All-(Generalized-)Interval(-System) Chords

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Abstract: *We survey the all-interval chords of small order and the interval systems in which they are situated. We begin with an examination of traditional all-interval chords in chromatic pitch-class spaces, and extend the notion of their structure to their counterparts in David Lewin's Generalized Interval Systems. Mathematically, we observe that these chords belong to three categories of difference sets from the field of combinatorics: $(v, k, 1)$ planar difference sets, $(v, k, 2)$ non-planar difference sets, and $(v, k, 1, t)$ almost difference sets. Further, we explore sets of all-interval chords in group-theoretical terms, where such sets are obtained as orbits under the action of the normalizer of the interval group. This inquiry leads to a catalog of the 11,438 all-interval chords of order k, where $2 \leqslant k \leqslant 8$. We conclude with remarks about future work and open questions.*

Keywords: *All-interval Chords. Generalized Interval Systems. Group Theory. Difference Sets.*

I. INTRODUCTION

The compactness and efficiency of all-interval chords have attracted the attention of composers and music theorists since the early years of the twentieth century. Such structures, which include one and only one of each interval in a given interval system, are rich compositional resources, as well as topics of theoretical interest to students of interval systems themselves. In particular, all-interval tetrachords in 12-tone chromatic space, represented in pitch-class set theory by the prime forms $[0, 1, 4, 6]_{12}$ and $[0, 1, 3, 7]_{12}$, [1] have received widespread application in the music of the major post-tonal composers. Among numerous notable examples, the first song in Arnold Schoenberg's *Das Buch der hängenden Gärten*, Op. 15, "Unterm Schutz von dichten Blättergründen," ends with a chord, $\{3, 5, 8, 9\}_{12}$, that is a member of set class $[0, 1, 4, 6]_{12}$ (see Figure 1). Likewise, the final song in Alban Berg's Vier Gesänge, Op. 2, "Warm die Lüfte," contains a passage (mm. 20-22) that consists exclusively of all-interval tetrachords, alternating members of set classes $[0, 1, 3, 7]_{12}$ and $[0, 1, 4, 6]_{12}$ (Figure 2). Indeed, entire compositions are constructed around all-interval tetrachords. For instance, Elliott Carter's First and Second String Quartets both incorporate these collections locally and structurally [1].

All-interval tetrachords also feature prominently in post-tonal theoretical writings. Each of the standard pitch-class-set-theoretical texts, beginning with Howard Hanson's 1960 *Harmonic Materials of Modern Music* [2], incorporates description and examples of these structures. [2] Relevant

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[1] As we consider sets of integers in a variety of moduli, we indicate the modulus as a subscript following a set. For example, $\{0, 1, 4, 6\}_{12}$ is the set of integers 0, 1, 4, and 6 (modulo 12).

[2] Hanson represents members of both set classes $[0, 1, 4, 6]_{12}$ and $[0, 1, 3, 7]_{12}$ not with pitch-class integers, but with his notation for their "interval analyses," *pmnsdt*, the letters of which indicate singular projections of the following intervals: perfect fifth (*p*), minor second (*m*), major second (*n*), minor third (*s*), major third (*d*), and tritone (*t*) [2, p. 22].
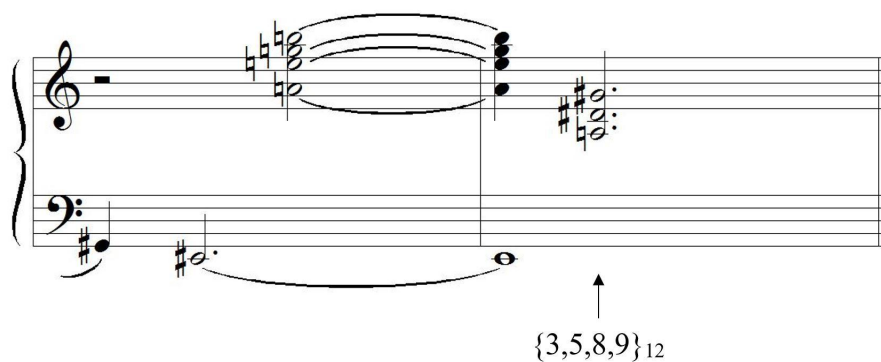
Figure 1: *Schoenberg, "Unterm Schutz von dichten Blättergründen," final chord.*
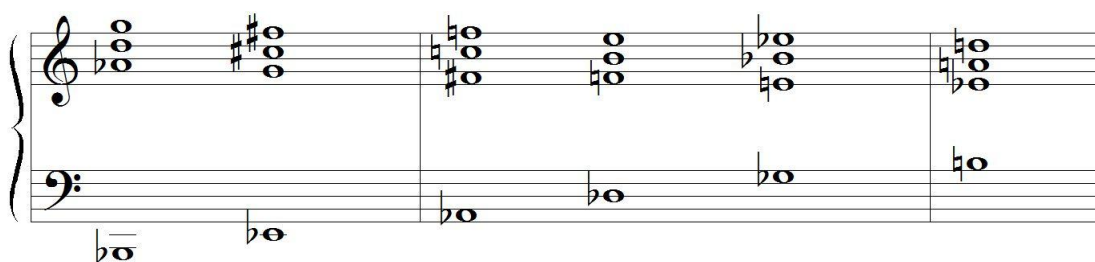

Figure 2: *Berg, "Warm die Lüfte," mm. 20-22 (voice-leading reduction).*

discussions appear in Allen Forte's *The Structure of Atonal Music* [3], John Rahn's *Basic Atonal Theory* [4], Robert Morris's *Composition with Pitch Classes* [5], George Perle's *Twelve-Tone Tonality* [6], Stefan Kostka's *Materials and Techniques of Post Tonal Music* [7], and Joseph Straus's *Introduction to Post-Tonal Theory* [8], among other sources. All-interval tetrachords have also been studied in terms of their transformational properties [9], and for their role in our understanding of the Z-relation [10].[3]

Treatments of all-interval chords in other chromatic spaces are comparatively rare in the literature. One significant work that addresses all-interval chords of varying sizes in microtonal systems is Carlton Gamer and Robin Wilson's "Microtones and projective planes" [11]. Gamer and Wilson present all-interval trichords, tetrachords, and hexachords in 7-, 13-, and 31-tone chromatic spaces, respectively, as difference sets, a concept from mathematical combinatorial theory. For their purposes, they define "a difference set (modulo n) to be a set of distinct integers $c_1, ..., c_k$ (modulo n) for which the differences $c_i - c_j$ (for $i \neq j$) include each non-zero integer (modulo n) exactly once" (p. 153). Mathematicians call such difference sets — wherein each difference appears exactly once — planar difference sets [12]. Such sets are a special type of all-interval chord: whereas every planar difference set is an all-interval chord, not every all-interval chord is a planar difference set.

This article moves beyond an examination of all-interval chords in chromatic systems to one of corresponding structures in David Lewin's *Generalized Interval Systems (GISs)* [13]. Specifically, it enumerates the isomorphism classes of all-interval k-chords of small order (i.e., those with $2 \leq k \leq 8$). In addition to the all-interval chords found in cyclic interval groups, such as those

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[3]Z-related pitch-class sets are those that possess the same interval vector, but which are not related by transposition and/or inversion.

above, we note their occurrence in certain non-cyclic abelian and non-abelian interval groups. Among these chords, we find planar and non-planar difference sets, as well as almost difference sets [14], a related concept that comes from combinatorics.

II.   Intervals and Interval Vectors in Chromatic Spaces

In this section, we discuss intervals and interval vectors in chromatic spaces, and extend relevant aspects to more general spaces in the following section. To define our concept of interval, it is necessary first to establish the context in which we find intervals. We call such a context a space: a universal set of musical objects, allowing that a path exists between any two members of the space. The examples above situate their intervals in v-tone (modular) chromatic spaces,[4] wherein the musical objects are pitch classes. Here, the notion of an interval between two pitch classes is construed as a distance, the number of chromatic steps as an integer modulo n. Intervals in such spaces may be directed or non-directed. Typically, melodic intervals are indicated as being directed: the distance from pitch class x to pitch class y. This type of interval is reckoned $y - x$ (modulo v). Harmonic intervals, on the other hand, are non-directed: the static distance between pitch classes x and y. The non-directed interval between pitch classes x and y is customarily represented by the lesser of $y - x$ (modulo n) or its inverse, $-(y - x) = x - y$ (modulo n). Throughout the remainder of this study, we refer to non-directed intervals simply as "intervals," whereas we always retain the qualifier "directed" when referring to directed intervals.

We are interested in the total interval content and total directed-interval content of a subset of a space. In the music-theoretical literature, one finds a distinction between tallies of a subset's interval content and those of its directed intervals. For subsets in 12-tone chromatic space, a tool that counts the number of occurrences of each directed interval is Lewin's 1960 interval vector [15], which we call a directed-interval vector or DIV. It consists of a 12-member array, in which the first coordinate lists the number of occurrences of directed intervals of length 0 (unisons); the second coordinate, directed intervals of length 1; the third, length 2; and so on through length 11. For a subset D of size k, the sum of the vector's coordinates is $k2$. For example, the DIV for the final chord $\{3, 5, 8, 9\}_{12}$ in "Unterm Schutz von dichten Blättergründen" (from Figure 1) is (411111211111): we find four unisons (e.g., between each pitch class and itself), two instances of directed intervals of length 6 (from pitch class 3 to 9, and from 9 to 3), and one of each of the remaining ten lengths, for a total of 42 = 16 directed intervals. DIVs can be adapted easily to other v-tone chromatic spaces. Essentially, one uses a v-member array, wherein the first coordinate lists the number of occurrences of directed intervals of length 0, the second coordinate directed intervals of length 1, the third length 2, and so on through length $v - 1$. Again, for a subset of size k, the sum of the vector's coordinates is $k2$. For instance, the $3^2 = 9$ directed intervals among the members of $\{0, 1, 3\}_7$ yield the DIV (3111111).

Tallies of non-directed interval content in subsets of 12-tone chromatic space are typically represented using Allen Forte's 1973 interval vector [3], also known as an interval-class vector or ICV. The ICV is a 6-member array in which the respective coordinates list the number of occurrences of each interval in a pitch-class set.[5] For a subset of size k, the sum of the coordinates in an ICV is the binomial coefficient $\binom{k}{2} = (k(k-1))/2$. For example, the ICV for the $\{3, 5, 8, 9\}_{12}$

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[4]Henceforth in this study, we use the variable v for the size of a space rather than the more customary n. This substitution is for consistency with our later incorporation of the standard notation (v, k, λ) for difference sets, in which v is the size of the group.

[5] It is significant to note that, whereas a DIV contains a coordinate (the first) that counts the number of unison intervals in a pitch-class set, an ICV does not count unison intervals.

tetrachord from Figure 1 is $[111111]$.[6] For v-tone chromatic spaces in general, the ICV is a $[\frac{v}{2}]$-member array that shows the number of occurrences of each interval class in a pitch-class set in order of ascending size from 1 to $[\frac{v}{2}]$.[7] As above, the sum of the vector's coordinates for a subset of size k is the binomial coefficient $t\binom{k}{2}$. For example, in a chromatic space of size 7, we find $[\frac{7}{2}] = 3$ interval classes. Accordingly, the ICV for the pitch class set $\{0, 1, 3\}_7$ is $[111]$. An important difference exists between Lewin's and Forte's vectors: whereas Lewin's vector counts the directed interval from pitch class x to pitch class y separately from the directed interval from y to x, Forte's vector counts the non-directed interval between pitch classes x and y only once. This distinction between Lewin's and Forte's vectors leads to significant results in later sections.

III. Intervals and Interval Vectors in Generalized Interval Systems

In this study, we conceptualize intervals in the manner of David Lewin's *Generalized Musical Intervals and Transformations (GMIT)* [13]; more precisely, Lewin's intervals agree with our notion of directed intervals. In this sense, a directed interval is a member of a mathematical group that has a simply transitive action on a space. Simple transitivity requires that (a) the action is transitive, i.e., each member of the space is related to every other member (and to itself) by some directed interval in the group; and (b) the action is free, a consequence of which is that one and only one directed interval relates a member of the space to any other member (or to itself). For instance, in each of the examples in the previous section, the interval group is the group of integers modulo v, $\mathbb{Z}_v$, which has a simply transitive action on the space S of v pitch classes. An interval exists between any two pitch classes x and y in S, and we find one and only one directed interval from x to y : yx (modulo v). Whereas we can generalize this situation to abstract cyclic groups and other types of group structures, Lewin's generalized intervals differ in significant ways from traditionally defined directed intervals. In particular, they do not possess qualities of distance and direction [16]. Instead, we are interested their functioning as "characteristic motions" among the members of a space [13, p. xxix].

GMIT does not address the notion of non-directed intervals. It is possible, however, to generalize these intervals in a manner that is consistent with Lewin's work. In particular, a non-directed interval (or interval) is an equivalence class (interval class) that contains a directed interval and its inverse. If a directed interval is equivalent to its inverse, such as is the case with an involution, then the interval class that includes it is a singleton. For instance, the interval class that includes $x \in \mathbb{Z}_{12}, x \neq 6$, also includes x (modulo 12), whereas the interval class that contains the involution $6 \in \mathbb{Z}_{12}$ consists of that element alone, as $6 \equiv -6$ (modulo 12).

As many of our subsequent examples involve non-abelian groups, our notation for intervals and directed intervals follows that of group elements in multiplicative groups (rather than the additive notation used with abelian groups). In compositions of such group elements, we incorporate right orthography (i.e., the product gh, where $g, h \in G$, means "do g first, then do h"). The composition gg is notated g^2, the inverse of an element g is indicated as g^{-1}, etc. Because a generalized directed interval g does not possess the quality of distance, we cannot merely label its interval class with

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[6]We use different bracket styles to distinguish between directed-interval vectors and interval-class vectors. For the former, we use parentheses, whereas we use square brackets for the latter.

[7]We use the floor function, $[\frac{v}{2}]$, in tabulating the number of interval classes in a cyclic group, on account of the variance in the number involutions in cyclic groups of even and odd orders. A standard result in group theory shows that even-order cyclic groups always have one involution, (e.g., 6 in $\mathbb{Z}_{12}$), whereas odd-order cyclic groups have none. As we will see below, the number of involutions helps determine the number of interval classes in a group.

the interval class's shorter constituent, g or g^{-1} (as is the custom in chromatic spaces). Hence, we designate the interval class that contains g and its inverse as $g^{\pm 1}$ (Accordingly, if $h = g^{-1}$, then $h^{\pm 1} = g^{\pm 1}$.)

In this study, we are concerned at times with subsets of spaces, and at other times with subsets of interval groups themselves. In particular, in GIS theory, it is sometimes more convenient to refer to elements of an interval group rather than those of a space. As a result of its simply transitive action, an isomorphism exists between an interval group and the space on which it acts. Technically, the space S on which a group G acts simply transitively is called a *G-torsor* [17] *S* is isomorphic to *G*, except that no point in S corresponds a priori to the identity element of *G*. However, once such an association is chosen — as in assigning the pitch class C to the identity element $0 \in \mathbb{Z}_{12}$ — the bijection of the remaining members in *S* to group elements in *G* is determined by right multiplication (for non-abelian groups; by addition for abelian groups).[8] We may therefore identify subsets of the space (i.e., chords) with subsets of the group (and vice versa). For instance, in the chromatic-space examples above, both the interval group of order v and the space of v pitch classes can be modeled with the integers modulo v, $\mathbb{Z}_v$. Having assigned the pitch class C to 0 as an origin in 12-tone chromatic space, we can interpret the members of a pitch-class set, such as $[0, 1, 4, 6]_{12}$, equally as pitch classes or as directed intervals from the origin.

Directed-interval vectors can be adapted to Lewin's GISs by replacing tallies of directed intervals of varying lengths with those of individual group elements, as long as it is made clear which coordinate in the vector represents the number of occurrences of which group element. As above, for an interval group of order v, the DIV has v coordinates, which sum to v^2. As with Lewin's interval vector, Forte's interval-class vector can also be adapted for use with interval groups in GISs. Again, it is necessary to establish which coordinate of the vector counts the occurrences of which interval class. For an interval group G, the number of coordinates in an ICV can be determined by the following formula, where w is the number of involutions in G, and $|G^\sharp|$ is the size of the set of non-identity elements in G.[9]

$$\frac{w + |G^\sharp|}{2}$$

As above, the sum of the ICV's coordinates for a set of size k equals the binomial coefficient $\binom{k}{2}$.

IV. All-Interval Chords

An all-interval chord is a subset of a space that possesses among its members at least one of every interval in the interval group that acts on that space [3]. Put another way, an all-interval chord is one which contains no 0s in its ICV (or, in the case of all-directed-interval chords, in its DIV). As we observe in §1, however, composers and music theorists have traditionally been interested in a special category of all-interval chords: those that contain *one and only one* of each non-unison interval, as such chords have the highest degree of intervallic efficiency. In these

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[8]One may also use left multiplication, yielding (for non-abelian groups) a *G-torsor* that is anti-isomorphic to the one determined by right multiplication. We incorporate right multiplication here for consistency with our use of right orthography.

[9]Using results from character theory [18], we determine the number w of involutions in a finite group G via the following:

$$w = \Sigma_\chi w(\chi)\chi(1)$$

where χ runs through the complex characters of G, and $w(\chi)$ is the Frobenius-Shur indicator of χ. Consequently, w is always either 0 or odd. That fact, together with the classical result that a group of odd order contains no involutions, means that $w + |G^\sharp|$ is always divisible by 2.

chords, the full set of available intervals is present in as small a subset as possible. Henceforth, when we refer to all-interval chords and all-directed-interval chords, we indicate chords with this particular property. A significant relationship exists between all-directed-interval chords and all-interval chords: any all-directed-interval chord is also all-interval, but the reverse is not necessarily true. For instance, a trichord with the prime form $[0, 1, 3, 7]_7$ possesses the DIV (3111111); it is accordingly an all-directed-interval chord. Its ICV, [111], indicates that this trichord is also all-interval. In contrast, a trichord with the prime form $[0, 1, 3, 7]_6$ is not an all-directed-interval chord in the specific sense described above. Its DIV is (311211), which contains a 2, not a 1, in its fourth coordinate; nevertheless, it is all-interval, as is evident from its ICV: [111].

Given the isomorphism from a *G-torsor* to the interval group *G*, we can define all-interval chords and all-directed-interval chords not only as subsets of a space, but as subsets of an interval group. In this connection, we examine various concepts from the theory of difference sets in the field of combinatorics. Most generally a difference set is a subset $D = (v, k,)$ of a group *G*, where *v* is the order of *G*, *k* is the size of *D*, and every non-identity element of *G* appears exactly λ times as compositions of elements gh^{-1}, where $g, h \in D$. Such subsets possess the quality of having a flat directed-interval distribution (i.e., all coordinates of non-unison directed intervals in their DIVs are equal to λ). For instance, let $G = \{P, I, R, RI\}$ be the group of basic twelve-tone row operations (prime [identity], inversion, retrograde, and retrograde-inversion), isomorphic to the Klein four-group, $\mathbb{Z}_2^2$. The trichordal subset $D = P, I, R$ of *G* is a $(4, 3, 2)$ difference set. *G* is of order 4; *D* is a three-element subset of *G*; and each non-identity element of *G* appears exactly twice as a product gh^{-1} of elements *g* and *h* in $D : PI^{-1} = IP^{-1} = I, PR^{-1} = RP^{-1} = R$, and $IR^{-1} = RI^{-1} = RI$, as is evident in the trichord's DIV (3222).

All-directed-interval chords are a particular category of difference set. A planar difference set is one in which $\lambda = 1$ (i.e., each non-unison directed interval appears exactly once). The $k = 4$ subset $\{0, 1, 3, 9\}_{13}$ of $\mathbb{Z}_{13}$ serves as an example; its DIV, (4111111111111), demonstrates the unary directed-interval distribution that distinguishes it as a planar difference set. A conjecture in the field of combinatorics [12, p. 421] states that if $\lambda = 1$, then $k - 1$ must be the power of a prime. That is, there are no planar difference sets of sizes 6, 10, 12. . . .

The familiar all-interval tetrachords of pitch-class set theory, $\{0, 1, 4, 6\}_{12}$ and $\{0, 1, 3, 7\}_{12}$, are not planar difference sets. In fact, they are not difference sets. As indicated by their shared DIV, (411111211111), these tetrachords do not possess flat directed-interval distributions. Rather, they are examples of almost difference sets. An almost difference set is a subset $D = (v, k, \lambda, t)$ of *G*, where *v* and *k* are defined as above; *t* non-identity elements of *G* appear exactly λ times as compositions of elements gh^{-1}, where $g, h \in D$; and the remaining $v - 1 - t$ non-identity elements of *G* appear $\lambda + 1$ times as gh^{-1} compositions. Hence, $\{0, 1, 4, 6\}_{12}$ and $\{0, 1, 3, 7\}_{12}$ are examples of $(12, 4, 1, 10)$ almost difference sets. Their DIVs possess ten coordinates that equal 1, and $12 - 1 - 10 = 1$ coordinate that equals $1 + 1 = 2$: the difference 6 (modulo 12).

Whereas planar difference sets are always all-directed-interval, and therefore also all-interval, non-planar difference sets and almost difference sets are never all-directed-interval. Furthermore, they are all-interval only if the following two circumstances are met. First, they must have $1 \leq \lambda \leq 2$; and, second, any element in *G* (i.e., directed interval) with $\lambda = 2$ must be an involution. For example, the trichord $\{P, I, R\}$ above is an example of a $(4, 3, 2)$ non-planar difference set. Every gh^{-1} composition has $\lambda = 2$ and is also an involution. Similarly, $\{0, 1, 4, 6\}_{12}$ and $\{0, 1, 3, 7\}_{12}$ are $(12, 4, 1, 10)$ almost difference sets, in which the gh^{-1} compositions in either set with $\lambda = 2$ are involutions, i.e., 6 (modulo 12). As we see below, these three categories of difference sets, planar and non-planar difference sets and almost difference sets, account for every all-interval chord (up to isomorphism) of small order (i.e., $2 \leq k \leq 8$).

V. Interval Groups and All-Interval Chords

A significant relationship exists between the number of intervals in a group G and the potential for its having all-interval subsets. A subset D of size k in G has a triangular number, $\binom{k}{2}$, of unordered dyads that can be labeled with intervals. Therefore, if D is to include one and only one occurrence of every interval in G, then G must have exactly $\binom{k}{2}$ intervals. For instance, a tetrachord contains $\binom{4}{2} = 6$ unordered dyads that can be labeled with intervals. For it to be an all-interval tetrachord, the group that contains these intervals must also have six interval classes, as does $\mathbb{Z}_{12}$. Two factors determine how many intervals are in a group: the order of the group itself (minus the identity) and the number of involutions that it contains. As we observe above in §3, the number of intervals is equal to the sum of number of involutions in the group plus half the number of elements of order > 2. If this number equals $\binom{k}{2}$ for some k, the group may potentially contain all-interval k-chords. As we demonstrate below, however, this condition is necessary — but not sufficient — for the existence of all-interval chords.

It is possible for a group to have $\binom{k}{2}$ intervals, and not to contain any all-interval k-chords. For example, the dicyclic group of order 12, $Dic_{12} = \langle x, y | x^6 = y^4 = 1, x^3 = y^2, y^{-1}xy = x^{-1} \rangle$, contains $\binom{4}{2} = 6$ intervals, the same as $\mathbb{Z}_{12}$, but it has no all-interval tetrachords. Dic_{12} has a cyclic subgroup of 6, generated by an element x, which yields three intervals: $x^{\pm 1}$, $x^{\pm 2}$, and $x^{\pm 3}$ (an involution). The remaining six elements of Dic_{12} are all of order 4, yielding three additional interval classes. Moreover, for any of these elements y of order 4, y^2 is equal to the single involution within the cyclic subgroup, x^3. Nevertheless, the existence of an all-interval tetrachord fails. It requires one interval labeled as $x^{\pm 3}$, such as the interval between x^0 (the identity element) and x^3, and one occurrence of y. However, the interval between x^0 and y and the interval between y and x^3 are the same, i.e., $y(x^0)^{-1} = x^3 y^{-1} = y$, resulting in more than one occurrence of that interval.

In terms of a musical representation, Dic_{12} is isomorphic to a particular transposition/skew-inversion group.[10] This group has a cyclic subgroup that consists of the six transposition operators with even indices (i.e., T_m, m is even), and the remaining six elements of order 4 are skew-inversions with odd indices (i.e., S_n, n is odd). Table 1 lists the cycles of the eleven non-identity elements of this group as they act on the set of twelve chromatic pitch classes. As any all-interval tetrachord in this group requires a tritone, let us select arbitrarily the tritone {1, 7} from the odd whole-tone collection. Further, such a tetrachord would also require at least one pitch class from the even whole-tone collection; we choose 6. From the cycles in the table, however, we see that the interval between 1 and 6 and the interval between 6 and 7 are the same, $S_1^{\pm 1}$, which is not allowed in an all-interval chord. As this situation occurs for any combination of a tritone from one parity's whole-tone collection and a single pitch class from the opposite parity's whole-tone collection, the existence of an all-interval tetrachord fails.

Certain limits exist on the size of groups that may include all-interval k-chords. The smallest groups that can potentially accommodate an all-interval k-chord are elementary abelian 2-groups, $\mathbb{Z}_2^n$, wherein every non-identity element is an involution. These groups contain a Mersenne number, $2n - 1$, of non-unison intervals. For example, the group of basic twelve-tone row operations is isomorphic to $\mathbb{Z}_2^2$, and it contains an all-interval trichord, e.g., P, I, RI. However, instances in which a triangular number $\binom{k}{2}$ equals some Mersenne number $2n - 1$ are rare.

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[10]Whereas an inversion is a reflection in pitch-class space, i.e., an operation of order 2, a skew-inversion is a pseudo-reflection of order 4 (i.e., a reflection of order > 2; see [19]). Under a skew-inversion, pitch classes of one parity, even or odd, map as normal inversions to their counterparts of the other parity, but when they reflect back to those in the original parity, they return a tritone away. Hence, four iterations of the cycle are required before returning to the original pitch classes. For instance, the operation on pitch classes $5x + y$ (modulo 12), is a skew-inversion for any odd y. Skew-inversions are similar to skew-Wechsels of neo-Riemannian theory (a category of contextual inversions), which are discussed in more detail in [20].

Table 1: *Non-trivial cycles of elements in the transposition/skew-inversion group $G \simeq Dic_{12}$.*

$T_2 := (0,2,4,6,9,10)(1,3,5,7,9,11)$	$T_{10} := (0,10,8,6,4,2)(1,11,9,7,5,3)$
$T_4 := (0,4,8)(1,5,9)(2,6,10)(3,7,11)$	$T_8 := (0,8,4)(1,9,5)(2,10,6)(3,11,7)$
$T_6 := (0,6)(1,7)(2,8)(3,9)(4,10)(5,11)$	
$S_1 := (0,1,6,7)(2,11,8,5)(3,4,9,10)$	$S_7 := (0,7,6,1)(2,5,8,11)(3,10,9,4)$
$S_3 := (0,3,6,9)(1,8,7,2)(4,11,10,5)$	$S_9 := (0,9,6,3)(1,2,7,8)(4,5,10,11)$
$S_5 := (0,5,6,11)(1,10,7,4)(2,3,8,9)$	$S_{11} := (0,11,6,5)(1,4,7,10)(2,9,8,3)$

Nevertheless, of the three known examples, two values of $2 \leq k \leq 8$ satisfy this condition, 3 and 6: $\binom{3}{2} = 2^2 - 1$, and $\binom{6}{2} = 2^4 - 1$.[11] In contrast, the largest groups that can potentially accommodate an all-interval k-chord contain no involutions. As groups of odd order contain no involutions, at least one isomorphism class — the cyclic group — exists for each odd order (and some odd orders contain additional isomorphism classes of groups). For instance, $\mathbb{Z}_7$ is the largest group to have all-interval trichords. It contains three intervals, none of which is an involution. However, it is useful to note again that the existence of a group with $\binom{k}{2}$ intervals — whether large, small, or in between — does not guarantee the existence of all-interval subsets.

VI. Sets of All-Interval Chords

Aside from $\mathbb{Z}_2$, which contains a single all-interval chord, $\{0,1\}$, interval groups that contain one all-interval chord also contain additional all-interval chords.[12] The set classes $[0,1,4,6]_{12}$ and $[0,1,3,7]_{12}$ in $\mathbb{Z}_{12}$ — orbits of these tetrachords under the action of the dihedral transposition and inversion group — are familiar examples of sets of all-interval chords. Each orbit is of size 24: twelve all-interval tetrachords that relate to one another by transposition, and twelve more that relate to those by inversion. As no other tetrachords in $\mathbb{Z}_{12}$ exist with the ICV (111111), these forty-eight forms constitute the full set of all-interval chords contained in this interval system.

The members of set-classes $[0,1,4,6]_{12}$ are related to those of $[0,1,3,7]_{12}$ by neither transposition nor inversion (and vice versa), and yet the members of both set-classes have the same intervallic content. Hence, we observe that these set classes are Z-related [3]. Their members also relate to one another's by the multiplicative operations M and MI: pitch-class multiplication by 5 and 7 (modulo 12), respectively [21]. This situation occurs in all cyclic interval groups: if D is an all-interval subset of $\mathbb{Z}_n$, then the affine transformations of D, $Dx + a = \{dx + a | d \in D, x \in \mathbb{Z}_n$ co-prime to $n, a \in \mathbb{Z}_n\}$, are also all-interval.[13] The reasoning is straightforward: affine transformations do not preserve distances, but they preserve ratios of distances. Accordingly, if D contains the set of all distances in the finite space $\mathbb{Z}_n$, a transformation that preserves the ratios of these distances results in a permutation on the set of distances.

The set of affine transformations on $\mathbb{Z}_n$ form a group, Aff($\mathbb{Z}_n$), the action of which on D yields a set of Z-related all-interval subsets. In general, Aff($\mathbb{Z}_n$) is equivalent to the normalizer of $\mathbb{Z}_n$ in the symmetric group on $\mathbb{Z}_n : N_{Sym(\mathbb{Z}_n)}\mathbb{Z}_n$.[14] This correspondence allows us to carry the

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[11]The next smallest value of k to satisfy this condition is 91, as $\binom{91}{2} = 2^{12} - 1 = 4095$. No further examples of reasonable size exist (and it is possible that no further examples exist at all).

[12]We might also say that the trivial group $\mathbb{Z}_1$ contain one all-interval chord, $\{0\}$, consisting of a single pitch class. It has one and only one occurrence of the sole interval in that group, the unison. However, we do not include this example, as we are concerned in this study only with non-unison intervals.

[13]The affine group Aff($\mathbb{Z}_n$) is the set of all transformations $xy + z$ (modulo n), where $y, z \in \mathbb{Z}_n$, and $x \in \mathbb{Z}_n$ is co-prime to n.

[14]The normalizer of a group G in another group H is the subgroup of elements in H that preserve G under conjugation.

Table 2: *An all-interval pentachord D in the interval system $G \simeq S_3 \times \mathbb{Z}_3$.*

	Sonority 1	Sonority 2	Sonority 3	Sonority 4	Sonority 5
High	C_3^5	B_3^5	C_3^5	C_3^6	A_4^6
Middle	B_3^5	C_3^5	A_3^5	B_3^6	C_4^6
Low	A_3^5	A_3^5	B_3^5	A_3^6	B_4^6

notion of affine transformations to other types of interval groups. If *G* is a group that acts simply transitively on a space *S*, then the normalizer of *G* in the symmetric group on *S*, $N_{Sym(S)}G$, serves as an analog for the affine group. As the action of $\mathrm{Aff}(\mathbb{Z}_n)$ on *D* yields a set of all-interval subsets in $\mathbb{Z}_n$, including Z-related subsets, the action of $N_{Sym(S)}G$ on *D* yields a set of all-interval subsets in any *S*, including GISZ-related subsets.[15] For all cases except one (as we discuss in §7), this action produces the full set of all-interval chords of small order for any interval system.

An example of a set of all-interval chords can be found in the following interval system. Define a 9-note sonority that is separated into three distinct registers: high, middle, and low. Put an A major triad in one register, a B major triad in a second register, and a C major triad in the third. In any one sonority, all three triads must appear in the same position: root position $\begin{bmatrix} 5 \\ 3 \end{bmatrix}$, first inversion $\begin{bmatrix} 6 \\ 3 \end{bmatrix}$, or second inversion $\begin{bmatrix} 6 \\ 4 \end{bmatrix}$. The space *S* of all possible configurations of such sonorities is of size eighteen: six permutations of the triads in three registers, and three positions in which the triads may appear. The interval group *G* that has a natural action on the space of these eighteen sonorities is isomorphic to the direct product of the symmetric group of degree 3 by the cyclic group of order 3, $S3 \times \mathbb{Z}_3$ (of order 18). This interval system has $\binom{5}{2} = 10$ interval classes: seven interval classes of invertible elements and three involutions. Moreover, it allows for all-interval 5-member subsets. For instance, the subset *D* of five sonorities that appear in Table 2 contains one and only one of each interval in this group.

The symmetric group Sym(*S*) on the space *S* of these sonorities is of size 18! = 6,402,373,705,728,000. Within this symmetric group, 216 operations normalize *G*. The orbit of D under $N_{Sym(S)}G$ is of size 108 (hence, each element of this orbit is stabilized by 216/108 = 2 members of the normalizer). These 108 pentachords constitute the full set of all-interval subsets for this interval system. We do not find GISZ-relations in this set particular of all-interval chords. All 108 pentachords in the orbit of D under the action of the normalizer are also in the orbit of D under the respective actions of the group of GIS-transposition and GIS-inversion operations and the group of interval-preserving operations.

VII.   All-Interval Chords of Small Order ($2 \leq k \leq 8$)

In this section, we catalog all the interval groups (up to isomorphism) that contain all-interval chords of orders $2 \leq k \leq 8$. For each chord size, we include remarks about the groups that produce all-interval chords, including the number of such chords that they contain and the types of difference sets that those chords exemplify, whether the groups are abelian or non-abelian, and other relevant information.

That is, $N_H G = h \in H | h^{-1} G h = G$.

[15]The GISZ relation is Lewin's adaption of the *Z* relation to the theory of Generalized Interval Systems. Sets that are GISZ-related to one another are related by neither GIS transposition nor GIS inversion (nor, in the non-abelian case, by members of the group of interval-preserving transformations) [22]

Bichords. The smallest all-interval chords include only one (non-unison) interval; hence, they are of size $k = 2$. Accordingly, the interval groups that contain these bichords must themselves have only one interval class. The smallest isomorphism class of such groups is the cyclic group of order 2, $\mathbb{Z}_2$. In this group, the single non-identity element is an involution. $\mathbb{Z}_2$ includes only one all-interval chord, the smallest non-empty set of all-interval chords for any group. This bichord is an example of a (2,2,2) non-planar difference set. The largest group with one interval class is the cyclic group of order 3, $\mathbb{Z}_3$, which possesses no involutions. $\mathbb{Z}_3$ contains three all-interval chords. As this group has no involutions, these all-interval chords are also all-directed-interval chords. The all-interval bichords in $\mathbb{Z}_3$ are examples of (3,2,1) planar difference sets, the smallest non-trivial class of these structures. Both the above groups are abelian.

Trichords. Groups with all-interval trichords must contain three interval classes. Three isomorphism classes of groups have the appropriate number: the Klein four-group, Z_2^2; the cyclic group of order 6, $\mathbb{Z}_6$; and the cyclic group of order 7, $\mathbb{Z}_7$. All three possess all-interval trichords. The three non-identity elements of Z_2^2 are all involutions, making it the smallest group to have all-interval trichords, as well as the smallest non-cyclic group to contain all-interval chords of any size. The four all-interval trichords in Z_2^2 are examples of (4,3,2) non-planar difference sets. $\mathbb{Z}_6$ contains two interval classes of invertible elements and one involution. Its twelve all-interval trichords are examples of (6,3,1,4) almost difference sets; as such, it is the smallest group to include all-interval chords with this type of structure. In contrast, $\mathbb{Z}_7$ contains no involutions. Its fourteen all-interval trichords are examples of (7,3,1) planar difference sets; hence, they are also all-directed interval. As with the bichords, the three groups that contain all-interval trichords are abelian.

Tetrachords. For a group to accommodate all-interval tetrachords, it must have six interval classes. Four groups satisfy this requirement: the dihedral group of order 8, D_8; the cyclic group of order 12, $\mathbb{Z}_{12}$; the dicyclic group of order 12, Dic_{12}; and the cyclic group of order 13, $\mathbb{Z}_{13}$. However, only three of these groups possess all-interval tetrachords: D_8, $\mathbb{Z}_{12}$, and $\mathbb{Z}_{13}$. (In §5, we examine the reasons why Dic_{12} fails to produce all-interval tetrachords.) D_8 is distinguished as being the smallest non-abelian group to contain all-interval chords of any size. Its sixteen all-interval tetrachords are examples of (8,4,1,2) almost difference sets. $\mathbb{Z}_{12}$ contains the canonical examples of the forty-eight all-interval tetrachords of pitch-class set theory, instances of (12,4,1,10) almost difference sets. $\mathbb{Z}_{13}$ has fifty-two all-interval tetrachords. As $\mathbb{Z}_{13}$ contains no involutions, these planar difference sets are also examples of all-directed-interval chords. These latter two groups are abelian.

Pentachords. As the size of the all-interval chords increases, the number of interval groups with appropriate numbers of interval classes that fail to produce all-interval chords also increases. All-interval pentachords require groups with ten interval classes. Whereas seven groups meet this condition, only four have all-interval pentachords: the semidihedral group of order 16, SD16 (also known as the quasidihedral group of order 16); the direct product of S_3 and $\mathbb{Z}_3$, $S_3 \times \mathbb{Z}_3$; the semidirect product of $\mathbb{Z}_7$ by $\mathbb{Z}_3$, $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$; and the cyclic group of order 21, $\mathbb{Z}_{21}$. SD_{16} has 128 all-interval pentachords, which are (16,5,1,10) almost difference sets, and the 108 all-interval pentachords in $S_3 \times \mathbb{Z}_3$ are (18,5,1,14) almost difference sets. Both these groups are non-abelian.

In the subsections above with $2 \leq k < 5$, we note the existence of almost difference sets in the cyclic group of order $(k-1)2 + (k-1)$ and planar difference sets in the cyclic group of order $(k-1)2 + (k-1) + 1$. With $k \geqslant 5$, this circumstance no longer holds. Specifically, we cease to find examples of all-interval k-chords in the cyclic group of order $(k-1)2 + (k-1)$. Whereas $\mathbb{Z}_{21}$ contains forty-two all-interval pentachords as (21,5,1) planar difference sets (hence, they are also all-directed-interval chords), no all-interval chords exist in $\mathbb{Z}_{20}$. The rationale for this situation is related to the non-existence of perfect Golomb rulers with five or more marks.[16] We find another

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[16] A Golomb ruler with k marks and length L has a different measurement between any two marks. A perfect Golomb

first occurrence with $k = 5$: the existence of a non-abelian group of order $v = (k-1)2 + (k-1) + 1$ that contains all-interval chords, $\mathbb{Z}_{v/3} \rtimes \mathbb{Z}_3$. Such a non-abelian group exists for every $k \equiv 2$ (modulo 3), where $k \geqslant 5$ [12, Theorem 18.68]. Like the cyclic group of order 21, $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$ contains no involutions. Hence, the 294 all-interval pentachords in this group are (21,5,1) planar difference sets (and they are also all-directed-interval chords), the same type of structure as those in $\mathbb{Z}_{21}$.

Hexachords. A hexachord has fifteen intervals; therefore, a group that contains all-interval hexachords must have that number of interval classes. Eight such groups exist, but only five of these groups contain all-interval chords: the direct product of four copies of the cyclic group of order 2, $\mathbb{Z}_2^4$; the direct product of the alternating group of degree 4 and the cyclic group of order 2, $A_4 \times \mathbb{Z}_2$; the direct product of $\mathbb{Z}_2^3$ and the cyclic group of order 3, $\mathbb{Z}_2^3 \times \mathbb{Z}_3$; the direct product of cyclic groups of order 14 and order 2, $\mathbb{Z}_{13} \times \mathbb{Z}_2$; and the cyclic group of order 31, $\mathbb{Z}_{31}$. Of these five groups, only $A_4 \times \mathbb{Z}_2$ is non-abelian. As with the sets of groups that contain all-interval bichords and trichords, we find all-interval hexachords in the smallest and largest possible groups to have the appropriate number of interval classes. $\mathbb{Z}_2^4$, in which all fifteen interval classes are involutions, is the smallest such group. Its 448 all-interval hexachords are examples of $(16,6,2)$ non-planar difference sets. In contrast, $\mathbb{Z}_{31}$ contains no involutions. It contains 310 all-interval hexachords as $(31,6,1)$ planar difference sets (i.e., all-directed-interval hexachords).

Two groups of order 24 exist with all-interval hexachords: one abelian with 1344 all-interval hexachords, $\mathbb{Z}_2^3 \times \mathbb{Z}_3$; and one non-abelian with 192 all-interval hexachords, $A_4 \times \mathbb{Z}_2$. As both these groups have seven involutions, all 1536 of these hexachords are instances of (24,6,1,16) almost difference sets. In the group $\mathbb{Z}_{14} \times \mathbb{Z}_2$ of order 28, we find more than one orbit of all-interval hexachords under the action of the normalizer of the group. Consequently, the GISZ relations among these hexachords do not derive from operations that are analogous to affine transformations. It is the only group with all-interval chords of size $2 \leqslant k \leqslant 8$ to have this property. Its 728 all-interval hexachords partition into three orbits: one orbit of size fifty-six, and two of size 336. All of these hexachords are examples of $(28,6,1,24)$ almost difference sets.

Heptachords. Of the sixteen groups with twenty-one interval classes, as required for all-interval heptachords, only one has this type of subset. Interestingly, it is not the cyclic group of order $(k-1)2 + (k-1) + 1 = 43$. As this number is prime, the cyclic group $\mathbb{Z}_{43}$ is the only isomorphism class of groups of that order. Further, it has no involutions, suggesting that it contains planar difference sets. However, we recall from §4 that $k - 1$ must be the power of a prime to yield planar difference sets (and in this case, $7 - 1 = 6$ is smallest integer that is not the power of a prime). Instead, the one isomorphism class of groups to produce all-interval heptachords is the extraspecial group $2(1 + 4)$ of minus type. This order-32 group is the central product of a dihedral group of order 8 and a quaternion group of order 8 that intersect in a central order-2 subgroup (i.e., all thirty-two members of the group commute with the members of this subgroup) [24]. The 512 all-interval heptachords found in this non-abelian group are all instances of (32,7,1,20) almost difference sets.

Octachords. As with the pentachords, $8 \equiv 2$ (modulo 3); hence, we find a non-abelian group of order 57, $\mathbb{Z}_{19} \rtimes \mathbb{Z}_3$, along with the cyclic group $\mathbb{Z}_{57}$. In fact, of the ten groups with twenty-eight interval classes, only these two produce all-interval octachords. $\mathbb{Z}_{57}$ contains 684 all-interval octachords, and $\mathbb{Z}_{19} \rtimes \mathbb{Z}_3$ has 6498. All 7182 are examples of (57,8,1) planar difference sets (i.e., all-directed-interval octachords).

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ruler is one in which every distance from 1 to L appears as such a difference [23]

VIII. Conclusions and Future Work

In total, we find 11,438 all-interval chords of sizes $2 \leqslant k \leqslant 8$ in twenty interval systems. The groups that these interval systems incorporate include abelian groups that are cyclic ($\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_6, \mathbb{Z}_7, \mathbb{Z}_{12}$, $\mathbb{Z}_{13}, \mathbb{Z}_{21}, \mathbb{Z}_{31}$, and $\mathbb{Z}_{57}$) and non-cyclic ($\mathbb{Z}_2^2, \mathbb{Z}_2^4, \mathbb{Z}_2^3 \times \mathbb{Z}_3$, and $\mathbb{Z}_{14} \times \mathbb{Z}_2$), as well as non-abelian groups ($D_8, SD_{16}, S_3 \times \mathbb{Z}_3, A_4 \times \mathbb{Z}_2, \mathbb{Z}_7 \rtimes \mathbb{Z}_3, 2^{(1+4)}$ [the central product of D_8 and Q_8], and $\mathbb{Z}_{19} \rtimes \mathbb{Z}_3$). Moreover, the all-interval chords themselves are of three general types: three classes of non-planar difference sets, seven classes of almost difference sets, and six classes of planar difference sets. The chords of the first two types are merely all-interval, whereas those of the third type also meet the stricter requirement of being all-directed-interval. The full sets of all-interval chords in these interval systems also vary in size and in terms of their GISZ-relations. Their sizes range from one chord ($\mathbb{Z}_2$) to 6498 chords ($\mathbb{Z}_{19} \rtimes \mathbb{Z}_3$). None of the sets with $k < 4$ have $Z-$ or GISZ-related members. Nineteen of the twenty sets are single orbits of all-interval chords under the action of the normalizer of the interval group. The GISZ relations among the chords in these sets derive from affine or affine-like transformations. The one remaining set — that of all-interval hexachords in $\mathbb{Z}_{14} \times \mathbb{Z}_2$ — is comprised of members in three such orbits. The GISZ relations among all-interval hexachords within different orbits of the normalizer of this group obtain from other, more obscure origins.

Within this diversity, we find some common threads that are of particular relevance to musical structure. First, abstract mathematical groups correspond to groups of symmetries. Whereas the generalized intervals we discuss here do not necessarily possess qualities of distance and direction, they do relate to symmetries. Further, the groups of symmetries to which these twenty interval systems correspond either contain simple symmetries themselves or are products (direct or semi-direct) of smaller groups that are composed of simple symmetries. Basic symmetries — such as translations, rotations, and reflections — surround us and shape our experience; they are found throughout nature, and they are commonplace in many human endeavors, including the visual arts, architecture, and music [25]. The cyclic groups $\mathbb{Z}_n$ agree with rotations of regular n-gons. These types of symmetries are used to model a variety of musical structures, including pitch-class transpositions and rhythmic translations in metric spaces. The dihedral groups D_{2n} add reflections to these rotational symmetries. In music, such reflections correspond to pitch-class inversions and rhythmic retrogrades. The Klein 4-group $\mathbb{Z}_2^2$ corresponds to a subgroup of symmetries of a square, or 2-cube. As we discuss in §4, these symmetries are those of the serial operations prime, inversion, retrograde, and retrograde-inversion. The larger elementary abelian 2-groups $\mathbb{Z}_2^n$ are isomorphic to particular subgroups of n-cube symmetries. These symmetries model nth roots of inversion and retrograde [26]. Similar associations exist for the other small groups that constitute these interval systems.

In addition to the interval systems, the all-interval chords they contain also have special musical significance. Their defining structure facilitates two important compositional processes: summation and deconstruction/development. The Schoenberg *Lieder* from Figure 1 illustrates the former process. This work — one of his first atonal compositions — is representative of the "emancipation of dissonance" that characterizes his atonal style. Rather than organizing pitch-class intervals in this song in terms of a hierarchy that is based on consonance and dissonance, Schoenberg treats all intervals equally. Thus, the final sonority of this piece, an all-interval tetrachord, serves as an economical summary of the song's intervallic content. The process of deconstruction/development is evident in the first two string quartets of Elliott Carter. Carter systematically deconstructs the all-interval tetrachords in these works into their constituent parts, exploring and developing each of the intervals in turn and in combination. Additional aspects of all-interval chords lend themselves to further musical interpretation. For instance, certain

transformational processes, such as the one in Figure 2, are possible because of the unique construction of these types of chords.

From a theoretical perspective, the completion of an existence theorem is perhaps the most significant open question. As we note in §5, an interval system cannot contain all-interval k-chords unless it has exactly $\binom{k}{2}$ interval classes. Satisfying this condition is necessary, but not sufficient, for the existence of all-interval chords. Does a single, unifying requirement exist that proves sufficiently the existence (or lack of existence) of all-interval chords in a given interval system? Much future work remains in the study of all-interval chords. The small-order structures investigated here can serve as departure points for new compositional designs and analytical investigations, and similar work may also be applied to larger-order all-interval chords.

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