Teaching Atonal and Beat-Class Theory, Modulo Small

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Abstract: *The paper advances a pedagogical program that models small cyclic systems before teaching the twelve-element chromatic system of atonal theory. The central properties, relations and protocols of atonal theory (complementation, inclusion, invariance, transpositional equivalence, set classification and labeling, maximal evenness) are introduced in the smallest cyclic system to which they apply. All cyclic systems of 2 to 9 elements have at least one familiar musical application, modeling beat-class (rhythmic) cycle, pentatonic and diatonic scales. By the time students have scaled up to a twelve-element universe, they are technically prepared to explore it, and to appreciate its special properties. Along the way, they have learned a model of meter, an otherwise under-theorized aspect of music pedagogy.*

Keywords: *Atonality. Musical set theory. Time cycles.*

I. The Challenge of Teaching Atonal Theory

The theory of atonality focuses on the cyclic universe of twelve elements (C12) interpreted as chromatic pitch classes, and on the properties of and relations among pitch-class sets in that universe. It is a standard curricular offering at both undergraduate and post-graduate levels in North America and elsewhere, is integrated into at least one comprehensive music-theory textbook [1], and is the sole focus of several dedicated ones [2],[3]. Atonal theory is a challenge to teach because of the unfamiliarity of atonal repertories, the level of abstraction, and the delay in the payoff. Even for students interested in atonal repertories and comfortable with mathematical modes of conception and representation, it takes some time before the techniques of atonal theory help students to achieve satisfying analytic insights. For students who are math-phobic, or not drawn to atonal repertories, or both, the curriculum can be frightening or alienating, at worst provoking hostility that generalizes to the entire project of thinking conceptually about music.

One part of the challenge can be confronted by broadening the range of repertory. C12 can be interpreted not only as a chromatic universe of pitch classes, but also as a twelve-beat cycle, which might be realized as a bar of $\frac{12}{8}$, two bars of $\frac{6}{8}$, four bars of $\frac{3}{4}$, a bar of $\frac{4}{4}$ with tripleted subdivisions, and so forth [4]. Knowledge of C12 can be used to analyze the rhythms of readily accessible and familiar repertories, such as West-African, Latino, global-popular, or minimalist music [5],[6],[7]. Students can learn such central topics as rotational equivalence, invariance, complementation, inclusion, and set classification and labelling, each of which has direct and revealing beat-class

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applications. Learning C12 in this environment, students acquire abstract knowledge ready for transfer to the isomorphic universe of pitch classes, and to atonal and serial repertories. There is a pedagogical advantage in staging the mathematical and musical challenge at different times. And there is a residual benefit: as they travel along the road to atonal theory, students are exposed to a theory of musical meter, which is otherwise neglected in the curriculum [8].

But this partial solution does not address the problem of abstraction. C12 is a complex system, with 4,096 distinct combinations, including 66 distinct pairs of elements, 220 triplets, and so forth. It is difficult to view such a system holistically, especially if one has not "scaled up" to it through simpler systems. The application of C12 to time cycles suggests a solution. There is only one chromatic universe of pitch classes, but there are many universes of beat classes, some quite small, that are familiar to musicians from an early stage of development. Larger-cardinality universes inherit the simple properties and relations of smaller ones, and add more complex ones. An incremental pedagogical progression from smaller to larger cyclic systems has some of the advantages of elementary mathematics curricula, such as the Japanese model for teaching arithmetic to children, which is initially restricted to the subtizable numbers [9].

Not all small beat-class universes are equally familiar. Musicians in the west typically learn systems based on 2 and 3 beats, their powers (4, 8, 9), and their composites (6,12), thus catching every number in the range from 2 to 9, with the exception of 5 and 7. Fortuitously, it is exactly these missing prime universes that form the most familiar small cyclic pitch-class systems, as pentatonic and diatonic scales respectively. Unlike the beat-class and chromatic universes, the elements of these scales are not distributed evenly. But they are even enough that we can disregard, or "reduce out," the distinction between tone and semitone in the diatonic case, and tone and minor third in the pentatonic one [10]. Indeed, musicians are accustomed to such reductions, as when we assign scale degrees using natural numbers from 1 to 8.

II. Outlines of a Pedagogical Program

My pedagogical program progresses in three stages. In the first stage, which can be worked through in about 30 minutes of class time, power sets of 0, 1, and 2 elements are briefly studied. Basic relations such as null set, complementation, and cardinality are introduced, as are the basic symbols of set theory. The second stage studies cyclic universes of 3, 4, and 5 elements, realized respectively as triple and quadruple meter in the rhythmic domain, and pentatonic scales in the pitch domain. Topics that can be introduced at this stage include modular arithmetic, rotational (transpositional) equivalence, classification and labelling procedures, interval class, interval content, set class, and invariance. The musical applications are not yet surprising. Students are learnng a new language for properties and relations about which they long cultivated deep intuitions. They may feel skeptical about being asked to pay money for a Howitzer in order to shoot a few sitting ducks, when a much simpler implement will suffice. I find it useful to tell them, more than once, that it is better to get to know your machinery in a simple environment than in the complex ones that lie just around the bend.

As the size of the cycle increases, intuitions about its structure diminish at roughly the rate that interest in its musical capacities grows. Thus conceptual and terminological grasp is not overwhelmed at the same moment that students are struggling to gain some intuitive traction on structures that are becoming exponentially more complex. C6 represents a pedagogical watershed: just small enough to be graspable as a Gestalt, but large enough to introduce curious features with unexpected musical ramifications. The universes from 6 to 9 all have distinctive

Table 1: *Pascal's Triangle, interpreted as number of sets |d| in a |c|-universe.*

c=	d= 0	1	2	3	4	5	6	7	8	9	10	11	12	$2^c =$
0	1													1
1	1	1												2
2	1	2	1											4
3	1	3	3	1										8
4	1	4	6	4	1									16
5	1	5	10	10	5	1								32
6	1	6	15	20	15	6	1							64
7	1	7	21	35	35	21	7	1						128
8	1	8	28	56	70	56	28	8	1					256
9	1	9	36	84	126	126	84	36	9	1				512
10	1	10	45	120	210	252	210	120	45	10	1			1024
11	1	11	55	165	330	462	462	330	165	55	11	1		2048
12	1	12	66	220	495	792	924	792	495	220	66	12	1	4096

features that underlie familiar and compelling musical properties. These universes are large enough that set-class enumeration and classification from scratch becomes a challenge, but small enough that it remains tractable. But enumeration is no longer the central focus at this level. That focus shifts toward the generation of larger sets by recursive stacking of a single interval. Each universe has a unique personality, with special musical ramifications, that results from its number of elements. Prime-numbered universes behave differently than composite ones, and power-numbered universes behave different from those that have multiple prime factors.

Three variable are used throughout this study. **c** counts the number of elements of in the cyclic universe, corresponding to the large-case variable in expressions such as "C12." **d** counts the number of elements of a pitch-class set [11],[12]. Thus, for a C-major scale drawn from a chromatic universe, $c = 12$ and $d = 7$, and for a C-major triad drawn from a diatonic one, $c = 7$, $d = 3$. Finally, **g** is a generating interval, and corresponds to one of the $[\frac{c}{2}]$ interval classes that exist within a universe of size c. Motion upward through the values of c corresponds to motion downward through the rows of Pascal's triangle, a skewed version of which is as Table 1. Each entry in the table presents $\binom{c}{d}$, the number of sets of cardinality d within a universe of c elements. The two highest and two lowest values of d, shaded in the table, which I shall refer to as "exterior cardinalities," are entirely predictable and are of little interest musically and mathematically. Our attention will be focused exclusively on the unshaded cardinalities in the interior of each row, once they begin to appear at $c = 4$. The final column gives the cardinality of the power set, 2^c, which sums the entries to its left.

III. Non-cyclic universes of 0 to 2 elements

Since c = 3 is the smallest cyclic universe, there is some sense in beginning there, but I have found it profitable to work quickly through the trivially small, non-cyclic universes first, establishing the most basic definitions (defined terms are here placed in italics). For $c = 0$, there is a single set that is both *null* (∅) and *universal* (U). For $c = 1$, these functions partition into two distinct and *complementary* sets of different *cardinality*. The formal definition of complementation ($B = U \setminus A$ iff $A \cup B = U$ and $A \cap B = \emptyset$) gives occasion to introduce some fundamental terms and symbols of set theory. The power set of $c = 2$ is completed by augmenting $\varnothing$ and $U = \{A, B\}$ with two singleton sets {A} and {B}, related both by complementation and by *cardinality equivalence*.

IV. Cyclic universes of 3 to 5 elements

With $c = 3$ we arrive at a properly cyclic domain, and the next several universes will present occasion to gradually introduce the most significant properties, relations, and representational protocols for musically realized cyclic spaces, including modular arithmetic, cyclic graphs, rotational (transpositional) equivalence, set class, abstract complementation and inclusion, invariance , interval class, interval vector, and interval generation. **c = 3** has eight sets: $\varnothing$, U, three singletons, and their three complements. Growth in the number of elements requires a more systematic labelling protocol, and so *integer labelling* of elements is introduced, from 0 to $c - 1$. Using integers as labels risks confusing the different "registers" in which numbers will be used – for labelling, counting, and measuring – and it is important at this stage to exhort vigilance about these distinctions.

This is the appropriate universe in which to introduce *modular arithmetic*, *cyclic graphs, and transpositional equivalence*. Figure 1a shows the cyclic graphs for the three sets for $c = 3$, $d = 2$, using filled circles to indicate the presence of an element, and unfilled circles to indicate its absence from the set. It can be readily seen that the sets are related by rotation. Figure 1b realizes the same three sets as rhythms in 3/4 meter. A musician can just as easily intuit how rotation is realized in this domain as in the graphic one.

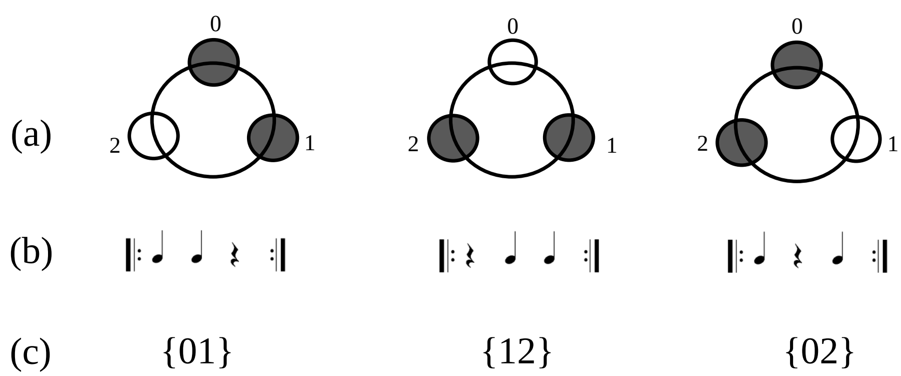

Figure 1: *Sets for $c = 3$, $d = 2$, represented in three different ways: (a) as cyclic graphs; (b) as repeating rhythms; (c) as integers.*

Finally,Figure 1c realizes the same three sets using integer notation. Here intuitions are less stable. How can {02}, whose elements seem separated, have the same structure as {01} and {12}, whose elements are explicitly adjacent? By comparing these representations with the cyclic graphs and rhythmic sets, students can gain their first foothold on the unfamiliar logic of modular arithmetic. It is through comparisons of this sort that students begin to explore the ways that graphic, arithmetic, and musical relations reciprocally model each, a process whose goal is to allow their pre-loaded intuitions about each of these three domains to inform their understanding of the other two.

c = 4 has sixteen sets, ten of which have exterior cardinalities that advance no new properties. We focus on the six sets where $c = 4$, $d = 2$. For the first time, we encounter cardinality-equivalent sets that are not equivalent by rotation (= transposition). These six sets represent two distinct pair-wise distances, or intervals, each corresponding to a *transposition class*, or T-class, which can be provisionally labelled as step (or adjacent pair) and leap (or diametric pair). The musical distinction between these two classes is represented by Figure 2, from Mozart, where each measure has four ♪beats, which can be labelled in order from 0 to 3. In the first measures, the beats are partitioned into complementary step-related pairs, {12} and {03}. Beginning at bar 5, beats are partitioned into complementary leap-related pairs, {02} and {13}.

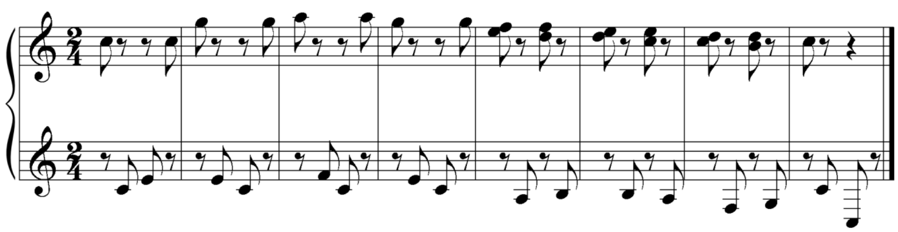

Figure 2: *From Mozart's Variations on "Ah, vous dirai-je, Maman." Measures 1 - 4 partition U4 as {03} and {12}, representing step class* [01] *Measures 5 - 7 partition U4 as {02} and {13}, representing leap class* [02].

The six dyads in C4 contains four steps but only two leaps, a curious circumstance that opens the door to the important topic of rotational *invariance*. Musicians easily intuit, from examining Figure 2, that there are more possible step-pairs than leap-pairs, since each leap-pair is indistinguishable from its two-unit rotation (or transposition). The invariance is easy to see in a cyclic graph, but more of a challenge to see when the same sets are represented using modulo-4 integers. It is in mastering challenges of this sort that students who have difficulty adjusting to modular arithmetic gain further traction.

The introduction of T-classes in C4 through the step/leap distinction gives an occasion to confront one of the perpetually confusing aspects of atonal set theory, the distinction between literal sets and abstract set classes. The project of set classification requires attaching labels to the classes, for book-keeping purposes. In 1973, Allen Forte assigned each set class a two-value label whose second value was an arbitrary number [13], but many atonal pedagogies now prefer *prime forms*, a procedure for selecting a member of the class to represent the class as a whole. Although this protocol eliminates the arbitrary relation between label and referent, there is still a pedagogical challenge: the same label evidently refers to objects at two different levels of abstraction. For example, in the usual chromatic/atonal interpretation of C12, {037} references a C minor triad, and [037] references the class of twelve minor triads. Unless the distinction between curly and square brackets is emphasized, considerable confusion arises. The step/leap distinction in $c = 4$ provides a pretext for introducing the prime-form protocol in an environment that is intuitive and contained: [01] and [02] are introduced as alternate labels for step and leap classes, respectively.

c = 5 has thirty-two sets, still small enough to explore and comprehend as a single *Gestalt*. Twenty of these sets are of intermediate cardinality: ten pairs ($c = 5$, $d = 2$), and their ten complements ($c = 5$, $d = 3$). Each cardinality class has two distinct rotation classes: steps and leaps for $c = 5$, $d = 2$, and their complements for $c = 5$, $d = 3$. The fact that 10 is a multiple of 5 further suggests that there are no transpositional invariances, a result of 5's status as a prime number.

I use $c = 5$ to introduce two significant concepts that are portable to larger cardinality universes. The first concept is total *interval content*, a property of set classes that is catalogued by *interval vectors*. Cataloguing the single interval of a two-element set is trivial work, but there is a payoff: cataloguing the three intervals of a $c = 5$, $d = 3$ set, and comparing that vector to that of its $c = 5$, $d = 2$ complement, exposes the intervallic affinities between complement-related sets, and leads to the introduction of a simple form of the complement algorithm. In addition to the gains realized by introducing complement relations in a small universe, there is also a pedagogical benefit to introducing the complement algorithm in a universe whose cardinality is odd. Because no interval divides the universe as the tritone does in C12, there are no invariances to complicate the complement/interval algorithm.

Because $c = 5$ is a prime universe, it is also a good place to introduce intervallic *generators (g)*, marking a shift from a static conception of a cyclic universe to a dynamic one, a space to navigate through time. Each set in $c = 5$ is generable either by step (g =1) or leaps ($g = 2$). Both values of g generate all sets of exterior cardinality. It is the intermediate cardinalities that are distinguishable by their generators: $g = 1$ generates steps of class [01] and their complements of class [012], and $g = 2$ generates leaps [02] and their complements [013].

$c = 5$ is musically realized as a pentatonic scale, whose intervals come in two chromatic sizes: "steps" are major seconds and minor thirds, and "leaps" are perfect fourths and a major third. Following John Clough [10], these chromatic distinctions are overlapped, and the pentatonic space is treated as if it were perfectly rather than maximally even. Step-generation involves a sequential pass through the scale, <C, D, E, G, A> ; leap-generation involves skipping scalar notes, <C, E, A, D, G>. There are dozens of pentatonic pieces that can be used as analytical illustrations of this universe, as compiled e.g in [14].

V. Cyclic Universes with 6 to 9 elements

As the half-way point between zero and twelve, C6 is the universe in which curious features with unexpected musical ramifications begin to arise. It is also the point where the size of the power set begins to get too large to control. The intermediate cardinalities consist of 15 pairs, 15 pair-complements, and 20 triplets. None of these numbers are multiples of c, indicating the presence of rotational invariances at each cardinality. There are three T-invariant diametric leaps [03] to go with the six steps [01] and six skips [02], and two T-invariant skip-generated triplets [024] to go with the the six step-generated clusters [012] and twelve ungenerated sets [013] and [014], which are discussed below.

Since six is the smallest number with two divisors, C6 is the smallest universe to have two distinct *perfectly even* sets. Realized as rhythms, these two perfectly even sets model the two meters available in a bar with six beats. Setting the beat to an eighth note, [03] suggests a bar of $\frac{6}{8}$ meter, and [024] a bar of $\frac{3}{4}$ meter with duple subdivisions. The successive juxtaposition of these meters models the Baroque pre-cadential hemiolas, and their superposition as [03] $\cup$ [024] = [0234] underlies the metric tug of a waltz, as well as the 3-against-2 cross-rhythms of West African and Afro-Caribbean repertories.

The complementary T-classes [013] and [014] introduce many new features that do not arise in smaller-cardinality universes. Neither set is inversionally symmetric; instead, the two classes are abstractly related to each other by inversion. Accordingly they have identical interval vectors: one instance of each interval, from which we see that C6 is the smallest universe that hosts non-trivial all-interval sets. The flatness of their interval vector is related to their ungenerability, as note above.

This is a good moment to make a systematic study on inversion (= reflection), showing how members of [013] and [014] invert into each other around a variety of axes. As in atonal pitch-class theory, questions of perceptibility immediately arise: can one hear inversionally related sets as "the same thing?" A study of Figure 3, from a Beethoven quartet, furnishes an opportunity to experience their **perceptual** non-equivalence. The cello and viola together sound beat classes {0, 3, 4}, which is a member of T-class [013]. The violin sounds the complementary beat classes,

Figure 3: *Beethoven, String Quartet no. 8, Op. 59 no. 2, Allegretto, bars 1 - 8. The violin attacks from mm. 2 - 7 are class [014]; the complementary attacks in the remaining parts are class [013].*

{1, 2, 5}, representing T-class [014]. For most listeners, these two beat-class sets project different meters. In general, when two adjacent time points are attacked but their immediately surrounding time points are tacit, listeners hear a phenomenal accent on the later attack of the pair[15]. The consequence is that set {1, 2, 5} reduces to {2, 5}, a member of [03] that bisects the measure, causing the violin to be heard in a displaced $\frac{6}{8}$ meter. By contrast, set 0, 3, 4 reduces to 0 4, which is filled out as 0, 2, 4 and thus the accompanying instruments project an undisplaced $\frac{3}{4}$ meter. In bar 8, all instruments articulate a [024] set, resolving the metric conflict at the cadence.

Both non-unit generators in C6 are divisors, and hence idempotent after several generations. There are also three generators in the prime universe of $c = 7$, but here each one retains its potency to generate the universe. These three generators have particular roles to play in the context of European diatonic tonality, the repertory at the core of most music-theory curricula: they respectively organize melody, harmony, and harmonic progression.

The three cyclic graphs of Figure 4 show the action of these generators on the diatonic collection. The unit generator ($g = 1$) forms scalar fragments, which are the basis of melodies. $g = 2$ generates tonal harmonies, the [024] diatonic triads and [0135] diatonic seventh chords. $g = 3$ generates the diatonic cycle of fifths, and thus its generated sets are the basis of progressions between successive harmonies. Most intermediate-cardinality set classes are generated by exactly one of these intervals, and thus can be seen to perform one of the three jobs. The ungenerated sets belong to the four inversionally asymmetric classes, [013], [023], [0124], and [0234], whose intervals have

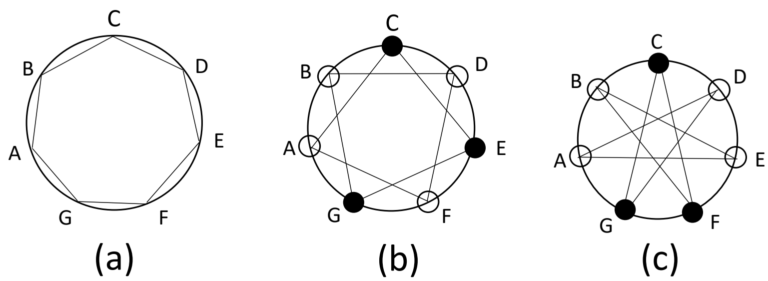

Figure 4: *Three generators on C7. (a) $g = 1$ produces a scale; (b)$g = 2$ produces chords such as triads (filled circles) and seventh chords (open circles). (c) $g = 3$ produces a cycle of fifths, connecting tonics to their dominants and subdominants.*

uniform multiplicities.

C7 is also an appropriate universe for introducing inclusion, *maximal evenness* (ME) and *Q-relations*. Inclusion follows naturally from a study of interval generation, since the sets from a single generator form an inclusion network. The ME property is held by fifths, triads, and seventh chords, the building blocks of classical tonality. The Q relation (my term) formalizes the parsimonious voice-leading relation between sets and set classes of equal cardinality [16]. All of these topics will have significant interpretations in the C12 chromatic universe.

$c = 8$ and $c = 9$ can be studied in order, but also in tandem, as both are powers of small primes. They are the smallest universes that have both divisor and non-unit prime generators [5], and the interaction of these two generator-classes is rich with dynamic potential in the context of metric cycles. The divisor generators for these universes (for $c = 8$, $g = (2, 4)$; for $c = 9$, $g = 3$) underlie the isochronous meters of the Western European tradition ($\frac{2}{4}$, $\frac{4}{4}$, and $\frac{9}{8}$ meter respectively). Non-unit prime intervals ($c = 8$, $d = 3$ and $c = 9$, $d = 2$) generate the non-isochronous meters [17] of vernacular repertories [5]. The rhytmic dialectic of concert and vernacular traditions is artfully exploited in a number of repertories beginning in the middle of the 19th century. In $\frac{4}{4}$ meter, Western pure-duple isochrony is juxtaposed with the Afro-Caribbean tresillo (Gottschalk, Joplin [18]) or paradiddle (Reich [19]). A similar juxtaposition is available in $\frac{9}{8}$ meter, where pure-triple isochrony is juxtaposed with characteristic Balkan (aksak) rhythms (Bartók [20], Brubeck [18]).

VI. Scaling up to C12

In the course I have developed for undergraduate music majors at Yale University, the progression to nine elements takes about six 75-minute classes, or three weeks of a thirteen-week semester. Since 10 and 11 have no familiar applications, in fourth week I begin the study of atonality proper, introducing the C12 chromatic pitch-class universe. Students are by now familiar with all of the foundational terms, concepts, and protocols of atonal theory. They can quickly generate a table of the 19 trichord classes, indicate which ones are inversionally symmetric, which two are inversionally paired, and recognize the special properties of the augmented triad; place trichord classes into a Q-relation network [16]; quickly assign trichords to prime forms; and identify abstract inclusion and complement relations with largercardinality sets.

Students also are in a position to appreciate what is special about living in a musical universe that has exactly twelve elements, and how different their world would be if that number were incrementally smaller or larger. Students who have studied the interaction of divisor generators

3 x 2 in C6 are primed to understand the C12 as a cross-product of augmented triads and diminished-seventh chords, and to appreciate the ways that atonal composers compound these generated cycles to create interactions between hexatonic, whole-tone, and octatonic scales [21]. Because they have studied how prime and divisor generators interact in C8 and C9, they are in a position to understand how diatonic and chromatic-cluster sets can be placed into opposition with divisor-generated ones in C12. Because they have studied transpositional invariance and maximal evenness in small-cardinality universes, they can identify sets with those properties in C12, and recognize their musical significance. There are also significant applications of C12 in the beat-class domain. For example, the relations of the divisor-generated perfectly even sets underlies the interaction of ($\frac{12}{8}$, $\frac{3}{4}$, and $\frac{3}{2}$ meters, as different ways to structure the interaction of incommensurate divisor generators [22],[23].

Along the way to C12, students have picked up intimate knowledge of pentatonic, diatonic, and time-cycle universes, and this knowledge has its own value to musicians. Much of the music that they perform, listen to, improvise, or compose is both diatonic and deeply metric. Students who have pursued this pedagogical path to atonality will have, along the way, acquired a mode that helps them explore the relationship between music with those ubiquitous properties, musical systems in which they participate, and the properties of the abstract universes that those systems instantiate.

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