimal dimension required for an isometric embedding in $\mathbb{R}^n$ is exactly $\operatorname{rank}(G) \leq n$.

Theorem 7. *Let* $X = Cay(\mathbb{Z}_{12}, \{\pm 1\})$ *be the cycle graph* C_{12} *equipped with the shortest path metric* $d_P \equiv d_C$*. Then,* X *is isometrically embeddable in* $\mathbb{R}^{11}$ (the trivial embedding) *but not in any* $\mathbb{R}^n$ *for* $n < 11$*.*

Proof. For the cycle graph C_{12}, this 12×12 Gram matrix has exactly one zero eigenvalue (due to connectedness) and eleven strictly positive eigenvalues, yielding $\text{rank}(G) = 11$. Consequently, the Cayley-Menger Embedding theorem provides an isometric embedding into $\mathbb{R}^{11}$ using the first 11 eigenvectors scaled by square-root eigenvalues but not in any lower dimension $n < 11$. Embedding into $\mathbb{R}^n$ with $n < 11$ is impossible: the 11-dimensional embedding is minimal, as projection onto fewer coordinates distorts the cyclic path distances due to the wrap-around effect ($0 \equiv 11$). Two- or three-dimensional representations like the chromatic circle preserve only topology, not the metric structure. ■

The rank(G) = 11 result reveals 12-TET geometry requires 11 dimensions for faithful embedding, confirming once again the non-flat structure beyond 2D/3D representations.

7. CONCLUSION

This establishes a discrete metric-topological framework for 12-TET, proving $(\mathbb{Z}_{12}, d_C)$ and $(\mathcal{C}_n, \rho_n)$ globally non-Ptolemaic, implying no Schoenberg embedding into Euclidean/Hilbert space due to intrinsic positive discrete curvature. Local Ptolemaicity explains diatonic Euclidean behavior (Diminished Seventh, Augmented Triad), while asymmetric chords occupy stable non-singular regions. Harmony emerges as a geometric consequence of the non-Euclidean 12-tone torus structure.

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