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Foreword

MusMat research group was founded in the year of 2013, departing from my personal meeting with the researcher and fellow professor Carlos Almada. The interlocution generated gave great strength to the research that I had been developing since 2003 on Musical Texture, which revealed several points of contact with the research on Progressive Variation and Grundgestalt of my colleague. Then the master student and composer Daniel Moreira joined the group, working with the Contour theory, thus forming our first team. Later on, composer, researcher, and professor Liduino Pitombeira was integrated to the group and continued to develop his research on Systemic Modeling.

Presently, with more than fifteen students, we have a live and consolidated work with strong participation in national and international congresses and publication of dozens of papers, related to the researches developed in the Undergraduate and Graduate Program in Music of the Federal University of Rio de Janeiro.

Dialogue with other centers was a natural consequence. So, today we have an extensive network of collaboration, involving researchers from all regions of Brazil and from outside the country as well.

In the year of 2016, when we held the first congress on Music and Mathematics in our country, we are immensely proud to present the first Brazilian electronic journal entirely dedicated to this field, involving first class researchers from Brazil and abroad.

The selection of articles in this volume covers several strands of mathematical and musical thought, bringing contrasting and complementary visions of their integration. Mathematics has now become much more than a means of representation - it is a tool and inspiration for musical creation and analysis.

In this way, we hope to contribute to the development of a theoretical, analytical, and creative thinking in Brazil and in the global communities.

Pauxy Gentil-Nunes
December 2016

Evolutionary Variation Applied to the Composition of CTG, for Woodwind Trio

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Abstract: *This paper integrates a broad research on musical variation. It specifically addresses an original concept, namely "evolutionary variation" (EV), resulted from an association between Schoenbergian principles of Grundgestalt and developing variation and some ideas from Genetics and Evolutionary Biology. The application of this concept in a compositional system (Gr-S) aims at the production of a large number of variants from a basic musical idea, covering a wide spectrum of similarity relationships. An application of EV in the composition of CTG, for woodwind trio, is described in the second part of the paper.*

Keywords: *Musical and Biological Evolution. Grundgestalt and Developing Variation. Computer-Aided Composition.*

I. Introduction

The present study aims to highlight special similarities between musical variation and genetic phenomena, which contributed to the elaboration of a new concept, *evolutionary variation*. The study integrates a broad research project intended basically to systematical approaches addressed to musical variation, under analytical and compositional perspectives. The research is theoretically based on two correlated principles elaborated by the Austrian composer Arnold Schoenberg, namely, developing variation and *Grundgestalt* (normally translated as "basic shape"), both resulted from an organicist conception of musical creation. Considered by Leonard Meyer as the most important extra-musical influence for the Romantic (especially Austro-German) composers [1], the trend of Organicism was decisively intensified in middle of the 19th century with the publication of works by Goethe (on the nature of plants) and, especially, Darwin, with his theory of evolution of the species [2]. Traces of this organic origins were deeply established in the Schoenbergian theory of *Grundgestalt*, as can be observed in his writings about this subject, being specially evidenced by the use of metaphors, like "seed", "organic growth", "internal development" and so on.[1]

In this research, the ties between music and biology are deliberately more tight and explicit. Concepts like *genomic/phenotypic* levels, *chromosomes*, *genealogy*, *ascendants/descendants*, *viruses*, among others, are not only part of the specific terminology, but also important components of adopted premises. Recently, a new branch was initiated taking as model Darwin's theories (and specially their modern unfolding, produced in the fields of Genetics and the so-called Neo-Darwinism [4], [5]). In this new approach, the concepts of "evolution" and "artificial selection"

[1] See, for example, [3].

were transposed from the realm of vegetal and animal life to the domain of music, which resulted in some prospective studies. One of them [6] describes an experiment for production of variants from a simple theme, through successive application of "micro-mutations" along 20 generations of descendants, with the help of a special computational software developed for the task. At each generation of 6 transformed "children", the most apt for continuing the lineage was selected becoming a"parent" for the next breeding (the adopted criteria for the selection were based on objective musical parameters: melodic contour, harmony and rhythmic configuration). This artificial selection strategy (similarly as that one used in farms for improving animal creation) can be associated to the developing variation techniques, but under an original and instigate perspective, since it promotes gradual divergences from the referential musical form according to some "evolutionary pressure", i.e., "directions" determined by the selective criteria ultimately adopted by the own composer. These findings constituted an important basis for the present study, associated to the systematical composition of organic musical pieces.

II. The Gr-System

The main motivation for the development of the Gr-System ("Gr" for *Grundgestalt*) was to investigate if it would be possible to produce music with a strict organic structure and in terms of maximal economy (i.e., with the use of the least possible external material). As one can easily see, this task would only be adequately accomplished with the aid of computational tools. Four programs were then idealized and grouped in a complex named *geneMus* (gM),[2] specifically destined to the production of variations in different operational levels. After a long phase of design and improvement, a more stable, versatile, and robust version of gM was recently consolidated. It is formed by four sequential and complementary modules, each one destined to a specific function, which are summarily described as it follows:[3]

- gM-01: produces abstracted variations (labeled in gM as "geno-theorems", or gTs) from a musical basic cell (the *Grundgestalt* or, in the research's terminology, the *axiom* [ax] of the system, the given external element). Two kinds of abstractions (identified as "chromosomes") are considered as referential in this process: the sequences of intervals (chromosome I) and temporal durations (chromosome R) that form the axiom. An indefinite number of generations of gTs (melodic and rhythmic) can be obtained through application of transformational operations to both chromosomes. Sequences of operations (inversion, augmentation, etc.) correspond in the system to "abstract" and progressive derivative procedures, or *developing variation of first order* (DV1);
- gM-02: is responsible for forming concrete musical building blocks (named as pheno-theorems, or pTs) through an exhaustively crossing-over of intervallic and rhythmic gTs produced in gM-01. Some filters (designed as fitness functions) were implemented in this module in order to select the "best" pTs (according the partiular intentions of the composer) and conversely eliminate the "ill-formed" ones, contributing for reducing the risk of overpopulation [8];
- gM-03: concatenates the selected pTs to form larger and more complex structures (similarly as motives are joined in the construction of themes), which are labeled as axiom-groups (axGrs). These constitute the referential forms for further derivation, performed in the next module;

[2]The four gM modules were implemented in the computational platform Matlab.
[3]For more detailed information, see [7].

- gM-04: yields generations of variants (theorem-groups, or thGrs) from the pre-produced axGrs, through application of transformational operations (analogously as happens in gM-01, but in this case involving concrete musical structures). This process is labeled as developing variation of second order (DV2);

The structure of gM and the correlations and specific functions of its four modules are summarized in Figure 1.

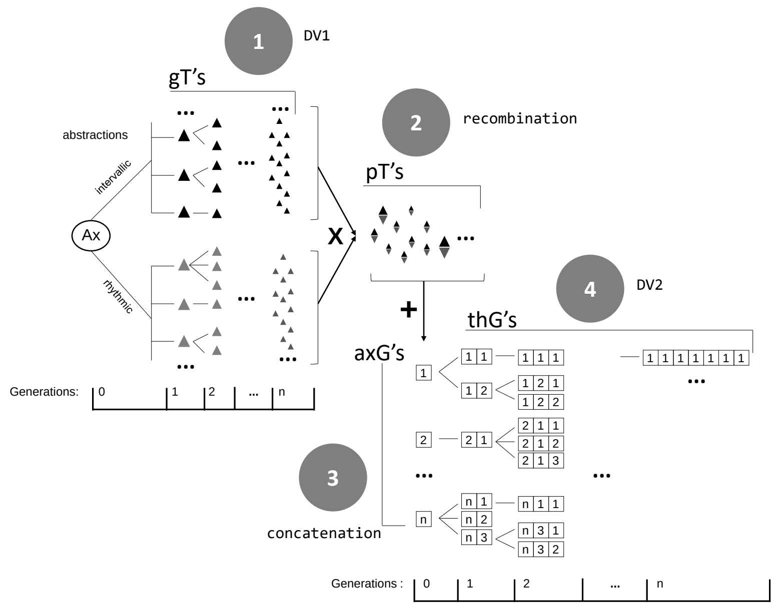

Figure 1: *Graphic representation of gM's structure and of its four modules (adapted from [7, p. 44]).*

The Gr-System depends on two important and correlated concepts: the coefficient of similarity and the derivative curve. The coefficient of similarity (Cs) of a given variant measures the "parenthood" degree between it and the referential form from which it ultimately derives,[4] being expressed as a real number between 0 (absolute contrast) and 1 (relation of identity).[5]

The derivative curve informs the fluctuations of Cs along the time. There are two types of curves: for planning and analysis. The former is used in the pre-compositional stage, with the finality to provide a broad idea of the intended derivative behavior of the musical material along the sections of a planned piece. For this reason, the planning derivative curve has an aspect rather of a sequence of rectangles than properly of a saw-like line. In turn, the analytical derivative curve aims at precisely describing the fluctuations of similarity relations in a finished piece, taking into account the involved voices at each beat.[6]

After concluding the production of the material (the abstract and concrete variants) and plotting the planning derivative curve, the compositional phase is properly ready to begin. A specific program, named *organiComposer*, was designed to aid the composer to structure a musical piece based on decisions and choices conditioned by his/her constructive intentions.

[4] The referential forms adopted for Cs measurement are of two kinds: the chromosomes I / R (in the case of the abstract derivation processes or, as above defined, DV1) and the axGrs (DV2). By definition, these forms are axiomatic in the system and have Cs = 1.

[5] The algorithms for calculating the coefficient of similarity are basically described in [9].

[6] For examples of the both types of derivative curves, see Figures 3 and 14.

Recently, a phase of practical tests for the system was initiated with the idealization of a compositional project entitled *Germinatas*, a cycle of 16 pieces, each one with four movements and specific instrumental formations (string quartet, piano, symphonic orchestra, etc.). *Germinata* I, for woodwind trio (oboe, clarinet and bassoon), is the first concluded piece of this project. The next part of the article is dedicated to examine the constructive processes employed in this work. Firstly, some general comments about the basic formal-derivative structure of the entire cycle, then a concise description of the production of the common material of *Germinata* I (i.e., the motivic substrate that is shared by the four movements of the piece), and finally a more specific and detailed exam of the last movement, entitled CTG.

III. The cycle *Germinatas*

The idealization of *Germinatas* was based on another link between the realms of music and genetics, considering specifically the four nucleotides that form the DNA, Adenine (A), Cytosine (C), Guanine (G) and Thymine (T). These elements were isomorphically associated to four profiles of similarity relations (Table 1). It is noteworthy to add that this association was purely arbitrary, with the only purpose of providing a useful means for structuring the pieces.

Table 1: *Correlations between nucleotides and similarity profiles in the Germinatas, considering four types of similarity relations measured in Cs values*

nucleotide	similarity relations	max./min. values for Cs
A	high similarity	1.00/.75
C	medium similarity	.74/.50
G	low similarity	.49/.25
T	high contrast	.24/.00

The internal organization of the movements in each of the 16 *Germinatas* also maps genetic processes, corresponding to a codon. Codons are structural elements that form the genetic code of a living form, consisting on nucleotide triplets (or three-letter "words") built by the combination of the bases A, C, G, and T. Since there are 64 possible triplets (4^3), the four movements of each one of the 16 *Germinatas* were structured as groups of four codons, in such a way that all the combinations without repetitions will be considered. Figure 2 presents a list of the 64 codons (and, isomorphically, the internal structures of the *Germinatas'* movements).

Four sequences was selected for *Germinata* I (see Table 2). Based on this selection, four planning derivative curves (one for each movement) were plotted (Figure 3). As previously stated, they represent only broadly the intended profiles of similarity for the movements,[7] but become an important and efficient basis for orienting the choice of the motivic-thematic material during the compositional phase.

IV. Production of common material for *Germinata* I

The composition in the Gr-System must begin with the choice of an axiom for becoming the very referential form for subsequent derivation. It can be an original musical fragment or be borrowed from an existent piece, as in the present case. For the axiom of *Germinata* I, intended to

[7] Since the primary aspect concerned in the graphs is the derivative behavior, the extensions of the three sections in each movement were roughly arbitrated as third parts (33%) of the total.

1	A	A	A	17	C	T	C	33	T	T	C	49	C	G	A
2	A	A	C	18	A	C	C	34	T	T	G	50	C	G	T
3	A	A	G	19	A	G	C	35	T	A	T	51	C	T	A
4	A	A	T	20	A	T	C	36	T	C	T	52	C	T	G
5	A	C	A	21	G	G	G	37	T	G	T	53	G	A	C
6	A	G	A	22	G	G	A	38	A	T	T	54	G	A	T
7	A	T	A	23	G	G	C	39	C	T	T	55	G	C	A
8	C	A	A	24	G	G	T	40	G	T	T	56	G	C	T
9	G	A	A	25	G	A	G	41	A	C	G	57	G	T	A
10	T	A	A	26	G	C	G	42	A	G	C	58	G	T	C
11	C	C	C	27	G	T	G	43	A	C	T	59	T	A	C
12	C	C	A	28	A	G	G	44	A	G	T	60	T	A	G
13	C	C	G	29	C	G	G	45	A	T	C	61	T	C	A
14	C	C	T	30	T	G	G	46	A	T	G	62	T	C	G
15	C	A	C	31	T	T	T	47	C	A	G	63	T	G	A
16	C	G	C	32	T	T	A	48	C	A	T	64	T	G	C

Figure 2: *List of the 64 possible codons/sectional sequences.*

Table 2: *Four-movement structure of Germinata I (hs: high similarity; ms: medium similarity; ls: low similarity; hc: high contrast)*

movement	corresponding codon	similarity relations
I	<ACG> [43]	hs-ms-ls
II	<TGA> [63]	hc-ls-hs
III	<CTA> [51]	ms-hc-hs
IV	<CTG> [52]	ms-hc-ls

be allusive to Brazilian musical popular genres, it was selected the anacrusis of the well-known choro *Carinhoso*, composed in 1917 by Pixinguinha and Braguinha (Figure 4).

The axiom (saved as a midi file) was then opened in gM-01, which abstracted its intervallic and rhythmic configurations, transcribing them as numeric strings, labeled as chromosomes I and R (Figure 5).

The derivative process was then ready to be initiated. Each chromosome was used as referential form for parallel production of 42 melodic and 21 rhythmic gTs. Some of these outcomes are shown in Figure 6.[8]

In gM-02 were then formed 882 pTs through mathematical combination of the 42 melodic and 21 rhythmic gTs. As previously stated, the pTs are basic concrete structures that approximately function as motives in the system (Figure 7 presents six of them).

The next step (in gM-03) was to form the referential axGrs. It was decided that each movement of *Germinata* I would be constructed based on a unique axGr (and, of course, their respective derivations). Thus, four axGrs were created through combination and concatenation of different pTs. This process involves three possible alternatives for each inserted pT: (1) transposition; (2) metrical displacement; (3) suppression of final notes. Needless to say that the composer, based on his own sense of form, must carefully consider the better combinations to do, selecting the

[8]For a detailed description of the functioning of gM-01, see [10].

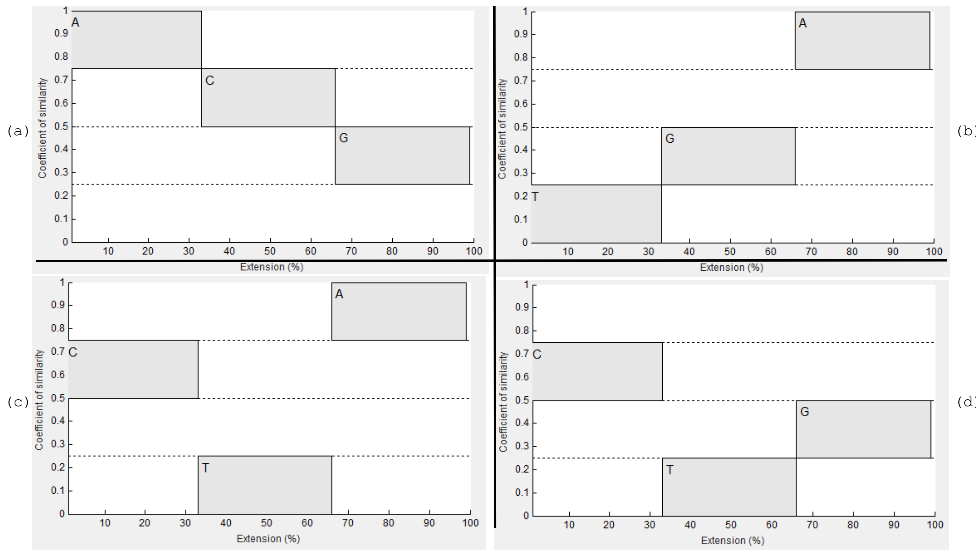

Figure 3: *Germinata I: Planning derivative curves of the four movements: ACG (a); TGA (b); CTA (c); CTG (d).*

Figure 4: *Pixinguinha and Braguinha: Carinhoso (anacrusis), taked as axiom of Germinata I.*

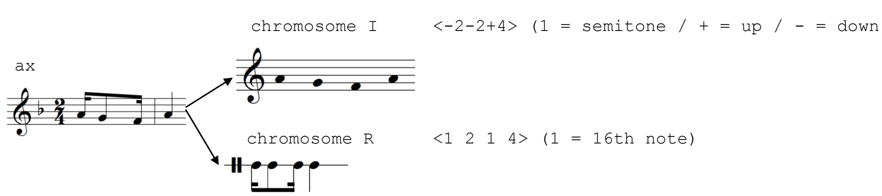

Figure 5: *Germinata I: axiom and chromosomes I and R.*

components according to their melodic contour and rhythmic configurations and contextual relations (as, for example, parallelism and contrast between segments). Figure 8 shows the formation of axGr-5 (used as referential form for the fourth movement, CTG),[9] identifying the pTs used as its components and the corresponding transformations which were applied, based on the three above mentioned options.

In gM-04 each axGr becomes a kind of patriarch of a lineage of variants, which are obtained through developing variation of second order (DV2). More precisely, in this phase it is properly initiated the process of evolutionary variation, as it will be detailed in the next sections, dedicated

[9]This apparent incongruence – i.e. the fourth movement's axGr is numbered with "5" and not "4" (as one could logically expect) – is due to the fact that, actually, there were produced for the piece in gM-03 six axGrs. Of these, only four were selected for the movements, respectively: axGr-1, axGr-3, axGr-4 and axGr-5.

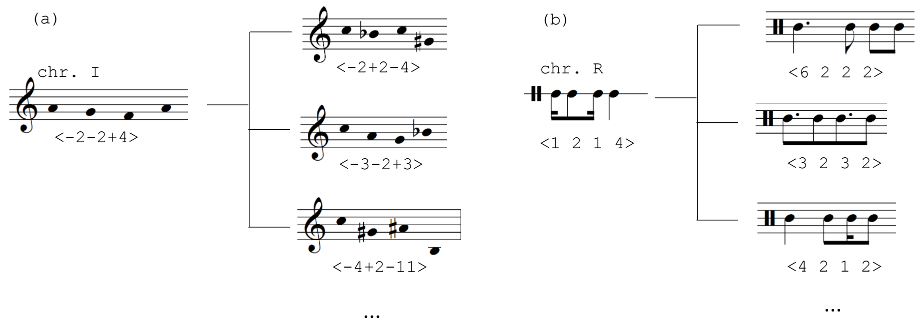

Figure 6: *Germinata I: some melodic and rhythmic gTs obtained from chromosomes I (a) and R (b) through developing variation of first order.*

Figure 7: *Germinata I: pTs numbers 1, 38, 77, 208, 314, and 536.*

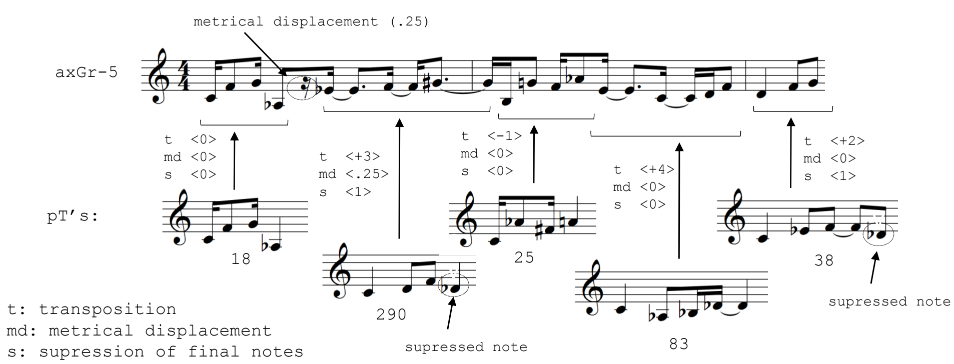

Figure 8: *Germinata I: formation of axGr-5, through combination of pT's numbers 18, 290, 25, 83, and 3.*

to describe the construction of CTG.

V. The evolutionary space of CTG

A specific motivic-thematic material of a piece built in the Gr-System consists essentially on the set formed by a given referential form (axGr-5, in the present case) and all the thGrs that derive from it. This set corresponds to the evolutionary space (ES) of the piece.[10] An ES is formed through a special type of developing variation process, namely, evolutionary variation (EV), which results

[10]The idea of evolutionary space was inspired by Richard Dawkins' concept of "genetic space" (see [4, pp. 81-91]).

from gradual and cumulative mutations (frequently quite subtle, almost imperceptible) along generations of variants. In a sense, EV can be compared (in a radically more compacted time-scale, of course) to the process of formation of the animal and vegetal species. Although the amount of modification between a given referential form and its immediate offspring is normally low, under a broader perspective, considering the whole derivative process and its multiple branching, divergence can reach extreme rates. In musical terms, this can provide a wide range of forms with distinct degrees of similarity in relation to the basic theme.

It is important to add that not necessarily all the elements of an ES must be present in the piece. Some of them may simply be created as intermediate stages in the formation of more meaningful structures (according to some constructive intentions). In sum, the "area" covered by the ES of a given piece depends directly on compositional needs and particular requirements, as it will be later evident. An ES is formally expressed as:

$ES_{\mathrm{p}} = \{axGr_{\mathrm{n}}, {}^{*}thGr_{\mathrm{n}}\}$, where ES_{p} is the evolutionary space of the piece p,$axGr_{\mathrm{n}}$ is the axiom-group of number n, and ${}^{*}thGr_{\mathrm{n}}$ is the total of theorem-groups derived from $axGr_{\mathrm{n}}$.

An ES_{p} is therefore formed in the module gM-04, dedicated to the production of thGrs from a given axGr. In the specific case of CTG, its ES comprised 53 elements ($axGr - 5plus52thGrs$), distributed into 7 main branches and 8 generations. A special function of gM-04 is used to produce a graphic representation of the EV in the format of a genealogical tree. Figure 9 shows the ES of CTG (ES_{CTG}): the nodes correspond to the thGrs and the lines represent their respective derivations. As can be observed, due to the fact that some thGrs are more "prolific" than others, it is considerably difficult to obtain a complete and clear visualization of the precise branching in some parts of the tree (as the cluster in the framed area in Figure 9a). In these cases, the program allows to magnify the view of the section in question, as it is shown in Figure 9b.

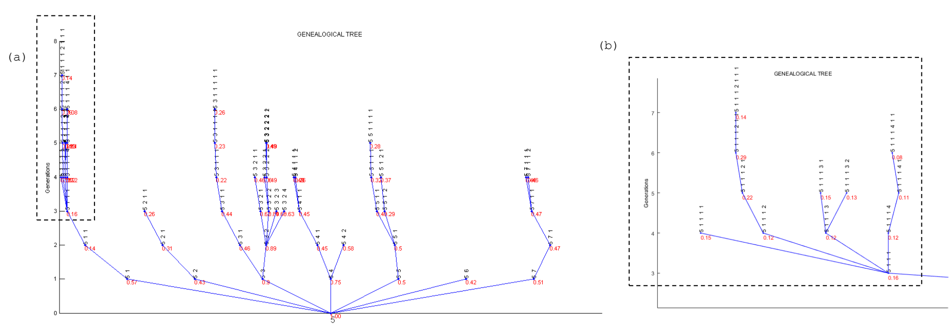

Figure 9: *GerminataI/CTG: ES as a genealogical tree (a); detailed view of the selected area (b). Nodes indicate variants (thGr's), their Gödel-vectors (see bellow) and coefficients of similaity are notated, respectively, in black and red.*

Each thGr receives a label corresponding to its lineage description, expressed as a vector, named *Gödel-vector* (Gv),[11] whose first element is the number of the considered axGr (5, in the present case). The remain components describe the genealogical trajectory of the form. Be, for example, the thGr, with Gv <5 1 1 2 3>. It corresponds to the following genealogical description (the vector content must be read backwards): "third descendant of the second descendant of the

[11]To more information about the elaboration of the *Gödel-vector*, the origins of its formulation and the algorithms involved in the genealogical description of variants in the system, see [11].

first descendant of the first descendant of the fifth 'patriarch'". Besides the Gv, the genealogical tree also informs the Cs indexes of the involved thGr (the numbers in red in Figure 9a).

A thGr (generically named as a "child") results from application of at least one transformational operation to a referential form (its "parent"). The operations can be intervallic (i.e. acting on the parentâĂŹs chromosome I), rhythmic (chromosome R) or intervallic-rhythmic (indistinctly applicable to the both structures). Table 3 presents a basic list of the available operations in gM-04.

Table 3: *Formal description of the operations, considering chromosome of application and corresponding algorithm, where <a> and <b> are, respectively, the numeric transcriptions of a "parent" and its "child", p is prime number between 2 and 11 and q = 1.5, 2 or 3.)*

operation	chromosome	algorithm
inversion (INV)	I	$\langle b \rangle = -1 * \langle a \rangle$
multiplication (MUL)	I	$\langle b \rangle = p * \langle a \rangle$
augmentation (AUG)	R	$\langle b \rangle = q * \langle a \rangle$
diminution (DIM)	R	$\langle b \rangle = 0.5 * \langle a \rangle$
expansion (EXP)	IR	$\langle b \rangle = \langle a \rangle + p$
contraction (CNT)	IR	$\langle b \rangle = \langle a \rangle - p$
retrogradation (RET)	IR	$< a_n, ... a_1, a_0 > = < a_0, ... a_{n-1}, a_n >$
rotation (ROT)	IR	$\langle b \rangle = < a_1, a_2 ... a_n, a_0 > = < a_0, a_1, ... a_{n-1}, a_n >$

There are two types of operational application: general and mutational. The former affects the entire content of the "parent", while in the second type just one element — randomly selected by the program — is modified.[12] This case emulates genetic, intracellular mutation, promoting a very gradual developmental process. The resulting child is always quite similar to its parent, but their own children (and the children of these, and so long) will certainly contribute to increase more and more the divergence, which, after some generations, causes a wide and varied spectrum of correlate musical ideas. With the purpose of expanding the range of possibilities it is also possible to combine two or more operations (general and/or mutational) in a single application. Figure 10 presents four of the innumerable alternatives for variant production from a hypothetical "parent".

The production of the thGrs that form ES_{CTG} was conditioned not only by needs of variety and contrast of the musical material, but also by the pre-established limits of similarity relationships, determined in the planning derivative curve (see Figure 3). Since these limits constrain the average Cs values (Cs_{av}) to be lower than 0.75 in all of the movement's three sections, the production of variants was directed to the selection of more divergent outcomes (which, in turn, would also breed descendants with low similarity in relation to axGr-5). It is relevant to add that in the Gr-System the selective process is always mediated by the composer's sense of form, which represents the ultimate and most decisive factor for the choice of the best (artistically speaking) results among a multitude of alternatives at each phase of the process. Figure 11 presents a subset of ES_{CGT}, informing their respective genealogical labels and Cs indexes.

[12]Frequently a mutational operation produces an ill-formed variant (with extremely large melodic range, for example). In this cases, the variant is simply discarded and, of course, leaves no descendants. It is noteworthy that such a situation is quite similar to what happens in organic life, since most of mutations that occur in the genome of a being imply in none advantage for it (if do not cause its death).

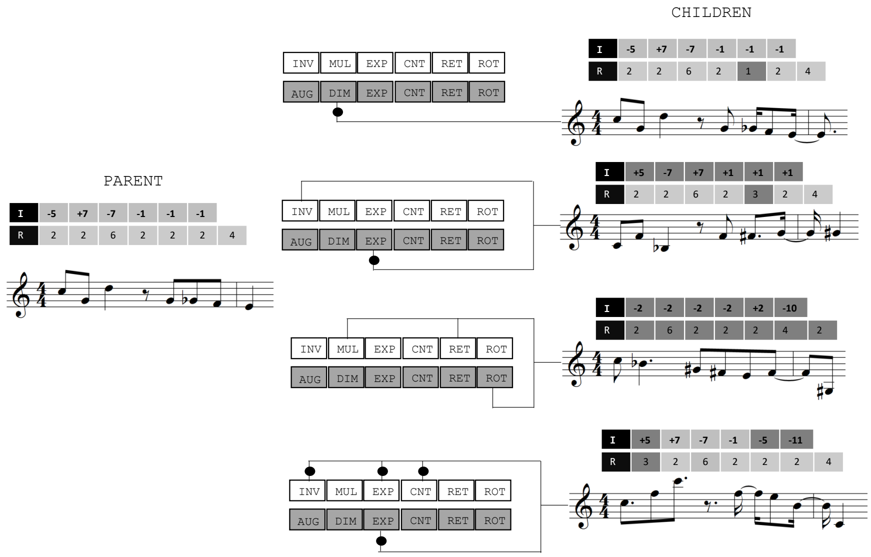

Figure 10: *Four thGr's ("children") derived from a hypothetical "parent" through application of single or combined transformational operations. Above each form is its "genetic code" (chromosomes I/R). The darker rectangles correspond to the modified characteristics. The black circles indicate mutational operation.*

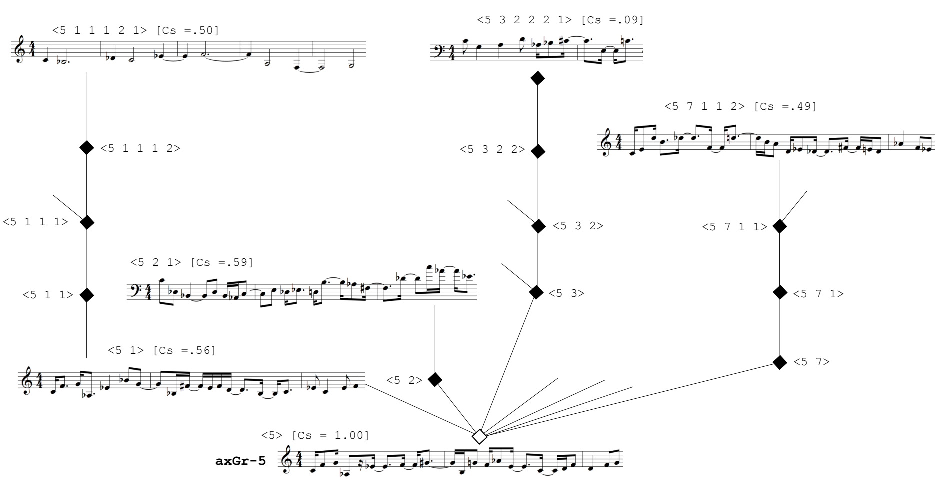

Figure 11: *Germinata I/CTG: topological diagram presenting five thG's derived from axGr-5 (genealogical label <5>) with their respective Cs values: <5 1> (first generation), <5 2 1> (second generation), <5 7 1 1 2> (fourth generation), <5 1 1 1 2 1> and <5 3 2 2 2 1> (fifth generation).*

VI. Composing CTG

In spite of its name, the program *organiComposer* (oC) was not designed with the purpose to substitute for the intellectual and creative work of a human composer.[13] Rather, it must simply be used as a tool for aiding him/her to structure a piece constructed according to Gr-System's parameters. The elements and functions of the program are present in its user interface (Figure 12).

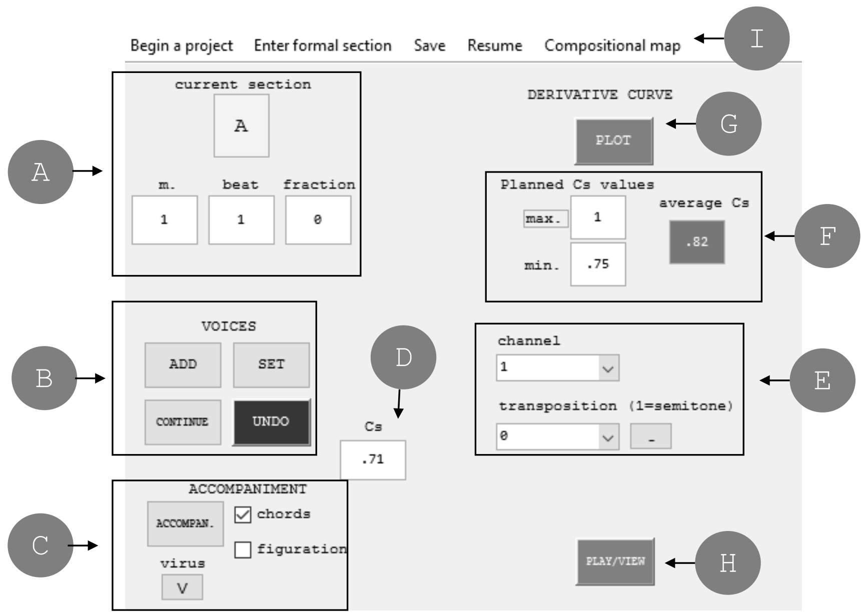

Figure 12: *oC's user interface: Information about localization: (A) section label ("A" in the present case) and positional data (measure number, beat and beat fraction) for entering a MI; (B) Buttons for opening a midi file corresponding to a MI and for inserting it in the coordinates established in A; (C) Buttons for selecting and inserting accompaniment (gTs / viruses); (D) Field corresponding to the Cs index of an inserted MI; (E) Information about the working voice (i.e., the voice selected for performing a selected MI): midi-channel and interval of transposition; (F) Fields that show the minimum and maximum values predetermined by the planning derivative curve and the Cs_{av} of the section; (G) Button for plotting the analytical derivative curve (see Figure 14a); (H) Button for hearing and viewing (in a piano-roll diagram) what was composed until that moment; (I)I. Menu tag for the compositional map (Figure 14b).*

In essence, the work with oC does not differ much from a conventional compositional process. At any instant the composer is faced with various decisions to make, concerning aspects like coherence, balance, changes of mood, tempo, texture, and so long. The program merely aids the user – literally – to *compose* (according to the Latin etymology of the verb: *com+ponere*, or put together) selected melodic/harmonic elements in selected metrical points transposed to selected pitch regions in the formation of selected textures.

[13]This program introduces a new and specific sense for the concept of "musical idea" (from now on referenced as MI). In this context, MI is a generic label used for naming one of the four possible melodic-harmonic materials employed in oC. Two of them are the forms components of the ES (axGr and thGrs), which constitute the most important elements for the composition. The remaining forms are used as subsidiary material constituting the accompaniment in homophonic textures, which can be formatted as consistent rhythmic figuration or chords: the melodic gTs (or MgTs produced in gM-01) as well the *viruses*. A virus is an external element (i.e., not produced in the system) that the composer (with whatever constructive reason) inserts in the piece (for example, a segment of the chromatic scale).

The insertions of elements are preserved in a temporary midi file, which allows the composer, at any moment, to verify the provisional results, edit the definitive score (including tempo indications and dynamic, articulation and expression marks) and, of course, to make eventual adjusts and corrections or even add ornamentation. In simpler terms, it provides a kind of *in natura* sketch to be elaborated and refined. An illustration of this adaptive process is shown in Figure 13, with a short passage of CGT in both versions.

Figure 13: *Germinata I/CTG (mm.46-50): sketch-like version (a); final version (b).*

Besides this evident practical aspect, the main and exclusive advantage of using oC for composing in the system is the possibility of registering two graphic schemes: the analytical derivative curve and the compositional map. These mutually complementary graphs provide a precise structural overview of a composed piece in the Gr-System.

The analytical derivative curve of CGT is shown in Figure 14a. As it can be observed, in each one of three sections, the Cs_{av} (represented by the red lines) was plotted within the pre-established C’âĂŹs limits (the dotted horizontal lines) of the respective planning derivative curve (Figure 3d), which therefore matches the similarity profile intended for the movement. The rectangles in the compositional map (Figure 14b) represent the various MIs used in each instrument/channel (1 for the oboe, 2 for the clarinet and 3 for the bassoon). The shades of gray are associated to the class of MI employed, from the darkest to white: axGr, thGr, MgT, and virus (observe that the genealogical label and Cs are added in the two first cases).[14] The combined visualization of both graphs provides an interesting perspective of the correlations between form, texture and similarity behavior.

[14]As it can be noted, in this movement there are only two brief uses of non-essential material (MgT and virus), both as punctuation at the end of section T. Their Cs indexes are not considered in the calculus of the Cs_{av}.

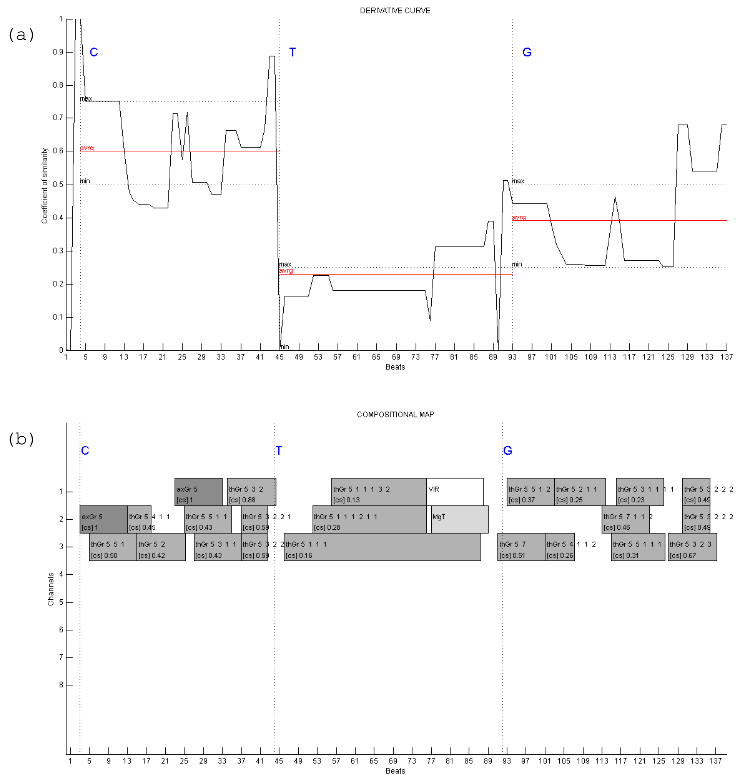

Figure 14: *Germinata I/CTG: analytical derivative curve (a) and compositional map (b).*

VII. Conclusions

This paper introduced the concept of evolutionary variation, associated to organic musical composition and to genetic processes, resulting from a new branch of a broad research project intended to systematically study variation in music. This branch is dedicated to the elaboration of a system (Gr-S) for organic and maximally economic composition aided by computational tools (gM and oC). The application of this concept in the composition of the fourth movement of *Germinata* I (CTG) yielded a large number of musical ideas within a wide spectrum of similarity in relation to the basic material, forming what the CTG's evolutionary space (ES_{CTG}). From this repository there were selected the musical ideas used to compose the piece, according to predetermined similarity profiles, an original sort of structuring strategy.

In spite of the fact that some parts of the programs that integrate the system still require adjustments and improvements, the results obtained in this study reveal its efficiency and flexibility.[15]

[15] Gr-S' flexibility can be clearly evidenced if we consider the following points: (a) a given axiom inserted in gM-01 can potentially yield an astronomic number of gTs; (b) The pTs produced through combination of gTs can be concatenated in a virtually infinite number of manners (considering the available options of permutation, transposition, metrical displacements and suppression of notes); (c) each one of the (infinite) possible axGrs thus formed can, in turn, produce infinitely wide evolutionary spaces. It is very instigating to speculate that even if we stipulated a closed structural planning and basic material (for example, the derivative curve of Figure 3d and ES_{CTG}) for, say, 100 different composers, we would certainly obtain 100 radically different pieces (although – and almost paradoxically – they would be closely related by their common origins), since these would depend largely on the individual skills and the myriad of decisions taken during their compositional processes.

Due to the several artificial selection strategies implemented in the gMs modules, the composer can control the entire derivative process, by selecting the variants according to his/her particular constructive intentions. Moreover, the compositional process, performed with the algorithmic aid of oC, is non-determinist and plainly dependent on the composer's imagination, sense of form and creativity.

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Teaching Atonal and Beat-Class Theory, Modulo Small

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Abstract: *The paper advances a pedagogical program that models small cyclic systems before teaching the twelve-element chromatic system of atonal theory. The central properties, relations and protocols of atonal theory (complementation, inclusion, invariance, transpositional equivalence, set classification and labeling, maximal evenness) are introduced in the smallest cyclic system to which they apply. All cyclic systems of 2 to 9 elements have at least one familiar musical application, modeling beat-class (rhythmic) cycle, pentatonic and diatonic scales. By the time students have scaled up to a twelve-element universe, they are technically prepared to explore it, and to appreciate its special properties. Along the way, they have learned a model of meter, an otherwise under-theorized aspect of music pedagogy.*

Keywords: *Atonality. Musical set theory. Time cycles.*

I. The Challenge of Teaching Atonal Theory

The theory of atonality focuses on the cyclic universe of twelve elements (C12) interpreted as chromatic pitch classes, and on the properties of and relations among pitch-class sets in that universe. It is a standard curricular offering at both undergraduate and post-graduate levels in North America and elsewhere, is integrated into at least one comprehensive music-theory textbook [1], and is the sole focus of several dedicated ones [2],[3]. Atonal theory is a challenge to teach because of the unfamiliarity of atonal repertories, the level of abstraction, and the delay in the payoff. Even for students interested in atonal repertories and comfortable with mathematical modes of conception and representation, it takes some time before the techniques of atonal theory help students to achieve satisfying analytic insights. For students who are math-phobic, or not drawn to atonal repertories, or both, the curriculum can be frightening or alienating, at worst provoking hostility that generalizes to the entire project of thinking conceptually about music.

One part of the challenge can be confronted by broadening the range of repertory. C12 can be interpreted not only as a chromatic universe of pitch classes, but also as a twelve-beat cycle, which might be realized as a bar of $\frac{12}{8}$, two bars of $\frac{6}{8}$, four bars of $\frac{3}{4}$, a bar of $\frac{4}{4}$ with tripleted subdivisions, and so forth [4]. Knowledge of C12 can be used to analyze the rhythms of readily accessible and familiar repertories, such as West-African, Latino, global-popular, or minimalist music [5],[6],[7]. Students can learn such central topics as rotational equivalence, invariance, complementation, inclusion, and set classification and labelling, each of which has direct and revealing beat-class

*I would like to acknowledge Andrew Jones for supplying information relating to this paper.

applications. Learning C12 in this environment, students acquire abstract knowledge ready for transfer to the isomorphic universe of pitch classes, and to atonal and serial repertories. There is a pedagogical advantage in staging the mathematical and musical challenge at different times. And there is a residual benefit: as they travel along the road to atonal theory, students are exposed to a theory of musical meter, which is otherwise neglected in the curriculum [8].

But this partial solution does not address the problem of abstraction. C12 is a complex system, with 4,096 distinct combinations, including 66 distinct pairs of elements, 220 triplets, and so forth. It is difficult to view such a system holistically, especially if one has not "scaled up" to it through simpler systems. The application of C12 to time cycles suggests a solution. There is only one chromatic universe of pitch classes, but there are many universes of beat classes, some quite small, that are familiar to musicians from an early stage of development. Larger-cardinality universes inherit the simple properties and relations of smaller ones, and add more complex ones. An incremental pedagogical progression from smaller to larger cyclic systems has some of the advantages of elementary mathematics curricula, such as the Japanese model for teaching arithmetic to children, which is initially restricted to the subtizable numbers [9].

Not all small beat-class universes are equally familiar. Musicians in the west typically learn systems based on 2 and 3 beats, their powers (4, 8, 9), and their composites (6,12), thus catching every number in the range from 2 to 9, with the exception of 5 and 7. Fortuitously, it is exactly these missing prime universes that form the most familiar small cyclic pitch-class systems, as pentatonic and diatonic scales respectively. Unlike the beat-class and chromatic universes, the elements of these scales are not distributed evenly. But they are even enough that we can disregard, or "reduce out," the distinction between tone and semitone in the diatonic case, and tone and minor third in the pentatonic one [10]. Indeed, musicians are accustomed to such reductions, as when we assign scale degrees using natural numbers from 1 to 8.

II. Outlines of a Pedagogical Program

My pedagogical program progresses in three stages. In the first stage, which can be worked through in about 30 minutes of class time, power sets of 0, 1, and 2 elements are briefly studied. Basic relations such as null set, complementation, and cardinality are introduced, as are the basic symbols of set theory. The second stage studies cyclic universes of 3, 4, and 5 elements, realized respectively as triple and quadruple meter in the rhythmic domain, and pentatonic scales in the pitch domain. Topics that can be introduced at this stage include modular arithmetic, rotational (transpositional) equivalence, classification and labelling procedures, interval class, interval content, set class, and invariance. The musical applications are not yet surprising. Students are learnng a new language for properties and relations about which they long cultivated deep intuitions. They may feel skeptical about being asked to pay money for a Howitzer in order to shoot a few sitting ducks, when a much simpler implement will suffice. I find it useful to tell them, more than once, that it is better to get to know your machinery in a simple environment than in the complex ones that lie just around the bend.

As the size of the cycle increases, intuitions about its structure diminish at roughly the rate that interest in its musical capacities grows. Thus conceptual and terminological grasp is not overwhelmed at the same moment that students are struggling to gain some intuitive traction on structures that are becoming exponentially more complex. C6 represents a pedagogical watershed: just small enough to be graspable as a Gestalt, but large enough to introduce curious features with unexpected musical ramifications. The universes from 6 to 9 all have distinctive

Table 1: *Pascal's Triangle, interpreted as number of sets $|d|$ in a $|c|$-universe.*

c=	d= 0	1	2	3	4	5	6	7	8	9	10	11	12	$2^c =$
0	1													1
1	1	1												2
2	1	2	1											4
3	1	3	3	1										8
4	1	4	6	4	1									16
5	1	5	10	10	5	1								32
6	1	6	15	20	15	6	1							64
7	1	7	21	35	35	21	7	1						128
8	1	8	28	56	70	56	28	8	1					256
9	1	9	36	84	126	126	84	36	9	1				512
10	1	10	45	120	210	252	210	120	45	10	1			1024
11	1	11	55	165	330	462	462	330	165	55	11	1		2048
12	1	12	66	220	495	792	924	792	495	220	66	12	1	4096

features that underlie familiar and compelling musical properties. These universes are large enough that set-class enumeration and classification from scratch becomes a challenge, but small enough that it remains tractable. But enumeration is no longer the central focus at this level. That focus shifts toward the generation of larger sets by recursive stacking of a single interval. Each universe has a unique personality, with special musical ramifications, that results from its number of elements. Prime-numbered universes behave differently than composite ones, and power-numbered universes behave different from those that have multiple prime factors.

Three variable are used throughout this study. **c** counts the number of elements of in the cyclic universe, corresponding to the large-case variable in expressions such as "C12." **d** counts the number of elements of a pitch-class set [11],[12]. Thus, for a C-major scale drawn from a chromatic universe, $c = 12$ and $d = 7$, and for a C-major triad drawn from a diatonic one, $c = 7$, $d = 3$. Finally, **g** is a generating interval, and corresponds to one of the $\lfloor \frac{c}{2} \rfloor$ interval classes that exist within a universe of size c. Motion upward through the values of c corresponds to motion downward through the rows of Pascal's triangle, a skewed version of which is as Table 1. Each entry in the table presents $\binom{c}{d}$, the number of sets of cardinality d within a universe of c elements. The two highest and two lowest values of d, shaded in the table, which I shall refer to as "exterior cardinalities," are entirely predictable and are of little interest musically and mathematically. Our attention will be focused exclusively on the unshaded cardinalities in the interior of each row, once they begin to appear at $c = 4$. The final column gives the cardinality of the power set, 2^c, which sums the entries to its left.

III. Non-cyclic universes of 0 to 2 elements

Since c = 3 is the smallest cyclic universe, there is some sense in beginning there, but I have found it profitable to work quickly through the trivially small, non-cyclic universes first, establishing the most basic definitions (defined terms are here placed in italics). For $c = 0$, there is a single set that is both *null* ($\varnothing$) and *universal* (U). For $c = 1$, these functions partition into two distinct and *complementary* sets of different *cardinality*. The formal definition of complementation ($B = U \setminus A$ iff $A \cup B = U$ and $A \cap B = \varnothing$) gives occasion to introduce some fundamental terms and symbols of set theory. The power set of $c = 2$ is completed by augmenting $\emptyset$ and $U = \{A, B\}$ with two singleton sets {A} and {B}, related both by complementation and by *cardinality equivalence*.

IV. Cyclic universes of 3 to 5 elements

With $c = 3$ we arrive at a properly cyclic domain, and the next several universes will present occasion to gradually introduce the most significant properties, relations, and representational protocols for musically realized cyclic spaces, including modular arithmetic, cyclic graphs, rotational (transpositional) equivalence, set class, abstract complementation and inclusion, invariance , interval class, interval vector, and interval generation. **c = 3** has eight sets: $\emptyset$, U, three singletons, and their three complements. Growth in the number of elements requires a more systematic labelling protocol, and so *integer labelling* of elements is introduced, from 0 to $c - 1$. Using integers as labels risks confusing the different "registers" in which numbers will be used – for labelling, counting, and measuring – and it is important at this stage to exhort vigilance about these distinctions.

This is the appropriate universe in which to introduce *modular arithmetic*, *cyclic graphs, and transpositional equivalence*. Figure 1a shows the cyclic graphs for the three sets for $c = 3$, $d = 2$, using filled circles to indicate the presence of an element, and unfilled circles to indicate its absence from the set. It can be readily seen that the sets are related by rotation. Figure 1b realizes the same three sets as rhythms in 3/4 meter. A musician can just as easily intuit how rotation is realized in this domain as in the graphic one.

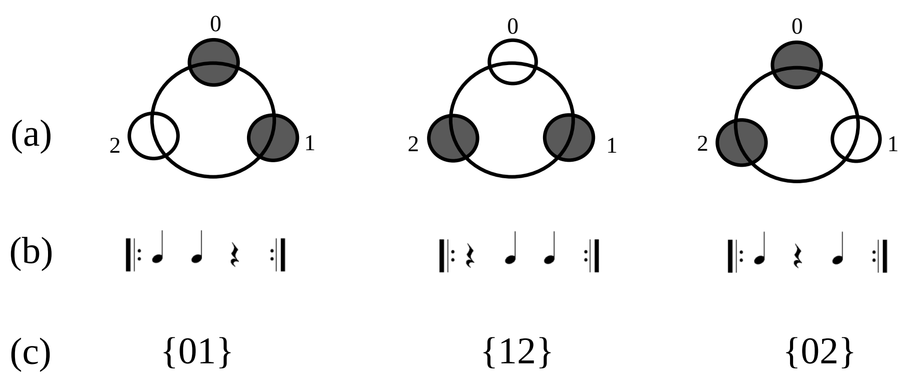

Figure 1: *Sets for $c = 3$, $d = 2$, represented in three different ways: (a) as cyclic graphs; (b) as repeating rhythms; (c) as integers.*

Finally,Figure 1c realizes the same three sets using integer notation. Here intuitions are less stable. How can {02}, whose elements seem separated, have the same structure as {01} and {12}, whose elements are explicitly adjacent? By comparing these representations with the cyclic graphs and rhythmic sets, students can gain their first foothold on the unfamiliar logic of modular arithmetic. It is through comparisons of this sort that students begin to explore the ways that graphic, arithmetic, and musical relations reciprocally model each, a process whose goal is to allow their pre-loaded intuitions about each of these three domains to inform their understanding of the other two.

c = 4 has sixteen sets, ten of which have exterior cardinalities that advance no new properties. We focus on the six sets where $c = 4$, $d = 2$. For the first time, we encounter cardinality-equivalent sets that are not equivalent by rotation (= transposition). These six sets represent two distinct pair-wise distances, or intervals, each corresponding to a *transposition class*, or T-class, which can be provisionally labelled as step (or adjacent pair) and leap (or diametric pair). The musical distinction between these two classes is represented by Figure 2, from Mozart, where each measure has four ♪beats, which can be labelled in order from 0 to 3. In the first measures, the beats are partitioned into complementary step-related pairs, {12} and {03}. Beginning at bar 5, beats are partitioned into complementary leap-related pairs, {02} and {13}.

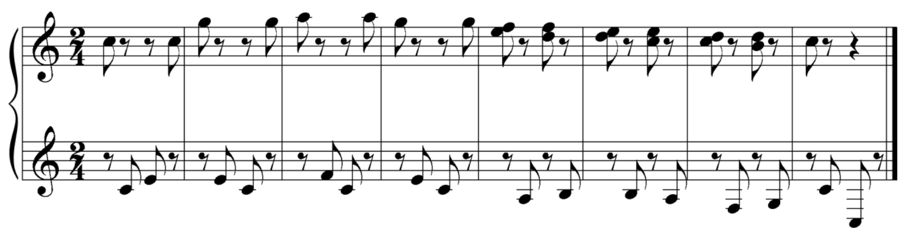

Figure 2: *From Mozart's Variations on "Ah, vous dirai-je, Maman." Measures 1 - 4 partition U4 as {03} and {12}, representing step class* [*01*] *Measures 5 - 7 partition U4 as {02} and {13}, representing leap class* [*02*]*.*

The six dyads in C4 contains four steps but only two leaps, a curious circumstance that opens the door to the important topic of rotational *invariance*. Musicians easily intuit, from examining Figure 2, that there are more possible step-pairs than leap-pairs, since each leap-pair is indistinguishable from its two-unit rotation (or transposition). The invariance is easy to see in a cyclic graph, but more of a challenge to see when the same sets are represented using modulo-4 integers. It is in mastering challenges of this sort that students who have difficulty adjusting to modular arithmetic gain further traction.

The introduction of T-classes in C4 through the step/leap distinction gives an occasion to confront one of the perpetually confusing aspects of atonal set theory, the distinction between literal sets and abstract set classes. The project of set classification requires attaching labels to the classes, for book-keeping purposes. In 1973, Allen Forte assigned each set class a two-value label whose second value was an arbitrary number [13], but many atonal pedagogies now prefer *prime forms*, a procedure for selecting a member of the class to represent the class as a whole. Although this protocol eliminates the arbitrary relation between label and referent, there is still a pedagogical challenge: the same label evidently refers to objects at two different levels of abstraction. For example, in the usual chromatic/atonal interpretation of C12, {037} references a C minor triad, and [037] references the class of twelve minor triads. Unless the distinction between curly and square brackets is emphasized, considerable confusion arises. The step/leap distinction in $c = 4$ provides a pretext for introducing the prime-form protocol in an environment that is intuitive and contained: [01] and [02] are introduced as alternate labels for step and leap classes, respectively.

c = 5 has thirty-two sets, still small enough to explore and comprehend as a single *Gestalt*. Twenty of these sets are of intermediate cardinality: ten pairs ($c = 5$, $d = 2$), and their ten complements ($c = 5$, $d = 3$). Each cardinality class has two distinct rotation classes: steps and leaps for $c = 5$, $d = 2$, and their complements for $c = 5$, $d = 3$. The fact that 10 is a multiple of 5 further suggests that there are no transpositional invariances, a result of 5's status as a prime number.

I use $c = 5$ to introduce two significant concepts that are portable to larger cardinality universes. The first concept is total *interval content*, a property of set classes that is catalogued by *interval vectors*. Cataloguing the single interval of a two-element set is trivial work, but there is a payoff: cataloguing the three intervals of a $c = 5$, $d = 3$ set, and comparing that vector to that of its $c = 5$, $d = 2$ complement, exposes the intervallic affinities between complement-related sets, and leads to the introduction of a simple form of the complement algorithm. In addition to the gains realized by introducing complement relations in a small universe, there is also a pedagogical benefit to introducing the complement algorithm in a universe whose cardinality is odd. Because no interval divides the universe as the tritone does in C12, there are no invariances to complicate the complement/interval algorithm.

Because $c = 5$ is a prime universe, it is also a good place to introduce intervallic *generators (g)*, marking a shift from a static conception of a cyclic universe to a dynamic one, a space to navigate through time. Each set in $c = 5$ is generable either by step (g =1) or leaps ($g = 2$). Both values of g generate all sets of exterior cardinality. It is the intermediate cardinalities that are distinguishable by their generators: $g = 1$ generates steps of class [01] and their complements of class [012], and $g = 2$ generates leaps [02] and their complements [013].

$c = 5$ is musically realized as a pentatonic scale, whose intervals come in two chromatic sizes: "steps" are major seconds and minor thirds, and "leaps" are perfect fourths and a major third. Following John Clough [10], these chromatic distinctions are overlapped, and the pentatonic space is treated as if it were perfectly rather than maximally even. Step-generation involves a sequential pass through the scale, <C, D, E, G, A> ; leap-generation involves skipping scalar notes, <C, E, A, D, G>. There are dozens of pentatonic pieces that can be used as analytical illustrations of this universe, as compiled e.g in [14].

V. Cyclic Universes with 6 to 9 elements

As the half-way point between zero and twelve, C6 is the universe in which curious features with unexpected musical ramifications begin to arise. It is also the point where the size of the power set begins to get too large to control. The intermediate cardinalities consist of 15 pairs, 15 pair-complements, and 20 triplets. None of these numbers are multiples of c, indicating the presence of rotational invariances at each cardinality. There are three T-invariant diametric leaps [03] to go with the six steps [01] and six skips [02], and two T-invariant skip-generated triplets [024] to go with the the six step-generated clusters [012] and twelve ungenerated sets [013] and [014], which are discussed below.

Since six is the smallest number with two divisors, C6 is the smallest universe to have two distinct *perfectly even* sets. Realized as rhythms, these two perfectly even sets model the two meters available in a bar with six beats. Setting the beat to an eighth note, [03] suggests a bar of $\frac{6}{8}$ meter, and [024] a bar of $\frac{3}{4}$ meter with duple subdivisions. The successive juxtaposition of these meters models the Baroque pre-cadential hemiolas, and their superposition as [03] $\cup$ [024] = [0234] underlies the metric tug of a waltz, as well as the 3-against-2 cross-rhythms of West African and Afro-Caribbean repertories.

The complementary T-classes [013] and [014] introduce many new features that do not arise in smaller-cardinality universes. Neither set is inversionally symmetric; instead, the two classes are abstractly related to each other by inversion. Accordingly they have identical interval vectors: one instance of each interval, from which we see that C6 is the smallest universe that hosts non-trivial all-interval sets. The flatness of their interval vector is related to their ungenerability, as note above.

This is a good moment to make a systematic study on inversion (= reflection), showing how members of [013] and [014] invert into each other around a variety of axes. As in atonal pitch-class theory, questions of perceptibility immediately arise: can one hear inversionally related sets as "the same thing?" A study of Figure 3, from a Beethoven quartet, furnishes an opportunity to experience their **perceptual** non-equivalence. The cello and viola together sound beat classes {0, 3, 4}, which is a member of T-class [013]. The violin sounds the complementary beat classes,

Figure 3: *Beethoven, String Quartet no. 8, Op. 59 no. 2, Allegretto, bars 1 - 8. The violin attacks from mm. 2 - 7 are class [014]; the complementary attacks in the remaining parts are class [013].*

{1, 2, 5}, representing T-class [014]. For most listeners, these two beat-class sets project different meters. In general, when two adjacent time points are attacked but their immediately surrounding time points are tacit, listeners hear a phenomenal accent on the later attack of the pair[15]. The consequence is that set {1, 2, 5} reduces to {2, 5}, a member of [03] that bisects the measure, causing the violin to be heard in a displaced $\frac{6}{8}$ meter. By contrast, set 0, 3, 4 reduces to 0 4, which is filled out as 0, 2, 4 and thus the accompanying instruments project an undisplaced $\frac{3}{4}$ meter. In bar 8, all instruments articulate a [024] set, resolving the metric conflict at the cadence.

Both non-unit generators in C6 are divisors, and hence idempotent after several generations. There are also three generators in the prime universe of $c = 7$, but here each one retains its potency to generate the universe. These three generators have particular roles to play in the context of European diatonic tonality, the repertory at the core of most music-theory curricula: they respectively organize melody, harmony, and harmonic progression.

The three cyclic graphs of Figure 4 show the action of these generators on the diatonic collection. The unit generator ($g = 1$) forms scalar fragments, which are the basis of melodies. $g = 2$ generates tonal harmonies, the [024] diatonic triads and [0135] diatonic seventh chords. $g = 3$ generates the diatonic cycle of fifths, and thus its generated sets are the basis of progressions between successive harmonies. Most intermediate-cardinality set classes are generated by exactly one of these intervals, and thus can be seen to perform one of the three jobs. The ungenerated sets belong to the four inversionally asymmetric classes, [013], [023], [0124], and [0234], whose intervals have

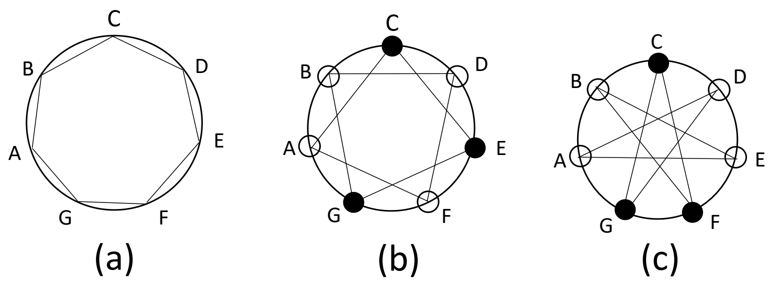

Figure 4: *Three generators on C7. (a) $g = 1$ produces a scale; (b)$g = 2$ produces chords such as triads (filled circles) and seventh chords (open circles). (c) $g = 3$ produces a cycle of fifths, connecting tonics to their dominants and subdominants.*

uniform multiplicities.

C7 is also an appropriate universe for introducing inclusion, *maximal evenness* (ME) and *Q-relations*. Inclusion follows naturally from a study of interval generation, since the sets from a single generator form an inclusion network. The ME property is held by fifths, triads, and seventh chords, the building blocks of classical tonality. The Q relation (my term) formalizes the parsimonious voice-leading relation between sets and set classes of equal cardinality [16]. All of these topics will have significant interpretations in the C12 chromatic universe.

$c = 8$ and $c = 9$ can be studied in order, but also in tandem, as both are powers of small primes. They are the smallest universes that have both divisor and non-unit prime generators [5], and the interaction of these two generator-classes is rich with dynamic potential in the context of metric cycles. The divisor generators for these universes (for $c = 8$, $g = (2, 4)$; for $c = 9$, $g = 3$) underlie the isochronous meters of the Western European tradition ($\frac{2}{4}$, $\frac{4}{4}$, and $\frac{9}{8}$ meter respectively). Non-unit prime intervals ($c = 8$, $d = 3$ and $c = 9$, $d = 2$) generate the non-isochronous meters [17] of vernacular repertories [5]. The rhytmic dialectic of concert and vernacular traditions is artfully exploited in a number of repertories beginning in the middle of the 19th century. In $\frac{4}{4}$ meter, Western pure-duple isochrony is juxtaposed with the Afro-Caribbean tresillo (Gottschalk, Joplin [18]) or paradiddle (Reich [19]). A similar juxtaposition is available in $\frac{9}{8}$ meter, where pure-triple isochrony is juxtaposed with characteristic Balkan (aksak) rhythms (Bartók [20], Brubeck [18]).

VI. Scaling up to C12

In the course I have developed for undergraduate music majors at Yale University, the progression to nine elements takes about six 75-minute classes, or three weeks of a thirteen-week semester. Since 10 and 11 have no familiar applications, in fourth week I begin the study of atonality proper, introducing the C12 chromatic pitch-class universe. Students are by now familiar with all of the foundational terms, concepts, and protocols of atonal theory. They can quickly generate a table of the 19 trichord classes, indicate which ones are inversionally symmetric, which two are inversionally paired, and recognize the special properties of the augmented triad; place trichord classes into a Q-relation network [16]; quickly assign trichords to prime forms; and identify abstract inclusion and complement relations with largercardinality sets.

Students also are in a position to appreciate what is special about living in a musical universe that has exactly twelve elements, and how different their world would be if that number were incrementally smaller or larger. Students who have studied the interaction of divisor generators

3 x 2 in C6 are primed to understand the C12 as a cross-product of augmented triads and diminished-seventh chords, and to appreciate the ways that atonal composers compound these generated cycles to create interactions between hexatonic, whole-tone, and octatonic scales [21]. Because they have studied how prime and divisor generators interact in C8 and C9, they are in a position to understand how diatonic and chromatic-cluster sets can be placed into opposition with divisor-generated ones in C12. Because they have studied transpositional invariance and maximal evenness in small-cardinality universes, they can identify sets with those properties in C12, and recognize their musical significance. There are also significant applications of C12 in the beat-class domain. For example, the relations of the divisor-generated perfectly even sets underlies the interaction of ($\frac{12}{8}$, $\frac{3}{4}$, and $\frac{3}{2}$ meters, as different ways to structure the interaction of incommensurate divisor generators [22],[23].

Along the way to C12, students have picked up intimate knowledge of pentatonic, diatonic, and time-cycle universes, and this knowledge has its own value to musicians. Much of the music that they perform, listen to, improvise, or compose is both diatonic and deeply metric. Students who have pursued this pedagogical path to atonality will have, along the way, acquired a mode that helps them explore the relationship between music with those ubiquitous properties, musical systems in which they participate, and the properties of the abstract universes that those systems instantiate.

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Music as a Carbon Language: A Mathematical Analogy and its Interpretation in Biomusicology

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Abstract: *Carbon dynamics influence human physiology, culture and social patterns. Along centuries, linguists had been sufficiently discussed how breathing and cardiovascular performance set preconditions for word segmentation, phrasing, repetition, iteration, variation and expressiveness. Less attention had been paid to this influence as reflected in music, due to the belief that music can be "purely instrumental", and therefore far away from speech. However music, dance, respiration and verbal language share common evolutionary grounds, as well as important physiological features and constraints related to the organic properties of carbon and to its role in biological evolution. In this context, this contribution interprets chemical proportions in bioorganic compounds as analogies of their musical parallels, with consequences to music theory. Mathematical evidence is suggested for sketching a* carbon hypothesis of music. *From this perspective, music is more a feature and a consequence of chemical and biological constraints (not exclusive of humans), than a product "purely social" or "uniquely cultural".*

Keywords: *Carbon. 1/f Noise. Zipf. Music Language Self-similarity.*

I. A finding of quantitative linguistics

Little known outside his technical domain, Luděk Hřebíček's (1934 -) research on speech self-similarity is a major contribution to quantitative linguistics including developments on semantic attractors, grammar structures as graphs and networks, and word-phrasing length variation as self-similar dynamics (particularly in [24, 25]). Making part of a new paradigm in language investigation, these theoretical devices fit strikingly well with their corresponding analogies in music, even when musical semantics and syntax preserve important differences in respect to verbal language.[1]

Hřebíček published his article "Fractals in Language" (1994) as a first attempt to explain the Menzerath-Altmann law, with relevance to the theory of phrase extension, in pursuit of a theory of sentence aggregates in natural language. This law applies to the discrete probability distribution in the frequency of data which can be syllables, words or phrases in a text, and is closely related to Frumkina's Law, a probabilistic model for the occurrence of linguistic units in text passages [21].

Linguist Reveka M. Frumkina (1931-) systematically investigated the distribution of words in text blocks of fixed length. Later, also the recurrence of syntactic structures and functions was analysed with similar results in [27]. Applying this probabilistic law in random samples of

[1] For a discussion on musical semantics and syntax, see the work of Raymond Monelle, David Lidov, Eero Tarasti, Kofi Agawu, and Lidia Goehr, among other semioticians involved with music theory. Their specialised developments are not referred to in this contribution.

texts and voice recordings, provides us with a useful tool for determining authenticity of styles in literature, but also, in very distinct contexts, for the identification of patterns in ecolects and idiolects (i.e. contextualised pragmatic variations of speech). An equivalent technique in music and sound analysis has had major applications in automata for the identification of musical style and authorship (e.g. [16]).

Music and speech diverge in a complexity of variables and outputs, however it suffices to compare intervals of duration, intonation, emphasis and repetition, in order to obtain useful information that ultimately can be interpreted in terms of rhythm, pitch, amplitude, timbre and texture. In fact, most samples of speech and music can be characterised as "melodies" containing these features; therein the interest that Hřebíček's approach may receive in musicology: whether in linguistics an "aggregate" denotes a group of sentences in a text containing a length-variating word or lexical unit (Hřebíček 1997:104), the analogous concept in music may reflect the structure of motive-phrase variation, and so forth the variation of melodic components (i.e. how a melody performance is enriched by second or third-level complexity, with variations of rhythm, pitch, amplitude, timbre and texture). Contrapuntal densities can also be conceived as relatively simultaneous melodies, because, as Julián Carrillo (1948/1967:152) thought: "counterpoint is the amalgam of melodies with same or different rhythms". In strict sense, musical counterpoint can be described by a set of rules acting under an *aggregates law*.

Complementarily, the Menzerath-Altmann law states that the longer an aggregate in a number of sentences, the shorter its sentences in their number of words; this implies a tendency to concentrate the periodic recurrences in the structure of a phrase, towards a compact group of structural units (something familiar to musicians thinking of motivic commonplaces). Although this concept is not verbatim mirrored in counterpoint theory, actually the variety of contrapuntal "species" portraits the historical intuition around this law (in its background, a *power law* in a thermodynamic context),[2] from first species to florid counterpoint. Furthermore, Schenkerian schematism also captures—in its fashion—the level of the most "compact group of structural units", over which successive structured layers arise. Then a musicological version of the Menzerath-Altmann law would state that, the longer a melody is obtained by a set of rules, there will also be a greater tendency to segmentation.

At this point, musicology and linguistics embodies the shape of a garden of connecting paths, since different theoretical approaches, proper of each discipline, start to unveil a single phenomenology. Thus, Hřebíček's theory of speech self-similarity may be traced by its connections with the Zipf's law of language *harmonicity* (in terms of physical, compositional trends), with the *law of least effort* (ethological empiricism that emphasizes constant power laws in biological behavioural economy), and with the balance between noise and syntax correctness in Markovian systems as probabilistic-statistic collections of codes under postulates of information theory.

Zipf's law is an empirical law employed in probability and statistics, which reflects an approach to a probabilistic distribution applicable to a variety of samples in many different fields including physics, biology and social sciences. The law, originally proposed by linguist George K. Zipf (1902-1950), states that in a generalized sample of verbal expressions, the most frequent word will occur approximately twice as often as the second most frequent word, which occurs twice as often as the fourth most frequent word, and so on (see [63, 64]). In the end only a few words are used very frequently, whilst most of the words are little used (an experience common to conscious processing of a new language acquisition, comparable to a musical repertoire learning process). This principle is summarized in the formula

$$P_i \sim P_1/i,$$

[2]See [50] for a general introduction to the concept of *power law*.

where P_i is the probability of selecting an object i, and P_1 is the probability which a first term of a series has to appear in a repertoire of objects. Given that the successive probabilities are positive and ordered ($p_1 \geq p_2 \geq ...$), and for all i, $p_i \leq 1/i$, it is suggested that a second term occurs approximately 1/2 as often as the first term does, whilst the third term occurs approximately 1/3 as often as the first one, and so on. From this law it is also conjectured that the most common words tend to be shorter, and that when they tend to be too short, then they are replaced by longer words [52, p. 27].

[42, p. 238-249] description of Zipf's law, focused on the idea of efficiency in terms of an ability to "emphasize some choices at the expense of others", has a special significance for music, assuming that many musical strategies for consistency are based on a same kind of efficiency. As a matter of fact, the structuring form of Zipf's equation, as the series

$$1 \sim 1, \quad 2 \sim 1/2, \quad 3 \sim 1/3, \quad 4 \sim 1/4, \quad 5 \sim 1/5...$$

constitutes a self-referential sequence, analogue to the aliquot division of an acoustic system, with a fundamental frequency and its natural harmonics. Such a self-referentiality can also be interpreted as a sequence with statistical self-similarity, i.e. not necessarily with "obvious" superficial self-similarity.

Acoustic (i.e. molecular) patterns in biology involve patterns of *activation-inhibition*, or *perturbation-absorption*, as happens in the mechanical vibrations of a string or a membrane. In a diversity of layers, these patterns tend to synchronization and consequent self-similarity, as it can be noticed in phenomena such as the intricate network of changing figures in a surface of water shaped by the wind; or in the Chladni figures of fine sand upon a plate periodically vibrating; but also in respiration and cardiac pulsation in individuals, and in a variety of practices coordinated among groups of individuals. The rhythms shaping these patterns cannot be explained uniquely as a linear physical interaction, because of the relative autonomy of physical sources participating within a same environment, as noted in [43]. Therefore the application of nonlinear methods – sometimes in cooperation with the linear ones - may be useful for systematic musicology, as explained below.

II. Music and speech as bioacoustic patterns

From evolutionary physiological and structural views, respiration is of first importance for the foundations of music and speech. As a physical pattern, respiration has a constant, quasi-periodic relationship with the brain's oxygenation, and thus, with the rhythm of mental functions and mind-performative processing. Accordingly, it is not surprising that cardiorespiratory quasi-periodicity synchronises with brain performance and the nervous system's functions involved with music and speech outputs.

Respiration has at least three analogous – although importantly differentiated-scalar levels, closely associated among them: (1) cardiopulmonary carbon dioxide release and absorption of oxygen from external environment, (2) cellular respiration which, by degrading glucose with O_2 participation, allows organisms to obtain energy, and (3) carbon self-structuring at DNA-bases sequences' recursion. Conversely, the electric features of carbon and carbon oxidation have a capital role building protein blocks, making up the sequences of the genetic code and self-repair of tissues and limbs. This major role of carbon also affects the endorhythmic coordination of organs, including motor and cognitive functions.

At each of these levels of cycles of energetic-chemical interaction, these cycles match at least two different systems; for example, the cardiopulmonary cycle is coupled to the muscular/locomotor cycle, but also to the cycle of involved brain/modular performance (in its turn, a system of electro-chemical couplings). Abundant literature [7, 8, 9, 23, 43, 51] reports natural synchronization of physical cycles in many animal species and in quite varied biological processes. The mathematical modelling of these coupled cycles usually employs the *circle mapping* representing quasi-periodicity of the organic cycles. Quasi-periodicity ratio in coupled cycles fall into the so-called *mode locking patterns* in the circle, helpful for measuring synchronization [43] (a concept somehow familiar to musicians, but in contexts of tuning, harmonic structure and physical empathy).

Mode locking in human cardiorespiratory patterns crucially include ratios 1/1, 1/2, 2/3, or 1/4, mentioned in [9 and 23]. Then it is not just casual that ratios 1/1, 1/2, 2/3 (i.e. triplet implication), 4/4, 2/4, 3/4 and 6/8 are commonplaces in music; particularly in dance music and its derivations. Since human body is nearly symmetric, and dance is usually a group practice reflecting human symmetry in a given space, dance steps are mirrors of bodily ratios such as the mentioned ones. Ancient Pythagoreans did notice that these simple symmetries easily coincide with small ratios they identified as "harmony" in acoustic phenomena; thus they discovered a *natural harmony* represented by a succession of rational numbers, where the smallest ratios were considered as more pleasant or "more harmonic", and conversely higher ratios would correspond to less pleasant and "less harmonic" musical sounds. Now we may add that prime numbers progressively larger would also progressively produce a feeling of *sound unprocessing*, a concept that Barlow (2001, p. 6-8) labels as "indigestibility of primes". Table 1 suggests this ratio progression, from higher symmetry and easier predictability, to levels increasingly embedding sub-symmetries and bigger prime factors, which may be enriched by introducing probabilistic variation of inner *accentuation* (in metric patterns) and *intervallic chord composition* – both musical concepts in Riemannian sense [48].

Songs and other vocal repertoire may reflect an interplay between bodily symmetries, such as the mentioned dance metres, and Pythagorean ratios. Congruently, vocal repertoire intuitively identifies simple ratios with the Zipf's law series described above, as the vocal functions and recurrences also follow the *law of least effort*. In fact, the use of the so-called "harmonic series" 1/1, 1/2, 1/3, 1/4, 1/5, 1/6..., when applied to tuning voices and instrumental practice, cannot ignore the Weber-Fechner law on the relationship between the physical magnitude of a stimulus and its perceived value. The effect of this law makes that, in musical experience, the harmonic series cannot mean a simple succession of abstract ratios, but instead may involve a deployment of epistemic-cognitive deviation reflected in experiential varied interpretation. Table 1 illustrates this correspondence among the simplest tuning intervals and the most used musical metres, suggesting a Weber-Fechner complexity for the human interpretation of progressively smaller intervals, gradually increasing anti-intuition, i.e. gradually going beyond the arrow in the lower row of the table (a progression that Figure 1 suggests in more detail).

Moreover, if we interpret the harmonic series as a probabilistic arrangement, we may say that – using the simplest example in tonal classical harmony – there is a total probability of 1/1 for a *fundamental pitch*, to be the ubiquitous signature in a tonal piece. Then we would have a probability of 1/2 for the second tonal hierarchy (commonly the so-called interval of *perfect fifth*), and 1/3 for the probability of a third interval (the interval of *third*), and so forth. Although this may be interesting for some aspects of tonal theory, it is evident that musical practice does not consist uniquely on arranging ranks of pitch-span probability according to the hierarchies of a rigid structure. This is why Barlow (2001, p. 4) theorizes on probability as a function of *musical priority*; nevertheless, one may ask what "musical priority" is exactly, as it directly concerns to form and style from the very foundations of music. Barlow (*ibid.*, p. 2-3) proposes that – for the sake of simplification, let the metrical one be an illustrative example – musical priority is distinguishing a diversity of *probabilistic weights* in a given metre: "In the case of ametric music, all the pulses are equally probable [...] But if you want to make the music more and more metric, you have to then decide how probable or how important the individual pulses ought to be. This assumes there might be a correlation between their importance and their probability" (*op. cit.*, p. 2).

For the Riemannian theory of musical metre and phraseology, the correlation between the diversity of musical weights and the extension of musical phrases and periods was already a big concern. In fact, Riemann (1903, p. 200-201) conceives musical metre as an *analogy* (i.e. proportionality) of musical harmony, since "it is clear that, as the proportions of a measurement grow, the answering member is increasingly likely to lead to an ever more noticeable resting point." Altogether with this notion, Riemann introduces *probability* ("likelihood") for estimating the weight and extension of musical notes, motifs, phrases and periods, in a fashion analogous to the linguistic concept of "aggregate" used by Hřebíček (1994, 1997). Thus *tension* and *extension* (the latter identified by [59, p. 337] as "complementary cadences" in a Neo-Riemannian context), as well as the function of what Riemann calls *resting points*, are aspects of a whole system common to speech and music practices, where a diversity of parameters are frequently correlated; therein the importance of measurement, contrast and punctuation, both in lyrical and musical traditions where balance between periodicity and aperiodicity has a capital structural and semiotic function. As a matter of fact, the Riemannian theory is an elaboration upon Koch's *Versuch einer Anleitung zur Composition* (1793), a work of enormous influence throughout 19th and 20th centuries, which emphasizes the analogies of *basic unit transformation*, *periodicity*, and *structure* in speech and music, as well as the intuition of symmetry, for verbal and musical composition and diversification.

In contrast to diversity, speech and music also necessarily seek for structural and functional economy. Since early times of systematic musicology (see [13, 44, 34, 35]), information theory receives particular attention for being helpful as a method estimating the balance between *noise* and *code*; between *randomness* and *meaning* within a musical system. Once again, we may invoke the relationship between *probability* and *priority*, as possible, often desirable, equilibration in tension and extension of periodic-aperiodic systems.

Whether the economy of the code and the gradual sophistication of the "message" are biological characteristics starting from chemical organic self-assembling, the *ordering function of the code* also gradually leads to structural coherence manifested as self-similarity within in a vast range of diversities. Self-similarity can be understood, then, as a mechanism preserving information at low cost (see [39, p. 209-210])). In music, a basic example of this is the geometric-arithmetic relationship $3/2 \leftrightarrow 12/8$ (see Table 1) which contains both the whole and the half-step of the diatonic system (i.e. its *diazeuctic feature*), and simultaneously allows the tonal system cycles, also represented in a two-dimensional periodic space by the *Tonnetz*, the Euler-Riemannian honeycomb lattice that characterizes tonal functions. The recurrence of this structural relationship, including all balances between periodic-aperiodic repetitions and extensions within a given grammar, guaranties the efficient economy of music as an information system, nonetheless *continuously productive* (i.e. *poietic*, both in biological and cultural senses). For organic chemistry this is not a new kind of systematic relationship; on the contrary, the cycles of periodic-aperiodic relationships lie at the bottom of chemical self-organisation, as noticed by crystallographers since the ending of 19th century.

III. Towards a *carbon hypothesis* of music and speech

The philosophical association between cardiorespiration and harmony delves into the darkness of antiquity. Plato's harmonic concept of cardiorespiration in his *Timaeus* (70 b-d) is just one example within an endless collection of historical sources. However this issue comes into clarity in relatively recent days. By mid-20th century, Carrillo wrote that "the vibrations our heart produces are of

Table 1: *Columns from left to right: musical ratios ordered by numerator size, starting at 1/1; decimal expansion of the same ratios, and their conventional denominations in Western music; correspondence to the harmonic series (acoustics); and, in the rightmost column, common metrical signatures in music and dance, e.g. from the single-beat bar* 1/1 *to the* 12/8, *common in distinct cultures albeit with different inner accentuation. Notice the* 12/8 *ratio closes the first cycle of music self- structuring towards* 3/2, *connecting the diapente (perfect fifth) with the diapason (octave), allowing the circle of fifths and expressing the diazeuctic feature of the diatonic set that embeds the chromatic scale. Complex metrics and harmony arise progressively going downwards in this conceptually endless table (continuation is suggested by an arrow and ellipsis for each column), which mathematical ordering is suggested further in Figure 1.*

ratio	**decimal expansion**	**harmonic interval (tonal degree)**	**harmonic equivalence**	**music & dance metre signature**
1/1	1	generator (*fundamental* or *tonic*)	1/1	1/1, 4/4, 2/2, 8/8
2/1	2	diapason (*octave*)	2/1	4/2, 8/4
3/2	1.5	diapason (perfect fifth, *dominant*)	1/2	2/4, 6/4, 4/8
4/3	1.333...	diatessaron (perfect fourth, subdominant)	1/3	1/3 : 4/4 (*triplet*)
5/3	1.666...	major sixth (s*ubmediant*)	2/3	1/4 : 3/4 (*quadruplet*)
5/4	1.25	fifth harmonic (major third, *mediant*)	1/4	1/4, 2/8
6/5	1.2	minor third	1/5	1/5 : 4/4 (*quintuplet*)
7/4	1.75	seventh harmonic (*subminor seventh*)	(1/2 + 1/4)	3/4, 6/8
7/6	1.1666...	septimal minor third	1/6	1/6 : 4/4 (*sextuplet*)
8/7	1.142857...	major second (*supertonic*)	1/7	1/7 : 4/4 (*septuplet*)
9/8	1.125	major tone (*epogdoon*)	1/8	9/8
10/9	1.111...	minor tone (*lesser tone*)	1/9	1/9 : 4/4 (*nontuplet*)
11/8	1.375	eleventh harmonic (*tritone*)	(1/4 + 1/8)	3/8
12/8	1.5	diazeuctic feature of the diatonic/chromatic proportion (12/8 : 3/2) and f*ifths cyclical,feature*	1/2	12/8
13/8	1.625	thirteenth harmonic (*tridecimal neutral sixth*)	(1/2 + 1/8)	5/8
15/8	1.875	major seventh (*subtonic* or *leading tone*)	(1/2 + 1/4 + 1/8)	7/8
↓	...	...	...	...

musical nature as they fall within the human acoustic thresholds and within the *ratios* of musical sounds [...] and they are the cause of *empathy* or lack of it, between human beings and animals of all species" [12, p. 167-169, 409] (my translation and emphases). In a parallel investigation, quick developing cardiology soon discovered and registered the "harmonic" patterns of cardiac behaviour, as extremely useful signs for understanding the heart as a dynamical system (see [5, 6, 7, 9, 14, 20, 23, 28, 32, 45, 49]).

In human cardiorespiratory performance, according to [32, p. 1] "the heart can act as a pacemaker for respiration". This is a mechanism of synchronization that in physical terms signifies that heart and lungs, and the whole cardiovascular system tend to adjust pressure and electric potentials within a same harmonic system with constant variation and re-adjustment. [32, p. 5] proposes a diversity of *tunings* — although not exactly using this term — of cardiovascular human synchronization that behaves as a system of harmonic couplings (in its physical sense). Whether brain oxygenation strongly depends on this process of synchronization, [29] provides arguments to hypothesize that Hebbian synaptic plasticity (the adaptation of neurons in the brain during memorising, learning and comparing processes) shares the same kind of proportionality. In few words, music would be an expression of empathy and coordination of a complex selfness, a connection and articulation of endorhythms and exorhythms oriented by carbon signals at different levels: from organic chemical bonds, to cellular coordination, and then to cardiorespiration that provides rhythmic assortment of oxygen and hydrogen to the brain and the emerging mind.

Carbon has an exceptional role in biochemistry: leading and performing electrochemical bonds and structures with the nitrogenous bases in RNA and DNA; with its structural-energetic self-organisation constructing organs and organisms; and with its central participation in cardiovascularity, respiration and brain-nervous system operation. At the genetic level, each nucleotide consists of three components: a five-carbon monosaccharide (*pentose*) called *ribose*, a phosphate group, and a nitrogenous base. At the muscular and locomotor systems, CO2 release comes from the breakdown of glucose — as said before. As well, besides carbon dioxide and water, aerobic respiration produces Adenosine Triphosphate (ATP), the *molecular unit of currency* in metabolism and the intracellular energy transfers, which has remarkable molecular plasticity thanks to its multi-faceted topological features including carbonic bonds (see [60]). Besides [14, p. 6] describes how *tyrosine-protein kinase* (an enzyme encoded by the Abelson-related human gene, localised in stress fibers and cardiocyte disks) does stimulate rhythmic pulsation of the cardiac system:

> Receptor tyrosine kinase protein phosphorylation plays a crucial role in a wide variety of cellular processes that control signal transduction [in the cardiac system]. Protein phosphorylation is a rapidly reversible process that regulates the intracellular signaling in response to a specific stress [...] Signaling by activated tyrosine kinase receptor protein is initiated by the phosphorylation of cytoplasmic proteins, which in turn potentiate the intracellular signaling cascade.

A tyrosine kinase (TK, a subclass of protein *kinase*; from Greek *kinein*, "to move") is then an enzyme that can transfer a phosphate group from ATP, produced in respiration, to a protein in a cell. TK and ATP are closely related in cardiovascular and respiratory processes where the phosphorylation of pentose sugar molecules (carbon atom ribose) directly participates in DNA synthesis and cellular oxygenation [20]. In this context, TK operates as a chemical *on-off* switch of cellular functions related to the patterns of activation-inhibition involved in motor and cognitive human functions with quasi-periodic behaviour [20, 23, 32].

[56] believes that DNA frequencies can be traced as chemical noise, and mentions that "individual base positions in DNA sequences' [...] measurements demonstrate the ubiquity of low frequency $1/f^{\beta}$ and long-range fractal correlations as well as prominent short-range periodicities."

[56, p. 7]. In this fashion, [56] associates $1/f$ noise (called "carbon noise" in electric circuits context), to "large averages over classifications in the Genetic Bank data bank [including] primate, invertebrate, plant [...] [with] systematic changes in spectral exponent β with evolutionary category." Sumarising, [55, 56] interpret the symbolic autocorrelation function for measuring DNA, in terms of low frequency $1/f$ noise. The musicological meaning of this "noise" is exhaustively studied in [39].[3]

i. Empirical evidence and theoretical expansion

Observing dynamics in carbon quasi-periodic cycles, analogous to physical dynamics modelling, we may assume that this model is useful to investigate a wide range of biological quasi-periodicity. From this analogous systematization, the circle mapping of a locally-constant rational rotation number that produces Arnold tongues (see Figure 1) emulates quasi-periodicity in physiological transduction (i.e. couplings of electric and mechanical systems) as happen in cardiorespiration and nervous dynamics.

Vaughn (1990) is probably the first author to report emergent physiological harmonic synchronization in a human being singing a melody. This and further research on the same topic employ time series analysis in order to estimate and describe emotional complexity in musical performance and self-perception. The obtained results illuminate the structure of physiological quasi-periodicity. [6, 29, 45] confirm the adequacy of this approach that connects cardiorespiration, and nervous-cerebral dynamics, with the analysis of speech and music employing the circle map and the Arnold tongues as a set of associated analogies.

The circle map exhibits certain regions of its parameters where it is locked to the driving frequency (*phase-locking* or *mode-locking* in the jargon of electronic circuits) in periodically forced nonlinear oscillators. The Arnold tongues is a resonance zone emanating out from rational numbers in a two-dimensional parameter space [46, p. 130-131, 217]. Within the Arnold tongues, the orbits of the circle map are periodic and they are called *mode* (or *frequency*)- *locked solutions* [47, p. 135], a feature useful for mapping rational periodic — and therefore musical — intervals.

Figure 1 (equation originally published in [3]) displays Arnold tongues obtained by iterating the function shown upper left in the diagram. Between the tongues asymptotically sprouting from K = 0, the dynamics are quasi-periodic, and the frequency ratio is irrational. As K increases, the Arnold tongues broadens, finally leading to an overlap between two tongues, and the system can display chaos ([2, p. 65], [51, p. 122]). However, as the tongues' broadness decreases in the lower part of the diagram, the rotational values of the function change until mapping the set of rational numbers (lower limit). Notice that a zoom-in between any of these intervals will display subsequent harmonic hierarchies nested among the infinite rational intervals contained within the tongues' limit. At this limit we find precise analogies with rational numbers as intervals of classical music from distinct harmonic regimes.[4] The distribution of these musical intervals is neither successively continuous, nor perfectly symmetric, but harmonically segmented in hierarchies; thus, for example, the interval 3/5 (perfect fifth) has a higher structural hierarchy than 4/3 (major fourth), and the latter has a higher hierarchy than 5/4 (major third), and so on, in a sense plotting a *generalised musical harmony*.

The lower part of the diagram in Figure 1 suggests a quasi-musical harmonic proportionality for biochemistry, with the ratio hydrogen-carbon as "generator interval" or first-order harmony,

[3] The relationship of $1/f$ noise reported from DNA autocorrelation, is a sequel from [57, 58], after the primitive "carbon noise" originally reported by [10]. Actually, the circle map equations (such as in Figure 1) also models the phase-locked loop in electronics, as typically happen in carbon circuits [22, 37, 53]. $1/f$ noise wavelets and signals, in scalar invariance, are related by their generalized self-similarity.

[4] For a more in-detail explanation of the Arnold tongues in the context of musicology, see [39, p. 354-371].

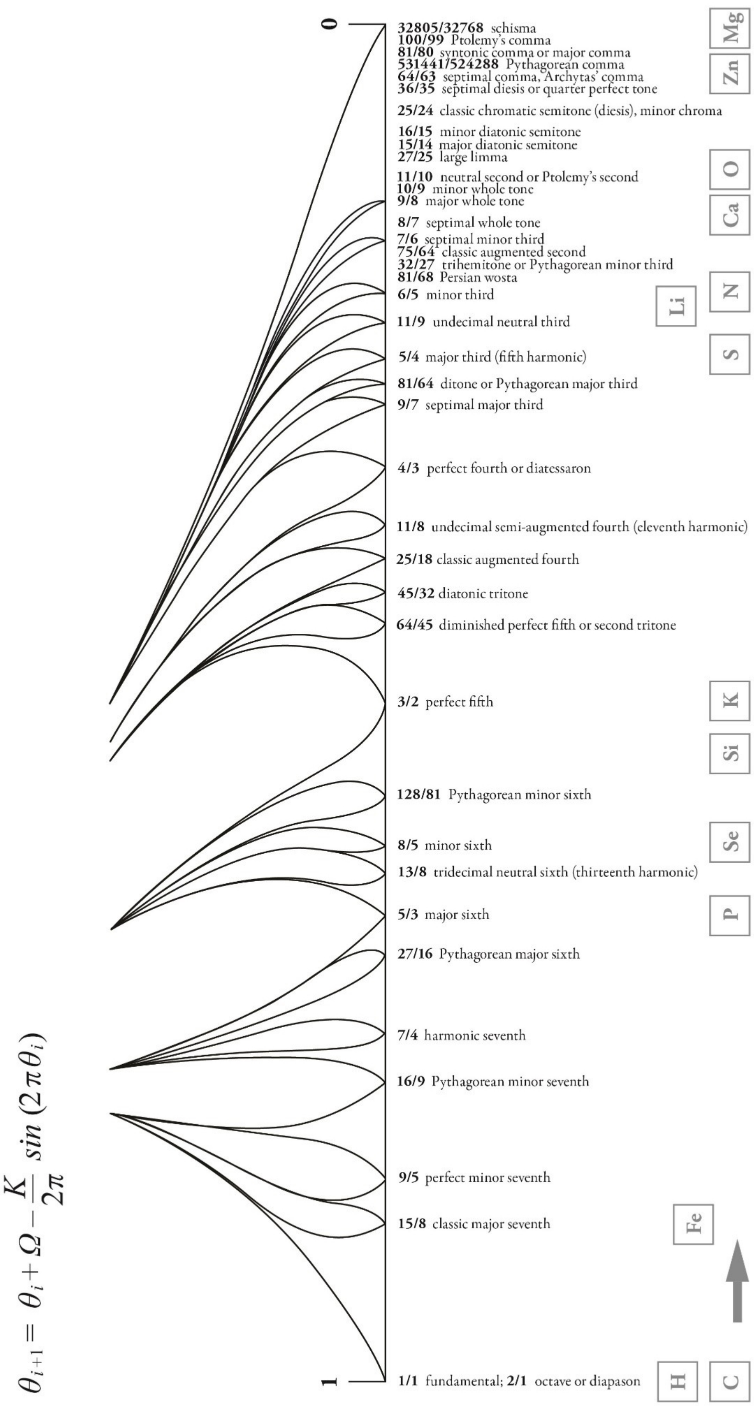

Figure 1: *Arnold tongues in the phase diagram for the continuous model, based on the difference equation shown upper left in the diagram (an equation originally published in Aubry, 1979). The vertical axis corresponds to the intensity measure of the model's periodic potential; and the horizontal limit corresponds to the family of rational numbers obtained by the function. The complete set of musical ratios can be hierarchically mapped in the interval [1, 0] (where maximal harmonicity corresponds to 1, and minimal to 0). Here the tongues limit matches a selection of the most important classical music intervals; however an infinite zoom detailing the limit structure, would reveal dense distribution of rationals. The rightmost section of this diagram, the interval 9/8 to 32805/32768 may be infinitely and systematically segmented in smaller harmonic "micro-intervals". The lower part of the scheme suggests harmonic analogies with biochemical self-structuring harmony (see main text for explanation).*

and potassium as a following hierarchy, before phosphorus, sulfur, nitrogen, oxygen, and lower harmonic hierarchies laddered in smaller intervals. These hierarchies are "visible" across the comparison between the tongues' areas, i.e. the blank areas between the sigmoid lines in the diagram. Table 2 includes corresponding values for this comparative harmony between music and biochemistry, displaying the magnitudes of atomic Larmor frequencies (the angular frequency of atoms). Since hydrogen has a Larmor frequency (Lf) of approximately (radians converted to) 42.5761 MHz, this measurement can be symbolised as a musical diatonic pitch high E (i.e. E + 1/3 of a tone). Congruently, whether carbon has Lf of 10.7058 MHz, then it also can be symbolised as a pitch high E + 1/3, although *two octaves* below the frequency of hydrogen. Thus, the H-C interval (i.e. generator-first subharmonic) should define a proportional arrangement with the following elements participating in biochemistry, as suggested in Figure 2.

The study of proportionality in organic chemistry is a consolidated field at least since Jacob Berzelius' (1779-1848) times. But what is relevant for the present study is rather the affinity between models of harmony: one in musical practice, another one in cardiorespiratory quasi-periodicity, and the mentioned one in biochemistry; all of them hypothetically leaded by the H-C interval, as suggested in Figure 1–2 and Table 2.

According to these diagrams, which compare musical and biochemical harmony, the interval between *generator* H-C, and its *major seventh*, iron, has an interesting congruence with the *diazeuctic feature* of the diatonic/chromatic proportion (12/8 : 3/2), and the *fifths cycle*. In simple words, the relationship (H-C : Fe) locks the cycle of *biochemical harmony* in a an arrangement similar to musical ratios 12/8 : 3/2 : 2/1. These proportions are equally related to the optimization of the topological features of both, music and biochemistry. Optimal two-dimensional geometry can be measured in graphene carbon hexagonal lattice, analogous to the Euler-Riemann lattice of tonal classic harmony, the *Tonnetz* (see Figure 3). Unlike most of chemical elements tending to perform linear bonds with other elements, carbon may perform complex periodic compositions with special properties, including graphene periodic tiling and variations upon this regular tiling in two dimensions and emerging complexity in three dimensions, as in *fullerene manifolds* topology. Analogous self-organization economy in music includes circle mapping in the torus of phases (a product of circles is a torus), as employed by [31, p. 105], and later by [1] and [62], among other recent research exploring the Fourier space in a musical context.

Bell Telephone Laboratories reported, in 1938, an "objectionable" and "burning" noise that "results in resistance, volume efficiency and carbon noise characteristics which [...] are essentially independent of [their] angular position." Along the 20th century's second third, this noise became to be known as $1/f$ "fractional noise" expressed on a log-log scale, usually measured in electric circuits. In recent years, $1/f$ circuit noise has been reduced by the employment of silicon dielectric materials, and is investigated in low-temperatures carbon dispensed semiconductors [36], although this line of research and its applications still in experimental stage

The two typical crystal structures of carbon in two dimensions: graphite simple hexagonal, and face-centred diamond-cubic, are analogous to tonal music self-structuring: the graphite simple-hexagonal in relation to the Tonnetz, and the cubic one as described by [19]. Three dimension analogies of carbon are also meaningful in music. This conception may accept *forced coupling* as physical emulation of harmonic fields, as studied in dynamical systems applied to music, as well as harmonic segmentation (*graphene zigzags*) and self-containment as occurs in fullerenes [see [15, pp. 48-50]], providing a more complete analogy with music.

Table 2: *Mathematical and musical values of biochemical harmony. Columns from left to right: musical ratios ordered by numerator size, starting at 1; next column lists corresponding classical denominations of musical intervals, followed by their analogies to elements participating in biochemistry, including their atomic Larmor frequencies (*converted from radians to megahertz), and an approximation to diatonic intervals, in the rightmost column.*

ratio	**harmonic interval**	**associated chemical element**	**Larmor frequency*** $(x10^6 Hz)$	**Approach to diatonic orbit**
1/1	generator	Hydrogen	42.5761706239	$E^{\uparrow 1/3}$
2/1	diapason'	Carbon	10.7058112488	$E^{\uparrow 1/3}$
3/2	diapente (fifth)	Potassium	1.98680993149	$B^{\uparrow 1/3}$
5/3	major sixth	Phosporus	17.235193094	C♯
5/4	major third	Sulfur	3.2681491128	G♯
6/5	minor third	Nitrogen	3.07672499835	G
8/5	minor sixth	Selenium	8.1199364118	C
9/7	septimal major third	Lithium	6.265495351128	$G^{\uparrow 1/4}$
10/9	minor whole tone	Calcium	2.86540299085	F♯
11/10	neutral second	Oxygen	5.7717722319	F♯
12/7	septimal major sixth	Silicon	8.4587326212	$C^{\uparrow 1/4}$
15/8	major seventh	Iron	40.06158770288	$D\sharp^{\uparrow 1/4}$
36/35	quarter perfect tone	Zinc	2.663870321	$E^{\uparrow 1/4}$
64/63	Archytas' comma	Magnesium	2.606304759	$E^{\uparrow 1/8}$
↓	...	...	...	...

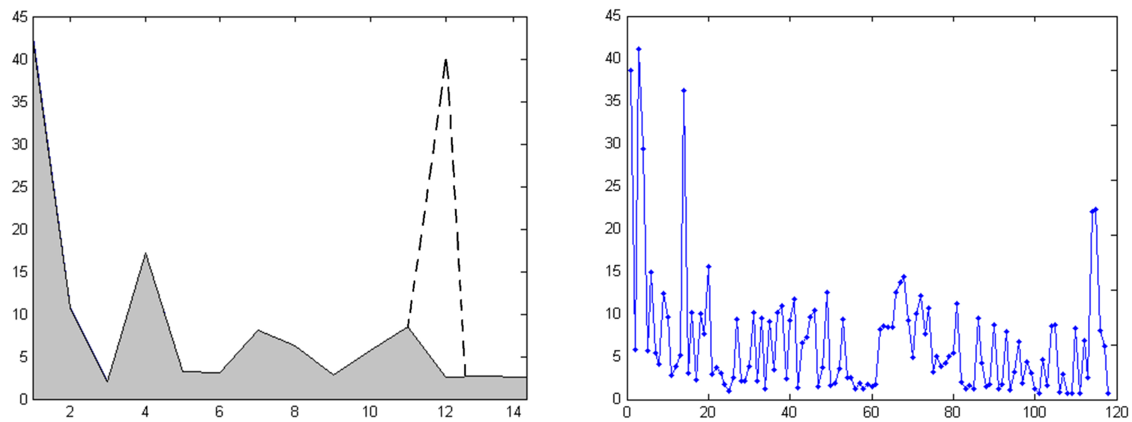

Figure 2: *Left: Harmonic profile of the main elements participating in biochemistry (those included in Table 2, above), ordered as musical harmonic intervals. The highest peak represents the hydrogen-carbon "generator" interval in Lf scale 0-45 ($\times 10^6$ Hz). Iron, with position 12, is represented both by its major seventh (broken lines) and minor second values respect to hydrogen. The complete length of the graph in its horizontal extension represents the interval of the biochemical octave. Right: A more complete picture of the elements following the periodic table's order (from left to right), where peaks represent Larmor frequencies (Lf) of atomic isotopes ranking from Hydrogen (^1H) to Uranium (^{235}U). The highest peak in leftmost area in the graph represents Lf of ^3H, Tritium, extremely rare on Earth. The next higher peak, at position 14, is iron (symbol Fe in Figure 1).*

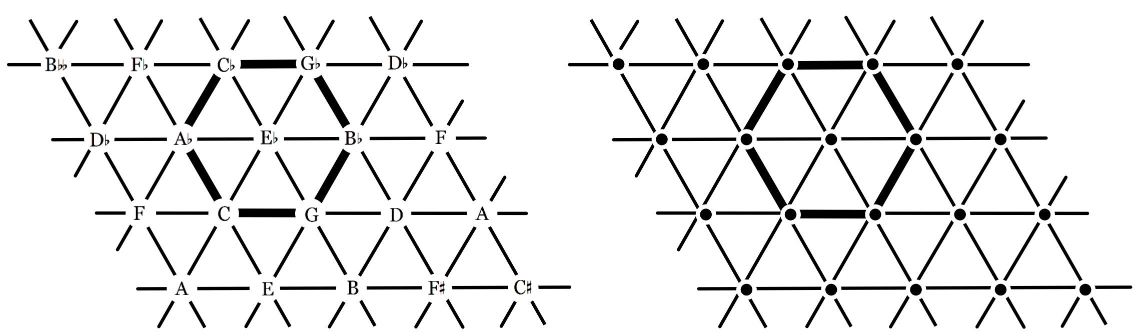

Figure 3: *Left: a hexagon lattice of tonal harmony can be rolled to form a torus, nesting and connecting the cycles of tonal functions [31, p. 105]. The torus is useful to map the space of musical intervals of optimal cardinality 12, in all of their possible harmonic relations and cyclical concatenation (diatonicity). Right: a hexagon lattice of graphite can be rolled to form a carbon tube with electrical conduction. Other shapes of carbon in two and three dimensions (including cube and truncated icosahedron) are also analogous to musical harmony.*

IV. Discussion

The harmonic distribution of intervals in the Arnold tongues follows a Farey tree consisting of a self-structured sequence of ordered proportions. A Farey sequence of order n is "the set of irreducible fractions between 0 and 1 with denominators less than n, arranged in increasing order" [33, p. 22]. We may theorize that human utterances and music following Zipf's law may be described as a set of *environmental resonances*; more precisely thermodynamic systems dissipating energy through a fluid medium (the air itself), [5] following a Farey tree self-hierarchisation. [6]

Harmonisation is a generalised feature of music in any cultural practice, not only in the most noticeable way as multiparametric proportionality of intervals (*resonance*), but as *phase synchronization* in acoustics and psychoacoustics. Acoustic synchronization, as explained in [43, p. xviii], is due to reciprocal influence and adaptation of mechanical systems to their interactions:

> Our surroundings are full of oscillating objects: violins in an orchestra, chemical systems exhibiting oscillatory variation of the concentration of reagents, a neural center that controls the contraction of the human heart and the heart itself [...] All these and many others systems have a common feature: they produce rhythms. Usually these objects are not isolated from their environment, but interact with other objects, in other words they are open systems. [...] This interaction can be very weak, sometimes hardly perceptible, but nevertheless it often causes qualitative transition: an object adjusts its rhythm in conformity with the rhythms of other objects.

Psychoacoustic synchronization is a complex, multi-layered and *transductive* phenomenon (i.e. that involves and correlates chemical, electrical and mechanical synchronization). Distinct systems can be analogous among them, replicating resonance patterns from their smaller layers to larger ones. Actually, societies can massively synchronize-mostly unaware-within a variety of physio-psychological phenomena [43, pp. xvii, 129]. Long term influence of this synchronization is hard to track in culture, nevertheless music seems to map resonance and synchronization over historical periods and trends. Accordingly, vast samples of music and speech from different contexts share features of spontaneous measurement and recurrence due to common physiological grounds and environmental conditions. Cardiorespiration, metabolism, nervous and cellular cycles, within their own feedback loops, express and transform the self-sustained and self-oscillatory characteristics of biochemical bonds.

The "preceptive rules" of musical traditions — somehow equivalent to syntax in speech — would be rather a cultural formalisation of "spontaneous" practices resulting from the evolution of systems within a specific context in resonance with common biochemical and physiological bases. This is a key concept for developing a hypothesis on the role of carbon in living organisms evolution, with its electrochemical features as precondition for the individual-context relationship where music and language arise.

Within this framework, caution is in order to avoid oversimplification, so it is crucially important to notice emerging stages and degrees of complexity, from the alluded carbon atomic correlations, to carbon-related functions among coordinated individuals and societies. Dynamical systems modelling of music does not look uniquely for specific objects, but particularly for systems

[5] For a physical modelling of this phenomenon in fluid mechanics, see [30, p. 44].

[6] A remarkable antecedent of this concept is Charles S. Peirce's (1839–1914) hierarchical structure of rational intervals, homologous to the Stern-Brocot tree (a comprehensive Farey binary sequence), in order to formulate a generalised sequentiation for the structure of human thought [41, pp. 277-280]. More recently composer and theorist Ervin Wilson (1928 -) adopted this model to construct a harmonic system on the grounds of the natural harmony first put forward by Novaro. Wilson (1994) emphasizes this musical model by its comparison with "growing systems [. . .] from crystals to living organisms" [61].

of relations. In this sense, dynamical systems "predicts that the perceived dynamics of tonal organization arise from the physics of non-linear resonance. Thus, non-linear resonance may provide the neural substrate for a substantive musical universal, [...] offering a direct link to neurophysiology" [29, p. 209]. We may add, *a direct link from music to physiology and biochemistry*.

Although direct analogy between chemical and acoustic intervals cannot be exact, because the former may be of *atomic*, and the latter of *molecular nature*, a generalised analogy is preserved in the context of biochemistry; namely, the leading and self-organizing resonance of the hydrogen-carbon interval, towards emergent multi-scalar complexity. Even when a huge amount of identical hydrogens may interact with other atomic forces spinning around their z-axis in different frequencies, *loss of phase coherence* and fall out of synchrony do not happen, since in many synchronizing interactions "phase differences don't have time to accumulate, so our signal [may] stay nice and strong despite the changing frequencies" ([17, pp. 115-116], within a biomolecular context).

Of course, many questions remain unanswered from this first-approach theorizing on a *biochemical harmony*. How a first, primary hydrogen did establish its current electromagnetic behaviour, is a question that also seems to implicate an intricate relationship in the universe's emergence of carbon. A question that goes much further from the initial purpose of the present study. Even "clear" assumptions from modelling a *biochemical harmony* entails very intriguing "findings". Two examples of this, in the context of Table 2, are:

1. Weather the iron ratio (15/8) is *responsible of magnetising the tonal octave*, as it produces a harmonic loop successively leading to a self-similar return to the H-C generator interval (1/1:2/1), then producing the *spiral of tonal harmony*, so invoked by scholars since Aristoxenus commentators up to Athanasius Kircher, and materialised in its modern physical modelling by Augusto Novaro [38].

2. Weather arsenic appears as an exact diminished fifth "in conflict" with the harmoni c hydrogen-carbon interval. In music theory this could be easily identified as the tritone "classical conflict" or 64/45 interval, disputing tonal orientation with perfect 3/2 consonant interval (see both intervals vicinity in Figure 1).

Furthermore, the *Arnold tongues interpretation* of musical harmony is useful not only as a set of classical proportionality, but particularly in terms of *music as an open system*. Then *probability*, a concept repeated in the initial pages of this contribution, does mean a guideline conducted by harmonic attractors (i.e. higher hierarchies in the Arnold tongues), which also generates intervallic possibilities in a discontinuous-dense set (the tongues' lower limit). Under this approach, a consistent theory of music is pending, from standard probability to fuzzy logic and uncertainty analogous to *quantum circles* as explained in [18], related to factual musical performance.

V. Conclusions

Whether the electronic behaviour of carbon sets preconditions for dynamical systems in terms of cycles of bioacoustic recurrence, such behaviour may have an effect in emergent patterns through human biology and socialisation, without disregarding cultural "development". This would explain, at least in part, how phase synchronization is expressed in music, speech and culture, as carbon-based correlated phenomena.

The *carbon hypothesis of music* (CHM) allows us to propose a dynamical and organic definition of music, as the set of psychoacoustic analogies of the human body, both individually and collectively, where the physical context dialogues with components of our evolutionary and actual existence. Besides, CHM also may be helpful to understand why music is a so common practice — if not obsessive — in human societies, independently of epochs and cultural contexts.

Music inherits and reflects psycho-physiological synchronization, where cardiorespiration acts as a leading force, with implications for the rhythms of brain/mind emerging complexity. In this sense, cyclical electric patterns of biochemical networks in quasi-periodic couplings, are statistically self-similar in respect to the quasi-periodic cycles of music. Such self-similarity is structurally related to the functional participation of carbon structures in psychoacoustic systems from the middle ear to the most complex brain electrical processing; but also to a smaller scale of *carbon noise*. This is how [55, p. 58] find $1/f^{\sim 1}$ noise ubiquity as a statistic footprint of carbonic self-similarity in speech (under Zipf's law) and music (e.g. in Riemannian aggregates), and later detect $1/f^{\beta}$ noise in DNA sequences, following "long-range fractal correlations as well as prominent short-range periodicities" [56, p. 7].

CHM allows us to answer the question of the *evolutionary motivation for vocalisation* in humans, transforming cycles of exhalation in potential socio-acoustic codification. Instrumental and progressively abstract music would originated from this, following the *law of least effort* and Weber-Fechner constraints. Then vocalization and music are literally an *expression* of the aerobic-carbonic biological dynamics.

Nevertheless, not only speech and musical vocalisation are expressive or communicational phenomena shaped by the *law of least effort*: since the carbon aerobic relationship strongly contributes to shape the brain/mind rhythms of communication and emerging analogies, such relationship also should involve symbolic and acoustic, verbal, non-verbal, and spatial epistemics (for instance, dance, walking, gesturing, and wider range proxemics). Human individual and collective bodies hear and respirate, and thus we are *physically synchronized* (as understood in [43]) with our communities and with the rhythms of our own culture and environment.

Finally we conclude that a new conception of music is needed in order to reflect the symmetrical pace from the conventional idea of music as a result of "high developed societies and culture", to the basic idea of music as the plural manifestation of a biochemical harmony in many other organisms than humans, including plants, fungi and bacteria. This post-anthropocentric definition of music should also stimulate other conceptual decentralisation processes in current musicology.

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All-(Generalized-)Interval(-System) Chords

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Abstract: *We survey the all-interval chords of small order and the interval systems in which they are situated. We begin with an examination of traditional all-interval chords in chromatic pitch-class spaces, and extend the notion of their structure to their counterparts in David Lewin's Generalized Interval Systems. Mathematically, we observe that these chords belong to three categories of difference sets from the field of combinatorics:* $(v, k, 1)$ *planar difference sets,* $(v, k, 2)$ *non-planar difference sets, and* $(v, k, 1, t)$ *almost difference sets. Further, we explore sets of all-interval chords in group-theoretical terms, where such sets are obtained as orbits under the action of the normalizer of the interval group. This inquiry leads to a catalog of the 11,438 all-interval chords of order k, where* $2 \leqslant k \leqslant 8$*. We conclude with remarks about future work and open questions.*

Keywords: *All-interval Chords. Generalized Interval Systems. Group Theory. Difference Sets.*

I. Introduction

The compactness and efficiency of all-interval chords have attracted the attention of composers and music theorists since the early years of the twentieth century. Such structures, which include one and only one of each interval in a given interval system, are rich compositional resources, as well as topics of theoretical interest to students of interval systems themselves. In particular, all-interval tetrachords in 12-tone chromatic space, represented in pitch-class set theory by the prime forms $[0, 1, 4, 6]_{12}$ and $[0, 1, 3, 7]_{12}$, [1] have received widespread application in the music of the major post-tonal composers. Among numerous notable examples, the first song in Arnold Schoenberg's *Das Buch der hängenden Gärten*, Op. 15, "Unterm Schutz von dichten Blättergründen," ends with a chord, $\{3, 5, 8, 9\}_{12}$, that is a member of set class $[0, 1, 4, 6]_{12}$ (see Figure 1). Likewise, the final song in Alban Berg's Vier Gesänge, Op. 2, "Warm die Lüfte," contains a passage (mm. 20-22) that consists exclusively of all-interval tetrachords, alternating members of set classes $[0, 1, 3, 7]_{12}$ and $[0, 1, 4, 6]_{12}$ (Figure 2). Indeed, entire compositions are constructed around all-interval tetrachords. For instance, Elliott Carter's First and Second String Quartets both incorporate these collections locally and structurally [1].

All-interval tetrachords also feature prominently in post-tonal theoretical writings. Each of the standard pitch-class-set-theoretical texts, beginning with Howard Hanson's 1960 *Harmonic Materials of Modern Music* [2], incorporates description and examples of these structures. [2] Relevant

[1] As we consider sets of integers in a variety of moduli, we indicate the modulus as a subscript following a set. For example, $\{0, 1, 4, 6\}_{12}$ is the set of integers 0, 1, 4, and 6 (modulo 12).

[2] Hanson represents members of both set classes $[0, 1, 4, 6]_{12}$ and $[0, 1, 3, 7]_{12}$ not with pitch-class integers, but with his notation for their "interval analyses," *pmnsdt*, the letters of which indicate singular projections of the following intervals: perfect fifth (p), minor second (m), major second (n), minor third (s), major third (d), and tritone (t) [2, p. 22].

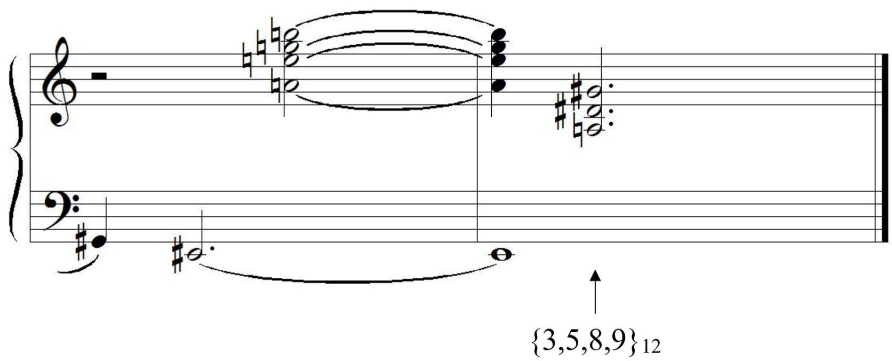

Figure 1: *Schoenberg, "Unterm Schutz von dichten Blättergründen," final chord.*

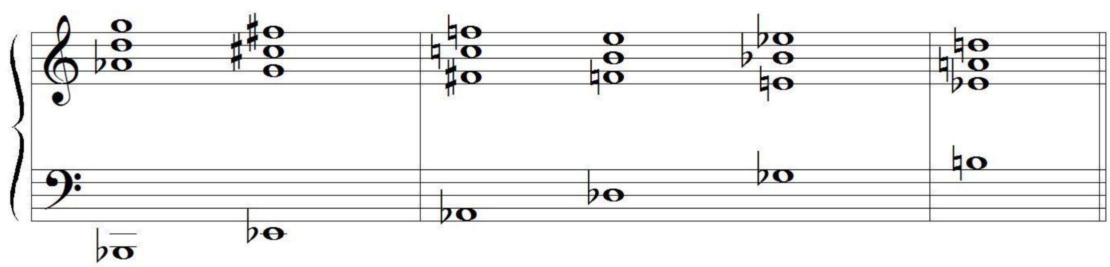

Figure 2: *Berg, "Warm die Lüfte," mm. 20-22 (voice-leading reduction).*

discussions appear in Allen Forte's *The Structure of Atonal Music* [3], John Rahn's *Basic Atonal Theory* [4], Robert Morris's *Composition with Pitch Classes* [5], George Perle's *Twelve-Tone Tonality* [6], Stefan Kostka's *Materials and Techniques of Post Tonal Music* [7], and Joseph Straus's *Introduction to Post-Tonal Theory* [8], among other sources. All-interval tetrachords have also been studied in terms of their transformational properties [9], and for their role in our understanding of the Z-relation [10].[3]

Treatments of all-interval chords in other chromatic spaces are comparatively rare in the literature. One significant work that addresses all-interval chords of varying sizes in microtonal systems is Carlton Gamer and Robin Wilson's "Microtones and projective planes" [11]. Gamer and Wilson present all-interval trichords, tetrachords, and hexachords in 7-, 13-, and 31-tone chromatic spaces, respectively, as difference sets, a concept from mathematical combinatorial theory. For their purposes, they define "a difference set (modulo n) to be a set of distinct integers $c_1, ..., c_k$ (modulo n) for which the differences $c_i - c_j$ (for $i \neq j$) include each non-zero integer (modulo n) exactly once" (p. 153). Mathematicians call such difference sets — wherein each difference appears exactly once — planar difference sets [12]. Such sets are a special type of all-interval chord: whereas every planar difference set is an all-interval chord, not every all-interval chord is a planar difference set.

This article moves beyond an examination of all-interval chords in chromatic systems to one of corresponding structures in David Lewin's *Generalized Interval Systems (GISs)* [13]. Specifically, it enumerates the isomorphism classes of all-interval k-chords of small order (i.e., those with $2 \leq k \leq 8$). In addition to the all-interval chords found in cyclic interval groups, such as those

[3]Z-related pitch-class sets are those that possess the same interval vector, but which are not related by transposition and/or inversion.

above, we note their occurrence in certain non-cyclic abelian and non-abelian interval groups. Among these chords, we find planar and non-planar difference sets, as well as almost difference sets [14], a related concept that comes from combinatorics.

II. Intervals and Interval Vectors in Chromatic Spaces

In this section, we discuss intervals and interval vectors in chromatic spaces, and extend relevant aspects to more general spaces in the following section. To define our concept of interval, it is necessary first to establish the context in which we find intervals. We call such a context a space: a universal set of musical objects, allowing that a path exists between any two members of the space. The examples above situate their intervals in v-tone (modular) chromatic spaces,[4] wherein the musical objects are pitch classes. Here, the notion of an interval between two pitch classes is construed as a distance, the number of chromatic steps as an integer modulo n. Intervals in such spaces may be directed or non-directed. Typically, melodic intervals are indicated as being directed: the distance from pitch class x to pitch class y. This type of interval is reckoned $y - x$ (modulo v). Harmonic intervals, on the other hand, are non-directed: the static distance between pitch classes x and y. The non-directed interval between pitch classes x and y is customarily represented by the lesser of $y - x$ (modulo n) or its inverse, $-(y - x) = x - y$ (modulo n). Throughout the remainder of this study, we refer to non-directed intervals simply as "intervals," whereas we always retain the qualifier "directed" when referring to directed intervals.

We are interested in the total interval content and total directed-interval content of a subset of a space. In the music-theoretical literature, one finds a distinction between tallies of a subset's interval content and those of its directed intervals. For subsets in 12-tone chromatic space, a tool that counts the number of occurrences of each directed interval is Lewin's 1960 interval vector [15], which we call a directed-interval vector or DIV. It consists of a 12-member array, in which the first coordinate lists the number of occurrences of directed intervals of length 0 (unisons); the second coordinate, directed intervals of length 1; the third, length 2; and so on through length 11. For a subset D of size k, the sum of the vector's coordinates is $k2$. For example, the DIV for the final chord $\{3,5,8,9\}_{12}$ in "Unterm Schutz von dichten Blättergründen" (from Figure 1) is (411111211111): we find four unisons (e.g., between each pitch class and itself), two instances of directed intervals of length 6 (from pitch class 3 to 9, and from 9 to 3), and one of each of the remaining ten lengths, for a total of 42 = 16 directed intervals. DIVs can be adapted easily to other v-tone chromatic spaces. Essentially, one uses a v-member array, wherein the first coordinate lists the number of occurrences of directed intervals of length 0, the second coordinate directed intervals of length 1, the third length 2, and so on through length $v - 1$. Again, for a subset of size k, the sum of the vector's coordinates is $k2$. For instance, the $3^2 = 9$ directed intervals among the members of $\{0,1,3\}_7$ yield the DIV (3111111).

Tallies of non-directed interval content in subsets of 12-tone chromatic space are typically represented using Allen Forte's 1973 interval vector [3], also known as an interval-class vector or ICV. The ICV is a 6-member array in which the respective coordinates list the number of occurrences of each interval in a pitch-class set.[5] For a subset of size k, the sum of the coordinates in an ICV is the binomial coefficient $\binom{k}{2} = (k(k-1))/2$. For example, the ICV for the $\{3,5,8,9\}_{12}$

[4]Henceforth in this study, we use the variable v for the size of a space rather than the more customary n. This substitution is for consistency with our later incorporation of the standard notation (v, k, λ) for difference sets, in which v is the size of the group.

[5] It is significant to note that, whereas a DIV contains a coordinate (the first) that counts the number of unison intervals in a pitch-class set, an ICV does not count unison intervals.

tetrachord from Figure 1 is [111111].[6] For v-tone chromatic spaces in general, the ICV is a $[\frac{v}{2}]$-member array that shows the number of occurrences of each interval class in a pitch-class set in order of ascending size from 1 to $[\frac{v}{2}]$.[7] As above, the sum of the vector's coordinates for a subset of size k is the binomial coefficient $t\binom{k}{2}$. For example, in a chromatic space of size 7, we find $[\frac{7}{2}] = 3$ interval classes. Accordingly, the ICV for the pitch class set $\{0,1,3\}_7$ is [111]. An important difference exists between Lewin's and Forte's vectors: whereas Lewin's vector counts the directed interval from pitch class x to pitch class y separately from the directed interval from y to x, Forte's vector counts the non-directed interval between pitch classes x and y only once. This distinction between Lewin's and Forte's vectors leads to significant results in later sections.

III. Intervals and Interval Vectors in Generalized Interval Systems

In this study, we conceptualize intervals in the manner of David Lewin's *Generalized Musical Intervals and Transformations (GMIT)* [13]; more precisely, Lewin's intervals agree with our notion of directed intervals. In this sense, a directed interval is a member of a mathematical group that has a simply transitive action on a space. Simple transitivity requires that (a) the action is transitive, i.e., each member of the space is related to every other member (and to itself) by some directed interval in the group; and (b) the action is free, a consequence of which is that one and only one directed interval relates a member of the space to any other member (or to itself). For instance, in each of the examples in the previous section, the interval group is the group of integers modulo v, $\mathbb{Z}_v$, which has a simply transitive action on the space S of v pitch classes. An interval exists between any two pitch classes x and y in S, and we find one and only one directed interval from x to $y : yx$ (modulo v). Whereas we can generalize this situation to abstract cyclic groups and other types of group structures, Lewin's generalized intervals differ in significant ways from traditionally defined directed intervals. In particular, they do not possess qualities of distance and direction [16]. Instead, we are interested their functioning as "characteristic motions" among the members of a space [13, p. xxix].

GMIT does not address the notion of non-directed intervals. It is possible, however, to generalize these intervals in a manner that is consistent with Lewin's work. In particular, a non-directed interval (or interval) is an equivalence class (interval class) that contains a directed interval and its inverse. If a directed interval is equivalent to its inverse, such as is the case with an involution, then the interval class that includes it is a singleton. For instance, the interval class that includes $x \in \mathbb{Z}_{12}, x \neq 6$, also includes x (modulo 12), whereas the interval class that contains the involution $6 \in \mathbb{Z}_{12}$ consists of that element alone, as $6 \equiv -6$ (modulo 12).

As many of our subsequent examples involve non-abelian groups, our notation for intervals and directed intervals follows that of group elements in multiplicative groups (rather than the additive notation used with abelian groups). In compositions of such group elements, we incorporate right orthography (i.e., the product gh, where $g, h \in G$, means "do g first, then do h"). The composition gg is notated g^2, the inverse of an element g is indicated as g^{-1}, etc. Because a generalized directed interval g does not possess the quality of distance, we cannot merely label its interval class with

[6] We use different bracket styles to distinguish between directed-interval vectors and interval-class vectors. For the former, we use parentheses, whereas we use square brackets for the latter.

[7] We use the floor function, $[\frac{v}{2}]$, in tabulating the number of interval classes in a cyclic group, on account of the variance in the number involutions in cyclic groups of even and odd orders. A standard result in group theory shows that even-order cyclic groups always have one involution, (e.g., 6 in $\mathbb{Z}_{12}$), whereas odd-order cyclic groups have none. As we will see below, the number of involutions helps determine the number of interval classes in a group.

the interval class's shorter constituent, g or g^{-1} (as is the custom in chromatic spaces). Hence, we designate the interval class that contains g and its inverse as $g^{\pm 1}$ (Accordingly, if $h = g^{-1}$, then $h^{\pm 1} = g^{\pm 1}$.)

In this study, we are concerned at times with subsets of spaces, and at other times with subsets of interval groups themselves. In particular, in GIS theory, it is sometimes more convenient to refer to elements of an interval group rather than those of a space. As a result of its simply transitive action, an isomorphism exists between an interval group and the space on which it acts. Technically, the space S on which a group G acts simply transitively is called a *G-torsor* [17] S is isomorphic to G, except that no point in S corresponds a priori to the identity element of G. However, once such an association is chosen — as in assigning the pitch class C to the identity element $0 \in \mathbb{Z}_{12}$ — the bijection of the remaining members in S to group elements in G is determined by right multiplication (for non-abelian groups; by addition for abelian groups).[8] We may therefore identify subsets of the space (i.e., chords) with subsets of the group (and vice versa). For instance, in the chromatic-space examples above, both the interval group of order v and the space of v pitch classes can be modeled with the integers modulo $v, \mathbb{Z}_v$. Having assigned the pitch class C to 0 as an origin in 12-tone chromatic space, we can interpret the members of a pitch-class set, such as $[0, 1, 4, 6]_{12}$, equally as pitch classes or as directed intervals from the origin.

Directed-interval vectors can be adapted to Lewin's GISs by replacing tallies of directed intervals of varying lengths with those of individual group elements, as long as it is made clear which coordinate in the vector represents the number of occurrences of which group element. As above, for an interval group of order v, the DIV has v coordinates, which sum to v^2. As with Lewin's interval vector, Forte's interval-class vector can also be adapted for use with interval groups in GISs. Again, it is necessary to establish which coordinate of the vector counts the occurrences of which interval class. For an interval group G, the number of coordinates in an ICV can be determined by the following formula, where w is the number of involutions in G, and $|G^\sharp|$ is the size of the set of non-identity elements in G.[9]

$$\frac{w + |G^\sharp|}{2}$$

As above, the sum of the ICV's coordinates for a set of size k equals the binomial coefficient $\binom{k}{2}$.

IV. All-Interval Chords

An all-interval chord is a subset of a space that possesses among its members at least one of every interval in the interval group that acts on that space [3]. Put another way, an all-interval chord is one which contains no 0s in its ICV (or, in the case of all-directed-interval chords, in its DIV). As we observe in §1, however, composers and music theorists have traditionally been interested in a special category of all-interval chords: those that contain *one and only one* of each non-unison interval, as such chords have the highest degree of intervallic efficiency. In these

[8]One may also use left multiplication, yielding (for non-abelian groups) a *G-torsor* that is anti-isomorphic to the one determined by right multiplication. We incorporate right multiplication here for consistency with our use of right orthography.

[9]Using results from character theory [18], we determine the number w of involutions in a finite group G via the following:

$$w = \Sigma_\chi w(\chi)\chi(1)$$

where χ runs through the complex characters of G, and $w(\chi)$ is the Frobenius-Shur indicator of χ. Consequently, w is always either 0 or odd. That fact, together with the classical result that a group of odd order contains no involutions, means that $w + |G^\sharp|$ is always divisible by 2.

chords, the full set of available intervals is present in as small a subset as possible. Henceforth, when we refer to all-interval chords and all-directed-interval chords, we indicate chords with this particular property. A significant relationship exists between all-directed-interval chords and all-interval chords: any all-directed-interval chord is also all-interval, but the reverse is not necessarily true. For instance, a trichord with the prime form $[0, 1, 3, 7]_7$ possesses the DIV (3111111); it is accordingly an all-directed-interval chord. Its ICV, [111], indicates that this trichord is also all-interval. In contrast, a trichord with the prime form $[0, 1, 3, 7]_6$ is not an all-directed-interval chord in the specific sense described above. Its DIV is (311211), which contains a 2, not a 1, in its fourth coordinate; nevertheless, it is all-interval, as is evident from its ICV: [111].

Given the isomorphism from a *G-torsor* to the interval group G, we can define all-interval chords and all-directed-interval chords not only as subsets of a space, but as subsets of an interval group. In this connection, we examine various concepts from the theory of difference sets in the field of combinatorics. Most generally a difference set is a subset $D = (v, k,)$ of a group G, where v is the order of G, k is the size of D, and every non-identity element of G appears exactly λ times as compositions of elements gh^{-1}, where $g, h \in D$. Such subsets possess the quality of having a flat directed-interval distribution (i.e., all coordinates of non-unison directed intervals in their DIVs are equal to λ). For instance, let $G = \{P, I, R, RI\}$ be the group of basic twelve-tone row operations (prime [identity], inversion, retrograde, and retrograde-inversion), isomorphic to the Klein four-group, $\mathbb{Z}_2^2$. The trichordal subset $D = P, I, R$ of G is a $(4, 3, 2)$ difference set. G is of order 4; D is a three-element subset of G; and each non-identity element of G appears exactly twice as a product gh^{-1} of elements g and h in $D : PI^{-1} = IP^{-1} = I, PR^{-1} = RP^{-1} = R$, and $IR^{-1} = RI^{-1} = RI$, as is evident in the trichord's DIV (3222).

All-directed-interval chords are a particular category of difference set. A planar difference set is one in which $\lambda = 1$ (i.e., each non-unison directed interval appears exactly once). The $k = 4$ subset $\{0, 1, 3, 9\}_{13}$ of $\mathbb{Z}_{13}$ serves as an example; its DIV, (4111111111111), demonstrates the unary directed-interval distribution that distinguishes it as a planar difference set. A conjecture in the field of combinatorics [12, p. 421] states that if $\lambda = 1$, then $k - 1$ must be the power of a prime. That is, there are no planar difference sets of sizes 6, 10, 12. . . .

The familiar all-interval tetrachords of pitch-class set theory, $\{0, 1, 4, 6\}_{12}$ and $\{0, 1, 3, 7\}_{12}$, are not planar difference sets. In fact, they are not difference sets. As indicated by their shared DIV, (411111211111), these tetrachords do not possess flat directed-interval distributions. Rather, they are examples of almost difference sets. An almost difference set is a subset $D = (v, k, \lambda, t)$ of G, where v and k are defined as above; t non-identity elements of G appear exactly λ times as compositions of elements gh^{-1}, where $g, h \in D$; and the remaining $v - 1 - t$ non-identity elements of G appear $\lambda + 1$ times as gh^{-1} compositions. Hence, $\{0, 1, 4, 6\}_{12}$ and $\{0, 1, 3, 7\}_{12}$ are examples of $(12, 4, 1, 10)$ almost difference sets. Their DIVs possess ten coordinates that equal 1, and $12 - 1 - 10 = 1$ coordinate that equals $1 + 1 = 2$: the difference 6 (modulo 12).

Whereas planar difference sets are always all-directed-interval, and therefore also all-interval, non-planar difference sets and almost difference sets are never all-directed-interval. Furthermore, they are all-interval only if the following two circumstances are met. First, they must have $1 \leq \lambda \leq 2$; and, second, any element in G (i.e., directed interval) with $\lambda = 2$ must be an involution. For example, the trichord $\{P, I, R\}$ above is an example of a $(4, 3, 2)$ non-planar difference set. Every gh^{-1} composition has $\lambda = 2$ and is also an involution. Similarly, $\{0, 1, 4, 6\}_{12}$ and $\{0, 1, 3, 7\}_{12}$ are $(12, 4, 1, 10)$ almost difference sets, in which the gh^{-1} compositions in either set with $\lambda = 2$ are involutions, i.e., 6 (modulo 12). As we see below, these three categories of difference sets, planar and non-planar difference sets and almost difference sets, account for every all-interval chord (up to isomorphism) of small order (i.e., $2 \leq k \leq 8$).

V. Interval Groups and All-Interval Chords

A significant relationship exists between the number of intervals in a group G and the potential for its having all-interval subsets. A subset D of size k in G has a triangular number, $\binom{k}{2}$, of unordered dyads that can be labeled with intervals. Therefore, if D is to include one and only one occurrence of every interval in G, then G must have exactly $\binom{k}{2}$ intervals. For instance, a tetrachord contains $\binom{4}{2} = 6$ unordered dyads that can be labeled with intervals. For it to be an all-interval tetrachord, the group that contains these intervals must also have six interval classes, as does $\mathbb{Z}_{12}$. Two factors determine how many intervals are in a group: the order of the group itself (minus the identity) and the number of involutions that it contains. As we observe above in §3, the number of intervals is equal to the sum of number of involutions in the group plus half the number of elements of order > 2. If this number equals $\binom{k}{2}$ for some k, the group may potentially contain all-interval k-chords. As we demonstrate below, however, this condition is necessary — but not sufficient — for the existence of all-interval chords.

It is possible for a group to have $\binom{k}{2}$ intervals, and not to contain any all-interval k-chords. For example, the dicyclic group of order 12, $Dic_{12} = \langle x, y | x^6 = y^4 = 1, x^3 = y^2, y^{-1}xy = x^{-1} \rangle$, contains $\binom{4}{2} = 6$ intervals, the same as $\mathbb{Z}_{12}$, but it has no all-interval tetrachords. Dic_{12} has a cyclic subgroup of 6, generated by an element x, which yields three intervals: $x^{\pm 1}, x^{\pm 2}$, and $x^{\pm 3}$ (an involution). The remaining six elements of Dic_{12} are all of order 4, yielding three additional interval classes. Moreover, for any of these elements y of order 4, y^2 is equal to the single involution within the cyclic subgroup, x^3. Nevertheless, the existence of an all-interval tetrachord fails. It requires one interval labeled as $x^{\pm 3}$, such as the interval between x^0 (the identity element) and x^3, and one occurrence of y. However, the interval between x^0 and y and the interval between y and x^3 are the same, i.e., $y(x^0)^{-1} = x^3 y^{-1} = y$, resulting in more than one occurrence of that interval.

In terms of a musical representation, Dic_{12} is isomorphic to a particular transposition/skew-inversion group.[10] This group has a cyclic subgroup that consists of the six transposition operators with even indices (i.e., T_m, m is even), and the remaining six elements of order 4 are skew-inversions with odd indices (i.e., S_n, n is odd). Table 1 lists the cycles of the eleven non-identity elements of this group as they act on the set of twelve chromatic pitch classes. As any all-interval tetrachord in this group requires a tritone, let us select arbitrarily the tritone {1, 7} from the odd whole-tone collection. Further, such a tetrachord would also require at least one pitch class from the even whole-tone collection; we choose 6. From the cycles in the table, however, we see that the interval between 1 and 6 and the interval between 6 and 7 are the same, $S_1^{\pm 1}$, which is not allowed in an all-interval chord. As this situation occurs for any combination of a tritone from one parity's whole-tone collection and a single pitch class from the opposite parity's whole-tone collection, the existence of an all-interval tetrachord fails.

Certain limits exist on the size of groups that may include all-interval k-chords. The smallest groups that can potentially accommodate an all-interval k-chord are elementary abelian 2-groups, $\mathbb{Z}_2^n$, wherein every non-identity element is an involution. These groups contain a Mersenne number, $2n - 1$, of non-unison intervals. For example, the group of basic twelve-tone row operations is isomorphic to $\mathbb{Z}_2^2$, and it contains an all-interval trichord, e.g., P, I, RI. However, instances in which a triangular number $\binom{k}{2}$ equals some Mersenne number $2n - 1$ are rare.

[10] Whereas an inversion is a reflection in pitch-class space, i.e., an operation of order 2, a skew-inversion is a pseudo-reflection of order 4 (i.e., a reflection of order > 2; see [19]). Under a skew-inversion, pitch classes of one parity, even or odd, map as normal inversions to their counterparts of the other parity, but when they reflect back to those in the original parity, they return a tritone away. Hence, four iterations of the cycle are required before returning to the original pitch classes. For instance, the operation on pitch classes $5x + y$ (modulo 12), is a skew-inversion for any odd y. Skew-inversions are similar to skew-Wechsels of neo-Riemannian theory (a category of contextual inversions), which are discussed in more detail in [20].

Table 1: *Non-trivial cycles of elements in the transposition/skew-inversion group $G \simeq Dic_{12}$.*

$T_2 := (0,2,4,6,9,10)(1,3,5,7,9,11)$	$T_{10} := (0,10,8,6,4,2)(1,11,9,7,5,3)$
$T_4 := (0,4,8)(1,5,9)(2,6,10)(3,7,11)$	$T_8 := (0,8,4)(1,9,5)(2,10,6)(3,11,7)$
$T_6 := (0,6)(1,7)(2,8)(3,9)(4,10)(5,11)$	
$S_1 := (0,1,6,7)(2,11,8,5)(3,4,9,10)$	$S_7 := (0,7,6,1)(2,5,8,11)(3,10,9,4)$
$S_3 := (0,3,6,9)(1,8,7,2)(4,11,10,5)$	$S_9 := (0,9,6,3)(1,2,7,8)(4,5,10,11)$
$S_5 := (0,5,6,11)(1,10,7,4)(2,3,8,9)$	$S_{11} := (0,11,6,5)(1,4,7,10)(2,9,8,3)$

Nevertheless, of the three known examples, two values of $2 \leq k \leq 8$ satisfy this condition, 3 and 6: $\binom{3}{2} = 2^2 - 1$, and $\binom{6}{2} = 2^4 - 1$.[11] In contrast, the largest groups that can potentially accommodate an all-interval k-chord contain no involutions. As groups of odd order contain no involutions, at least one isomorphism class — the cyclic group — exists for each odd order (and some odd orders contain additional isomorphism classes of groups). For instance, $\mathbb{Z}_7$ is the largest group to have all-interval trichords. It contains three intervals, none of which is an involution. However, it is useful to note again that the existence of a group with $\binom{k}{2}$ intervals — whether large, small, or in between — does not guarantee the existence of all-interval subsets.

VI. Sets of All-Interval Chords

Aside from $\mathbb{Z}_2$, which contains a single all-interval chord, $\{0,1\}$, interval groups that contain one all-interval chord also contain additional all-interval chords.[12] The set classes $[0,1,4,6]_{12}$ and $[0,1,3,7]_{12}$ in $\mathbb{Z}_{12}$ — orbits of these tetrachords under the action of the dihedral transposition and inversion group — are familiar examples of sets of all-interval chords. Each orbit is of size 24: twelve all-interval tetrachords that relate to one another by transposition, and twelve more that relate to those by inversion. As no other tetrachords in $\mathbb{Z}_{12}$ exist with the ICV (111111), these forty-eight forms constitute the full set of all-interval chords contained in this interval system.

The members of set-classes $[0,1,4,6]_{12}$ are related to those of $[0,1,3,7]_{12}$ by neither transposition nor inversion (and vice versa), and yet the members of both set-classes have the same intervallic content. Hence, we observe that these set classes are Z-related [3]. Their members also relate to one another's by the multiplicative operations M and MI: pitch-class multiplication by 5 and 7 (modulo 12), respectively [21]. This situation occurs in all cyclic interval groups: if D is an all-interval subset of $\mathbb{Z}_n$, then the affine transformations of D, $Dx + a = \{dx + a | d \in D, x \in \mathbb{Z}_n$ co-prime to n, $a \in \mathbb{Z}_n\}$, are also all-interval.[13] The reasoning is straightforward: affine transformations do not preserve distances, but they preserve ratios of distances. Accordingly, if D contains the set of all distances in the finite space $\mathbb{Z}_n$, a transformation that preserves the ratios of these distances results in a permutation on the set of distances.

The set of affine transformations on $\mathbb{Z}_n$ form a group, Aff($\mathbb{Z}_n$), the action of which on D yields a set of Z-related all-interval subsets. In general, Aff($\mathbb{Z}_n$) is equivalent to the normalizer of $\mathbb{Z}_n$ in the symmetric group on $\mathbb{Z}_n$: $N_{Sym(\mathbb{Z}_n)}\mathbb{Z}_n$.[14] This correspondence allows us to carry the

[11] The next smallest value of k to satisfy this condition is 91, as $\binom{91}{2} = 2^{12} - 1 = 4095$. No further examples of reasonable size exist (and it is possible that no further examples exist at all).

[12] We might also say that the trivial group $\mathbb{Z}_1$ contain one all-interval chord, {0}, consisting of a single pitch class. It has one and only one occurrence of the sole interval in that group, the unison. However, we do not include this example, as we are concerned in this study only with non-unison intervals.

[13] The affine group Aff($\mathbb{Z}_n$) is the set of all transformations $xy + z$ (modulo n), where $y, z \in \mathbb{Z}_n$, and $x \in \mathbb{Z}_n$ is co-prime to n.

[14] The normalizer of a group G in another group H is the subgroup of elements in H that preserve G under conjugation.

Table 2: *An all-interval pentachord D in the interval system $G \simeq S_3 \times \mathbb{Z}_3$.*

	Sonority 1	Sonority 2	Sonority 3	Sonority 4	Sonority 5
High	C_3^5	B_3^5	C_3^5	C_3^6	A_4^6
Middle	B_3^5	C_3^5	A_3^5	B_3^6	C_4^6
Low	A_3^5	A_3^5	B_3^5	A_3^6	B_4^6

notion of affine transformations to other types of interval groups. If G is a group that acts simply transitively on a space S, then the normalizer of G in the symmetric group on S, $N_{Sym(S)}G$, serves as an analog for the affine group. As the action of Aff($\mathbb{Z}_n$) on D yields a set of all-interval subsets in $\mathbb{Z}_n$, including Z-related subsets, the action of $N_{Sym(S)}G$ on D yields a set of all-interval subsets in any S, including GISZ-related subsets.[15] For all cases except one (as we discuss in §7), this action produces the full set of all-interval chords of small order for any interval system.

An example of a set of all-interval chords can be found in the following interval system. Define a 9-note sonority that is separated into three distinct registers: high, middle, and low. Put an A major triad in one register, a B major triad in a second register, and a C major triad in the third. In any one sonority, all three triads must appear in the same position: root position $[\begin{smallmatrix}5\\3\end{smallmatrix}]$, first inversion $[\begin{smallmatrix}6\\3\end{smallmatrix}]$, or second inversion $[\begin{smallmatrix}6\\4\end{smallmatrix}]$. The space S of all possible configurations of such sonorities is of size eighteen: six permutations of the triads in three registers, and three positions in which the triads may appear. The interval group G that has a natural action on the space of these eighteen sonorities is isomorphic to the direct product of the symmetric group of degree 3 by the cyclic group of order 3, $S3 \times \mathbb{Z}_3$ (of order 18). This interval system has $\binom{5}{2} = 10$ interval classes: seven interval classes of invertible elements and three involutions. Moreover, it allows for all-interval 5-member subsets. For instance, the subset D of five sonorities that appear in Table 2 contains one and only one of each interval in this group.

The symmetric group Sym(S) on the space S of these sonorities is of size 18! = $6,402,373,705,728,000$. Within this symmetric group, 216 operations normalize G. The orbit of D under $N_{Sym(S)}G$ is of size 108 (hence, each element of this orbit is stabilized by $216/108 = 2$ members of the normalizer). These 108 pentachords constitute the full set of all-interval subsets for this interval system. We do not find GISZ-relations in this set particular of all-interval chords. All 108 pentachords in the orbit of D under the action of the normalizer are also in the orbit of D under the respective actions of the group of GIS-transposition and GIS-inversion operations and the group of interval-preserving operations.

VII. All-Interval Chords of Small Order ($2 \leq k \leq 8$)

In this section, we catalog all the interval groups (up to isomorphism) that contain all-interval chords of orders $2 \leq k \leq 8$. For each chord size, we include remarks about the groups that produce all-interval chords, including the number of such chords that they contain and the types of difference sets that those chords exemplify, whether the groups are abelian or non-abelian, and other relevant information.

That is, $N_H G = h \in H | h^{-1} G h = G$.

[15]The GISZ relation is Lewin's adaption of the Z relation to the theory of Generalized Interval Systems. Sets that are GISZ-related to one another are related by neither GIS transposition nor GIS inversion (nor, in the non-abelian case, by members of the group of interval-preserving transformations) [22]

Bichords. The smallest all-interval chords include only one (non-unison) interval; hence, they are of size $k = 2$. Accordingly, the interval groups that contain these bichords must themselves have only one interval class. The smallest isomorphism class of such groups is the cyclic group of order 2, $\mathbb{Z}_2$. In this group, the single non-identity element is an involution. $\mathbb{Z}_2$ includes only one all-interval chord, the smallest non-empty set of all-interval chords for any group. This bichord is an example of a (2,2,2) non-planar difference set. The largest group with one interval class is the cyclic group of order 3, $\mathbb{Z}_3$, which possesses no involutions. $\mathbb{Z}_3$ contains three all-interval chords. As this group has no involutions, these all-interval chords are also all-directed-interval chords. The all-interval bichords in $\mathbb{Z}_3$ are examples of (3,2,1) planar difference sets, the smallest non-trivial class of these structures. Both the above groups are abelian.

Trichords. Groups with all-interval trichords must contain three interval classes. Three isomorphism classes of groups have the appropriate number: the Klein four-group, Z_2^2; the cyclic group of order 6, $\mathbb{Z}_6$; and the cyclic group of order 7, $\mathbb{Z}_7$. All three possess all-interval trichords. The three non-identity elements of Z_2^2 are all involutions, making it the smallest group to have all-interval trichords, as well as the smallest non-cyclic group to contain all-interval chords of any size. The four all-interval trichords in Z_2^2 are examples of (4,3,2) non-planar difference sets. $\mathbb{Z}_6$ contains two interval classes of invertible elements and one involution. Its twelve all-interval trichords are examples of (6,3,1,4) almost difference sets; as such, it is the smallest group to include all-interval chords with this type of structure. In contrast, $\mathbb{Z}_7$ contains no involutions. Its fourteen all-interval trichords are examples of (7,3,1) planar difference sets; hence, they are also all-directed interval. As with the bichords, the three groups that contain all-interval trichords are abelian.

Tetrachords. For a group to accommodate all-interval tetrachords, it must have six interval classes. Four groups satisfy this requirement: the dihedral group of order 8, D_8; the cyclic group of order 12, $\mathbb{Z}_{12}$; the dicyclic group of order 12, Dic_{12}; and the cyclic group of order 13, $\mathbb{Z}_{13}$. However, only three of these groups possess all-interval tetrachords: D_8, $\mathbb{Z}_{12}$, and $\mathbb{Z}_{13}$. (In §5, we examine the reasons why Dic_{12} fails to produce all-interval tetrachords.) D_8 is distinguished as being the smallest non-abelian group to contain all-interval chords of any size. Its sixteen all-interval tetrachords are examples of (8,4,1,2) almost difference sets. $\mathbb{Z}_{12}$ contains the canonical examples of the forty-eight all-interval tetrachords of pitch-class set theory, instances of (12,4,1,10) almost difference sets. $\mathbb{Z}_{13}$ has fifty-two all-interval tetrachords. As $\mathbb{Z}_{13}$ contains no involutions, these planar difference sets are also examples of all-directed-interval chords. These latter two groups are abelian.

Pentachords. As the size of the all-interval chords increases, the number of interval groups with appropriate numbers of interval classes that fail to produce all-interval chords also increases. All-interval pentachords require groups with ten interval classes. Whereas seven groups meet this condition, only four have all-interval pentachords: the semidihedral group of order 16, SD16 (also known as the quasidihedral group of order 16); the direct product of S_3 and $\mathbb{Z}_3$, $S_3 \times \mathbb{Z}_3$; the semidirect product of $\mathbb{Z}_7$ by $\mathbb{Z}_3$, $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$; and the cyclic group of order 21, $\mathbb{Z}_{21}$. SD_{16} has 128 all-interval pentachords, which are (16,5,1,10) almost difference sets, and the 108 all-interval pentachords in $S_3 \times \mathbb{Z}_3$ are (18,5,1,14) almost difference sets. Both these groups are non-abelian.

In the subsections above with $2 \leq k < 5$, we note the existence of almost difference sets in the cyclic group of order $(k-1)2 + (k-1)$ and planar difference sets in the cyclic group of order $(k-1)2 + (k-1) + 1$. With $k \geqslant 5$, this circumstance no longer holds. Specifically, we cease to find examples of all-interval k-chords in the cyclic group of order $(k-1)2 + (k-1)$. Whereas $\mathbb{Z}_{21}$ contains forty-two all-interval pentachords as (21,5,1) planar difference sets (hence, they are also all-directed-interval chords), no all-interval chords exist in $\mathbb{Z}_{20}$. The rationale for this situation is related to the non-existence of perfect Golomb rulers with five or more marks.[16] We find another

[16] A Golomb ruler with k marks and length L has a different measurement between any two marks. A perfect Golomb

first occurrence with $k = 5$: the existence of a non-abelian group of order $v = (k-1)2 + (k-1) + 1$ that contains all-interval chords, $\mathbb{Z}_{v/3} \rtimes \mathbb{Z}_3$. Such a non-abelian group exists for every $k \equiv 2$ (modulo 3), where $k \geqslant 5$ [12, Theorem 18.68]. Like the cyclic group of order 21, $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$ contains no involutions. Hence, the 294 all-interval pentachords in this group are (21,5,1) planar difference sets (and they are also all-directed-interval chords), the same type of structure as those in $\mathbb{Z}_{21}$.

Hexachords. A hexachord has fifteen intervals; therefore, a group that contains all-interval hexachords must have that number of interval classes. Eight such groups exist, but only five of these groups contain all-interval chords: the direct product of four copies of the cyclic group of order 2, $\mathbb{Z}_2^4$; the direct product of the alternating group of degree 4 and the cyclic group of order 2, $A_4 \times \mathbb{Z}_2$; the direct product of $\mathbb{Z}_2^3$ and the cyclic group of order 3, $\mathbb{Z}_2^3 \times \mathbb{Z}_3$; the direct product of cyclic groups of order 14 and order 2, $\mathbb{Z}_{13} \times \mathbb{Z}_2$; and the cyclic group of order 31, $\mathbb{Z}_{31}$. Of these five groups, only $A_4 \times \mathbb{Z}_2$ is non-abelian. As with the sets of groups that contain all-interval bichords and trichords, we find all-interval hexachords in the smallest and largest possible groups to have the appropriate number of interval classes. $\mathbb{Z}_2^4$, in which all fifteen interval classes are involutions, is the smallest such group. Its 448 all-interval hexachords are examples of $(16, 6, 2)$ non-planar difference sets. In contrast, $\mathbb{Z}_{31}$ contains no involutions. It contains 310 all-interval hexachords as $(31, 6, 1)$ planar difference sets (i.e., all-directed-interval hexachords).

Two groups of order 24 exist with all-interval hexachords: one abelian with 1344 all-interval hexachords, $\mathbb{Z}_2^3 \times \mathbb{Z}_3$; and one non-abelian with 192 all-interval hexachords, $A_4 \times \mathbb{Z}_2$. As both these groups have seven involutions, all 1536 of these hexachords are instances of (24,6,1,16) almost difference sets. In the group $\mathbb{Z}_{14} \times \mathbb{Z}_2$ of order 28, we find more than one orbit of all-interval hexachords under the action of the normalizer of the group. Consequently, the GISZ relations among these hexachords do not derive from operations that are analogous to affine transformations. It is the only group with all-interval chords of size $2 \leqslant k \leqslant 8$ to have this property. Its 728 all-interval hexachords partition into three orbits: one orbit of size fifty-six, and two of size 336. All of these hexachords are examples of $(28, 6, 1, 24)$ almost difference sets.

Heptachords. Of the sixteen groups with twenty-one interval classes, as required for all-interval heptachords, only one has this type of subset. Interestingly, it is not the cyclic group of order $(k-1)2 + (k-1) + 1 = 43$. As this number is prime, the cyclic group $\mathbb{Z}_{43}$ is the only isomorphism class of groups of that order. Further, it has no involutions, suggesting that it contains planar difference sets. However, we recall from §4 that $k-1$ must be the power of a prime to yield planar difference sets (and in this case, $7 - 1 = 6$ is smallest integer that is not the power of a prime). Instead, the one isomorphism class of groups to produce all-interval heptachords is the extraspecial group $2(1 + 4)$ of minus type. This order-32 group is the central product of a dihedral group of order 8 and a quaternion group of order 8 that intersect in a central order-2 subgroup (i.e., all thirty-two members of the group commute with the members of this subgroup) [24]. The 512 all-interval heptachords found in this non-abelian group are all instances of (32,7,1,20) almost difference sets.

Octachords. As with the pentachords, $8 \equiv 2$ (modulo 3); hence, we find a non-abelian group of order 57, $\mathbb{Z}_{19} \times \mathbb{Z}_3$, along with the cyclic group $\mathbb{Z}_{57}$. In fact, of the ten groups with twenty-eight interval classes, only these two produce all-interval octachords. $\mathbb{Z}_{57}$ contains 684 all-interval octachords, and $\mathbb{Z}_{19} \rtimes \mathbb{Z}_3$ has 6498. All 7182 are examples of (57,8,1) planar difference sets (i.e., all-directed-interval octachords).

ruler is one in which every distance from 1 to L appears as such a difference [23]

VIII. Conclusions and Future Work

In total, we find 11,438 all-interval chords of sizes $2 \leqslant k \leqslant 8$ in twenty interval systems. The groups that these interval systems incorporate include abelian groups that are cyclic ($\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_6, \mathbb{Z}_7, \mathbb{Z}_{12}, \mathbb{Z}_{13}, \mathbb{Z}_{21}, \mathbb{Z}_{31}$, and $\mathbb{Z}_{57}$) and non-cyclic ($\mathbb{Z}_2^2, \mathbb{Z}_2^4, \mathbb{Z}_2^3 \times \mathbb{Z}_3$, and $\mathbb{Z}_{14} \times \mathbb{Z}_2$), as well as non-abelian groups ($D_8, SD_{16}, S_3 \times \mathbb{Z}_3, A_4 \times \mathbb{Z}_2, \mathbb{Z}_7 \rtimes \mathbb{Z}_3, 2^{(1+4)}$ [the central product of D_8 and Q_8], and $\mathbb{Z}_{19} \rtimes \mathbb{Z}_3$). Moreover, the all-interval chords themselves are of three general types: three classes of non-planar difference sets, seven classes of almost difference sets, and six classes of planar difference sets. The chords of the first two types are merely all-interval, whereas those of the third type also meet the stricter requirement of being all-directed-interval. The full sets of all-interval chords in these interval systems also vary in size and in terms of their GISZ-relations. Their sizes range from one chord ($\mathbb{Z}_2$) to 6498 chords ($\mathbb{Z}_{19} \rtimes \mathbb{Z}_3$). None of the sets with $k < 4$ have $Z-$ or GISZ-related members. Nineteen of the twenty sets are single orbits of all-interval chords under the action of the normalizer of the interval group. The GISZ relations among the chords in these sets derive from affine or affine-like transformations. The one remaining set — that of all-interval hexachords in $\mathbb{Z}_{14} \times \mathbb{Z}_2$ — is comprised of members in three such orbits. The GISZ relations among all-interval hexachords within different orbits of the normalizer of this group obtain from other, more obscure origins.

Within this diversity, we find some common threads that are of particular relevance to musical structure. First, abstract mathematical groups correspond to groups of symmetries. Whereas the generalized intervals we discuss here do not necessarily possess qualities of distance and direction, they do relate to symmetries. Further, the groups of symmetries to which these twenty interval systems correspond either contain simple symmetries themselves or are products (direct or semi-direct) of smaller groups that are composed of simple symmetries. Basic symmetries — such as translations, rotations, and reflections — surround us and shape our experience; they are found throughout nature, and they are commonplace in many human endeavors, including the visual arts, architecture, and music [25]. The cyclic groups $\mathbb{Z}_n$ agree with rotations of regular n-gons. These types of symmetries are used to model a variety of musical structures, including pitch-class transpositions and rhythmic translations in metric spaces. The dihedral groups D_{2n} add reflections to these rotational symmetries. In music, such reflections correspond to pitch-class inversions and rhythmic retrogrades. The Klein 4-group $\mathbb{Z}_2^2$ corresponds to a subgroup of symmetries of a square, or 2-cube. As we discuss in §4, these symmetries are those of the serial operations prime, inversion, retrograde, and retrograde-inversion. The larger elementary abelian 2-groups $\mathbb{Z}_2^n$ are isomorphic to particular subgroups of n-cube symmetries. These symmetries model nth roots of inversion and retrograde [26]. Similar associations exist for the other small groups that constitute these interval systems.

In addition to the interval systems, the all-interval chords they contain also have special musical significance. Their defining structure facilitates two important compositional processes: summation and deconstruction/development. The Schoenberg *Lieder* from Figure 1 illustrates the former process. This work — one of his first atonal compositions — is representative of the ″emancipation of dissonance″ that characterizes his atonal style. Rather than organizing pitch-class intervals in this song in terms of a hierarchy that is based on consonance and dissonance, Schoenberg treats all intervals equally. Thus, the final sonority of this piece, an all-interval tetrachord, serves as an economical summary of the song's intervallic content. The process of deconstruction/development is evident in the first two string quartets of Elliott Carter. Carter systematically deconstructs the all-interval tetrachords in these works into their constituent parts, exploring and developing each of the intervals in turn and in combination. Additional aspects of all-interval chords lend themselves to further musical interpretation. For instance, certain transformational processes, such as the one in Figure 2, are possible because of the unique construction of these types of chords.

From a theoretical perspective, the completion of an existence theorem is perhaps the most significant open question. As we note in §5, an interval system cannot contain all-interval k-chords unless it has exactly $\binom{k}{2}$ interval classes. Satisfying this condition is necessary, but not sufficient, for the existence of all-interval chords. Does a single, unifying requirement exist that proves sufficiently the existence (or lack of existence) of all-interval chords in a given interval system? Much future work remains in the study of all-interval chords. The small-order structures investigated here can serve as departure points for new compositional designs and analytical investigations, and similar work may also be applied to larger-order all-interval chords.

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On the Relation of Quality and Quantity in the Context of Musical Composition

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***Abstract**: With this paper we aim to highlight the connection between quality and quantity, from a musical point of view. For this, we heuristically sketch a typology of musical qualities. Every quality offers a gamut of gradations. Each degree inside this range can be indexed as a value, making a range of quantities available. The changes of a musical quantity over time is represented as a list of values. This list can be manipulated through a variety of mathematical operations. Such approach can be applied to any musical quality (thus, encouraging students to face the elements assembled in a composition from the start). Some of these operations are presented here as functionalities of J-Syncker, an assistant software for the generation of pre-compositional material.*

***Keywords**: Music Composition. Musical Qualities. Musical Quantities. Lists. J-Syncker.*

I. Introduction

Listening reveals nuances inside its own realm. Basically, we perceive a sound by means of its intensity (strong or weak), duration (long or short), pitch (high or low) and timbre (smooth or rough). Furthermore, each one of this basic specialties (or qualities) signals variations to the attentive listener.

In a composition, musical qualities also describe changes in their values over time. We represent these values (or quantities) using the concept of list. For instance, we can use a list to depict the quantities associated to dynamics or melodic contour (respectively, variations of intensities and pitches).

Values of the same list can be used to quantify different musical qualities. On the one hand, this helps achieve coherence. On the other, it communicates dynamism to the craft of composition.

Employing operations on lists can be strategically used for musical ends, producing a wealth of pre-compositional material. It is also a way to summarize composition techniques, suggesting another manner of formalizing knowledge to the field of music composition. Once quantitative variation described by musical qualities is numerically represented, it can be subjected to a variety of mathematical resources. Still, this approach can be applied to any musical quality.

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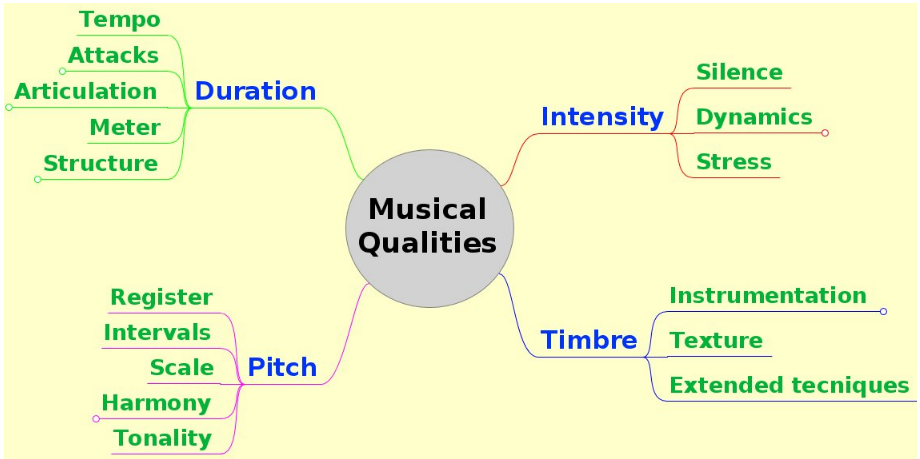

Figure 1: *An heuristic mapping of Musical Qualities.*

It is highly desirable for such an approach to be assisted by a computer application, through which a composer can listen to and choose from output results. In the proceedings of this paper we will talk about a specific application, written as a proof of concept. We will describe some of its functionalities in the context of referred operations on lists. For now, let us go back to the musical qualities.

II. Musical Qualities

In Figure 1 we map[1] a first generation of musical qualities: intensity, timbre, pitch, and duration. Intensity relates with the volume of musical sounds. Timbre is referred as the tone color (or that which differentiates one sound source from another). Pitch is related to sound frequencies. Duration refers to the length of events[2].

It is possible to subdivide these first branches, reaching new subtleties in the auditory sense. Within intensity we may locate silence (absence of intensity); dynamics (changes in intensity) and stress (a way to differentiate some events from others, bringing them to the foreground of listening). Within timbre, instrumentation deals with the possibilities of timbristic combinations; texture deals with the number of voices (or instruments) sounding in a given moment. Within pitch we locate register (a range of notes possible to be played in a instrument); intervals; scale; harmony and tonality. Within duration we have tempo (speed of execution); the size of attacks; articulations (the nuances in the lengths of notes); meter (a way to group beats in measures); and structure (rhythm of sections[3]). Also in this domain is the total duration of a given work.

Furthermore, we witness a combination of them sprouting new entities. Rhythm is in itself a complex combination of properties related to duration along with stress. Melody could be seen as how pitches change as function of durations and intensities. Orchestration is the variation of instrumentation and texture along with structure; the same with modulation, i. e., sectional articulation of tonalities.

[1]The software used to this end was Freeplane [11].

[2]Extra musical elements are beyond the scope of this work, such as lyrics, written descriptions, other medias (choreography, video, etc.), soggetto cavato, mood markings and so on.

[3]The idea of structure as a rhythm of sections was an insight shared by Professor Ilya Levinson (Columbia College Chicago) in a personal communication with the author.

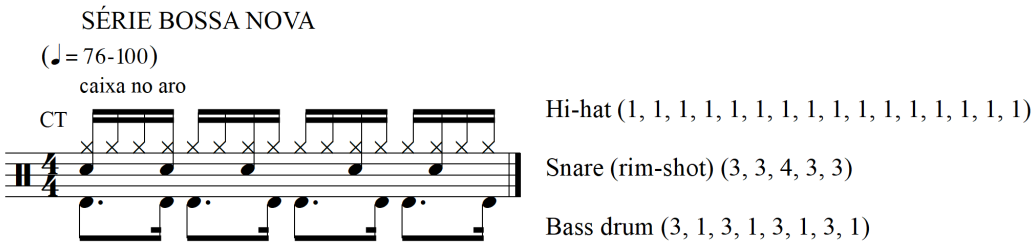

Figure 2: *An instance of Bossa Nova rhythm to drums [13, p. 60].*

Musically speaking, it is possible to notice that listening reveals nuances inside its own realm. We may identify sub-senses tracking a variety of stimuli. Music can invite a listening expansion in which different qualities may be perceived residing even within the auditive experience.

III. Musical Quantities

In a composition, musical qualities change their values over time. Pitches go up and down. Durations become longer and shorter. Intensities get strong and soft. The same may happen with timbre, texture, tonality and so on. These changes define ranges of possibilities even inside a very quality. It seems reasonable to relate a scale to each musical quality, i. e., a gamut of varieties provided by every domain [4].

Let us use a list to collect the values involved in a musical quality, changing over time. We employ the term list here as a sequence of quantities derived from a given musical domain. Next, we describe some attributes of a list. Take for instance a list of some composers's birthdays (6,1,19,12,2,15). The size (also referred to as length) gives the number of elements in the list. In this case, the size is 6. A list has an order of elements. Thus, an element has a position in a list. The birthday 12 is in the forth position. Also, an element may have a specif type, i. e., numeric, alphabetic, alphanumeric, etc. Our list has only numeric values. Let total be the sum of all numeric elements. In our case, total is 55.

Now we will use a list to represent the quantities associated to a musical quality. We will represent attack durations, for instance. Consider the information displayed in Figure 2.

Let us take only the snare part. The list (3,3,4,3,3) was generated taking into account the durations of each attack. The basic unity in this case is the 16th note. So, "1" is represented by a 16th note, "2", by a 8th note, "3", as a dotted 8th note, "4", by a quarter note and so on. (It is also possible to obtain different notations by tying music figures in different combinations.)

Let us consider some of the attributes of this list from the point of view of the rhythm. The size of the list (or the number of elements) gives the number of attacks. The one designated to the snare has 5 elements, thus, 5 attacks. The list sum gives the total duration of the rhythm (which is the duration of all attacks together). It is equal to 16 units. As we already know, the value of an element gives the duration of a single attack. The first attack takes 3 units, the second 3, the third 3, and so on. So, we can say that this specific rhythm of 16 units were partitioned into 5 attacks. This information will be represent as 16P5. However, notice that the sequence of 3, 3, 4, 3 and 3 units is unique. Just imagine that (4,3,2,3,4) is also another member of a 16P5. We will cover more about partitioning in the next section.

[4] This approach lies in the heart of the system of musical composition devised by Schillinger [15]

Next, we will use the same list to quantify different musical qualities. Besides rhythm itself, we will provide an example for structure and intervals.

In terms of structure, we perform a one-to-one correspondence between two lists, the second being a list of sections:

$$(3,3,4,3,3) \rightarrow (A,B,C,B,A) = (3A,3B,4C,3B,3A)$$
$$=(\mathrm{A,A,A,}\mathbf{B},\mathbf{B},\mathbf{B},\mathrm{C,C,C,C,}\mathbf{B},\mathbf{B},\mathbf{B},\mathrm{A,A,A})$$

Thus, an example of one list controlling the number of repetitions of the members of another list. Here, we can say that the first was used as a list of "coefficients of recurrence" [15].

In terms of intervals, we apply the values of the list as intervals (in semitones) to generate a scale of non neighbor pitches.

$$3\ 3\ 4\ 3\ 3$$
$$F\sharp ACEGB\flat$$

Likewise, we could keep applying the same list over other domains. Thus, rhythm may be represented as a list of durations. A structure, as a list of sections. A scale, as a list of pitches (or a list of pitch intervals).

Up to this point we have described how to represent quantities related to a musical quality. In the next section we will employ operations to produce changes in the quantities of such qualities. This will be very useful to aid the process of obtaining variety out of the same musical seed.

IV. Operations on lists

In the previous section we described a way to represent two important attributes of a list. They are sum and size. We represented them with the expression [sum][Part][size]. So, 16P5 means that a total of 16 were partitioned in five elements (also referred to as parts, terms, summands or addends). We use this device only with integers. For instance, we depict in the Figure 3 an example of partitions of the integer 8.

Notice how the representation [sum][Part][size] changes according to the place in the partitions generations. In this case it goes from 8P1 to 8P8, revealing 8 generations.

The number of partitions grows very quickly as bigger integers are used. Thus, we apply a restriction criteria, depicting only results with greater values placed on the left[5].

This family holds coherence once it is performed systematically within the bounds of its parent integer. Such coherence is very useful when applied to musical purposes. For instance, its members can be used in succession but also simultaneously, producing rhythmic polyphony.

Next, we are going to use a list to quantify rhythmic durations. Let us reuse the pattern executed by the snare in Figure 2, now represented as the list (3,3,4,3,3) – a member of 16P5. In Figure 4 we depict some operations on it, producing variants as results. (Notice also how the partitions change by means of such operations.)

We will describe each of these operations, as they are numbered in clockwise direction.

1. Reverse returned the original list. This is due to the list elements being palindromic, i. e., the same thing can be read back and forth. Such a non-retrogradable rhythm was referred as "the charm of impossibilities" by Messiaen [10];

[5] When permuted results are taking into account, the term Composition is used instead of Partition.

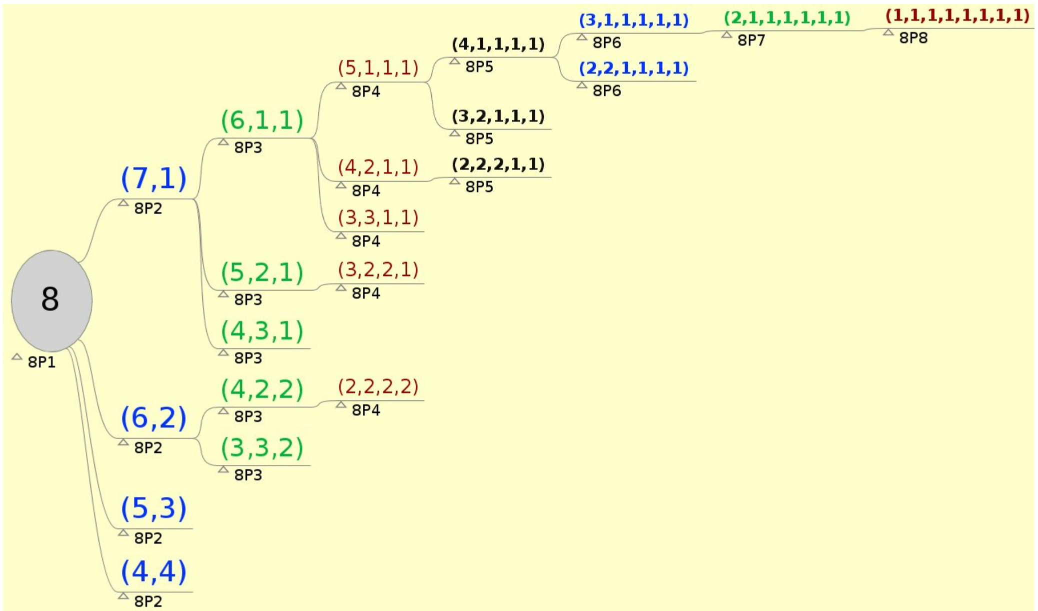

Figure 3: *Partitions of the integer 8.*

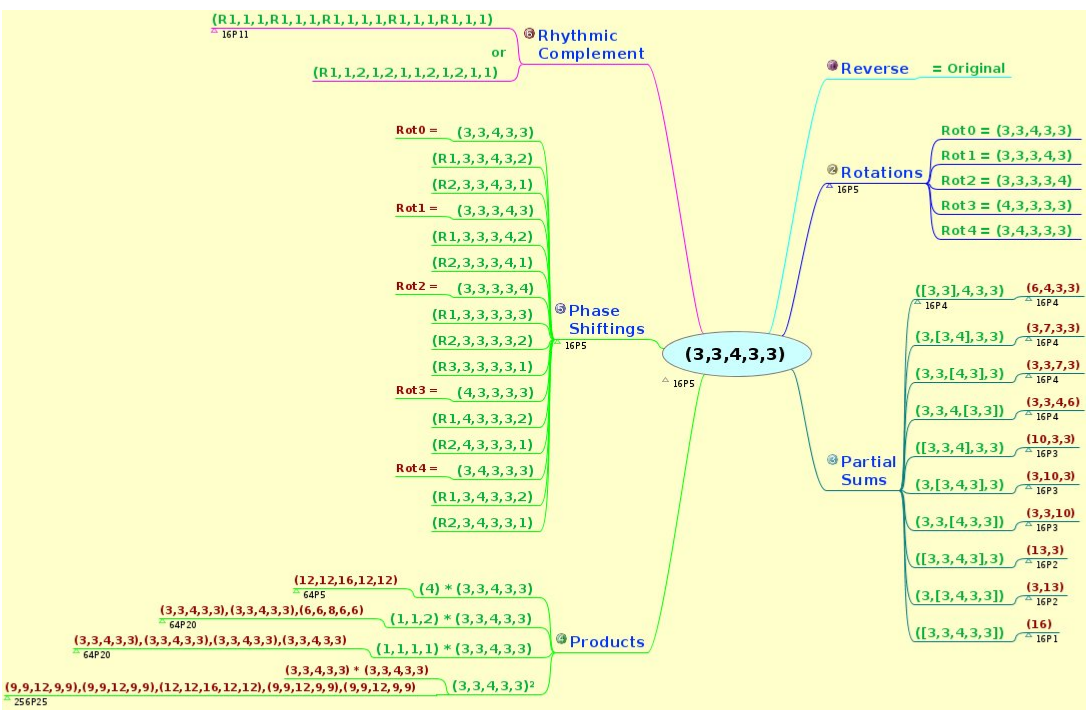

Figure 4: *Operations on a list of rhythmic durations (3,3,4,3,3).*

2. Rotations can be obtained imagining the list tied in its extremities and rotated clockwise or anticlockwise (the latter being notated with negative indexes, i. e., Rot_{-2}). The number of possibilities equals the number of elements (considering that the rotation zero is the original list). This operation is referred to as circular permutation [15, p. 116];

3. Partial sums performs a series of adjacent additions. The brackets within a list (depicted in Figure 4) contain the elements to be added. They are displaced to the right in each turn of sums until they contain the last element. These additions start with two elements then three, four and so on until the brackets contain all elements to be added, thus returning the total sum. In each turn the resulted lists have fewer elements. Such an operation generates "familiar" rhythms, since the results are tight connected to the original seed list. A similar operation is referred to as "Summation Series" by Schillinger [15, p. 119]. This operation also can be seen as an inverse operation of partition. (Notice that other rounds of partial sums could be performed with the results obtained in the example depicted here, giving rise to new generations. Some results of such recursion are redundant however. Exclusion of identical lists would be needed in these cases.);

4. Products. A list can be multiplied by an integer, becoming expanded by this factor. If multiplied by 2, for example, it will get 2 times expanded, i. e., each of its elements will increase by a factor of 2. (Which is the numeric representation of the augmentation technique, used as a contrapuntal device. We will cover this correspondent relationship later in this paper.) Furthermore, our example list can be multiplied by another list. (We choose to use partitions of 4 in this example: (1,1,2), which is a form of 4P3 and (1,1,1,1), a form of 4P4). In this case we apply the distributive property, so that each value of the first list multiplies all the elements of the second list. Notice that this operation may not be commutative for all cases, i. e., $(1,1) * (2,3,4)$ is different to $(2,3,4) * (1,1)$. Finally, a list can be raised to an exponent. The squared list is depicted, which is the product of it by itself;

5. Phase shifting. Each entrance of the pattern is shifted one rhythmic unity to the right. In this case the basic unit is the 16th note. (Notice that some results coincide with rotations.) See for instance an interesting video[6] with a graphic representation of Steve Reich's Clapping Music. In this work Steve Reich uses a shifted result (Clap 2) against the original (Clap 1), causing a kind of musical interference of attacks. For other compositional purposes the shifted result can be also used as a new durational pattern;

6. Rhythmic complement. A second rhythm sounds in the empty spaces of the first. It can be seen as a boolean negation as the durations are represented as a series of 1 and 0. 1 for an attack unit and 0 for the unities between them. Then, each 1 becomes a 0 and vice versa. In Table 1 we show the steps of such operation.

Line 1 depicts the elements of the list $(3,3,4,3,3)$ as a grouping inside the 16 durations. Line 2 holds its representation using 1 for attack and 0 for non attack. Line 3 is the negation of line 2, i. e., the result of the binary NOT operation. Line 4 converts back from 1 and 0 notation (using R to symbolize rests in the place of zeros). Line 5 depicts a optional form, "tying" rests with its previous durations[7].

In Table 2 we suggest some of the operations on lists that can be strategically used for musical ends.

[6]"Steve Reich – Clapping Music (Scrolling)". Gerubach (Youtube channel). Available at <`https://www.youtube.com/watch?v=lzkOFJMI5i8`>. Access 10/01/2016.

[7]This operation can be correlated to bit shifts, a kind of bitwise operation.

Table 1: *Rhythmic complement operation steps*

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	3			3			4				3			3		
2	1	0	0	1	0	0	1	0	0	0	1	0	0	1	0	0
3	0	1	1	0	1	1	0	1	1	1	0	1	1	0	1	1
4	R1	1	1	R1	1	1	R1	1	1	1	R1	1	1	R1	1	1
5	R1	1	2		1	2		1	1	2		1	2		1	1

It could even be desirable to search for a set of "primitive" (low level) operations, such as subtract, that can be combined in order to generate other (high level) operations, such as derivate (which return the difference of the values of adjacent numerical elements). List operations also can be used nested. For instance, consider the nesting operations depicted below:

$$reverse(rotate_n(product(\mathrm{f}; list)))$$

Table 2: *A sample of operations on lists*

accumulate	GCD–LCM	permute
append	intersect	power
(bitwise operations)	max	reverse
combine	min	riffle
derivate	mod	rotate
divide	mode	sort
factor	multiply	subtract
find	partition	sum

Table 3: *Some correspondences between composition techniques and list operations*

Augmentation	product(n;*list*)
Rotation	rotate(index;*list*)
Expansion	product(n;*list*)
[Pitch] Inversion (chromatic)	complement(12;list)
Retrograde	reverse(*list*)
Rhythmic complement	NOT(*list*)
[Phase] Shifting	bitShifts(index;*list*)
Transposition	add(n;list)

Where f is the factor that multiplies the list elements. Product's result are passed to Rotate, where n is the index of rotation. Finally, the result is retrograded by Reverse.

Working with operations on lists is also a way to summarize musical composition techniques. For example, take the correspondences depicted in Table 3.

Members in the right column represent the techniques used in music composition. Members in the left represent operations to perform analogous tasks. Notice that some of them need extra arguments, such as numbers and indexes (beyond the list itself). In the cases of `NOT` and `bitShifts` operations, the numeric list passed as argument should be first converted to a list of ones and zeros (see Table 1, steps 1 and 2).

It is convenient to open the use of the same operations on different musical qualities. In this way, much of the same set of operations can be applied over durations, intensity, pitch and timbre related lists.

Thus, we presented an attempt to gather an expanding collection of operations on lists. These lists are representations of values changing in time related to one or more musical qualities. Our goal is the generation (and variation) of pre-compositional material. Furthermore, to lay foundation for compositional pedagogy. Such an effort also casts a glimpse towards a creative theory of music composition.

So far we have been giving a systematic approach to Musical Composition, formalizing knowledge of its field by means of mathematical modeling. In extension, this also leads towards a next step in formalization. Namely, it is highly desirable that such an approach be assisted by a computer application (through which a composer can listen to and choose from computed results). In the next section, we will explore that possibility.

V. Synckers

My research project is entitled "Generation of pre-compositional material based on a computational interpretation of the Schillinger System of Musical Composition (SSMC)" [15]. Implementations based on the SSMC are feasible, for it also makes use of a mathematical approach[8] . A computer application was originally conceived as the main result of this research. It is devised to facilitate understanding of a variety of devices shown in SSMC. The application aims to benefit students and scholars who can make use of it according to their demands and musical purposes.

Oposmodus, Symbolic Composer and OpenMusic are among some of the softwares that could fit in the category we propose. Their disadvantages are due they are proprietary and do not offer versions to the Linux[9] operating system. From the academic context, we name similar works in implementing SSMC techniques as [12],[8], [5]and [4].

Two solutions are being developed in parallel. One application, called J-Syncker uses some of the devices presented in SSMC, making them available through a user friendly interface. J-Syncker is available to download at <`http://j-syncker.weebly.com/`>. Pd-Syncker is another solution developed by our team. Its development runs in parallel with J-Syncker. It is programmed in Pure Data. The advantage of this language is to be dedicated to multimedia processing (audio, video, Internet and MIDI devices). Pd-Syncker is available to download at <`http://pd-syncker.weebly.com/`>. Both solutions have been used as tools in Compositions Workshop courses given at School of Music of UFRN.

For the sake of space, next we will focus only to J-Syncker, showing how it works and how to perform some of the operations listed previously.

i. J-Syncker

The application is called J-Syncker for three reasons. First, the program was written in the Java programming language. It is common to see programs written in this language starting with the letter "J", for example, JFugue, Jedit, JabRef, etc. Second, the program deals primarily with the

[8] The Mathematical Basis of the Arts (MATHBART) is his published *magnum opus* [14]. It turns out that the SSMC is the musical branch of MATHBART.

[9] The Linux platform have being preferred for it is open source, free software, shared worldwide and benefit people who do not have affordable access to proprietary technology – offering an alternative to barriers imposed by such paradigm.

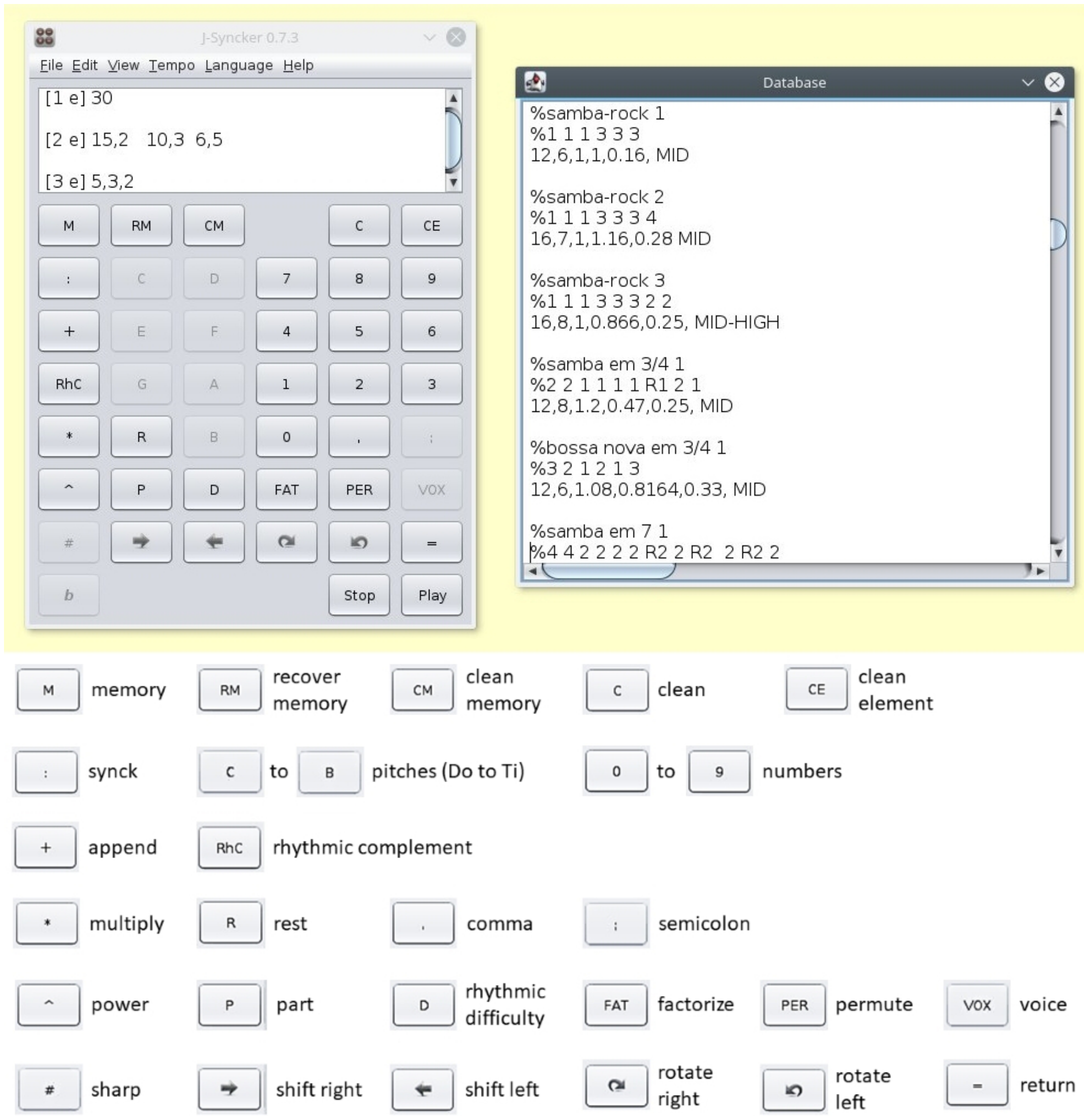

Figure 5: *J-Syncker's GUI, elements description and Database window.*

concept of synchronization (see [17]). Syncker was the short word chosen for synchronizer. Third, and in a more veiled way, the name of the application suggests a rhyme with Joseph Schillinger – or just J. Schillinger. [18] As it is written in Java, J-Syncker is multiplatform. It is free and open source.

J-Syncker has a user-friendly interface. Its design is inspired by the idea of the scientific calculator. This format is somewhat familiar and demonstrates features available on the GUI. Additionally, this kind of interface suggests the mathematical character underlying a compositional process (something that, from the outset, is not always apparent in music composition).

Next we will describe some of the functionalities from J-Syncker's GUI, depicted in Figure 5.

To begin with, it is possible to enter any durations list, typing directly in the format of a **comma separated list**. This means that one comma separates each two operands (integers, in this case). Use R button (or type "r") to enter a rest, followed by its duration.

To **clean** elements or the entire display, click the buttons CE and C respectively (or backspace and delete keys respectively).

To select, use the mouse to click and drag (double click or Ctrl+A selects all). Copying (Ctrl+C), cutting (Ctrl+X) and pasting (Ctrl+V) are also allowed.

To choose from **presets lists** of rhythmic durations, go to View in menu bar then choose Database (or simply Shift+D). Such lists are based on rhythms found in [13], [15] and [3]. (This database is used to evaluate rhythmic complexity. The complexity of a given numerical list representing rhythmic durations may be evaluated by pressing the "D" button. For more on this see [2])

Click the button (or type the capital letter) "M" and a number (from 0 to 9) to store data in **memory**. The same procedure can be used to recover and clean memories, using RM button (or Alt+R) and the CM button (or Alt+C) respectively.

To **append** one list to another, enter the first, click the plus button (or type the "+" character) then click the equal button (or hit Enter key).

The partition operation is performed by entering an integer and pressing P button (or typing letter "p") then hitting Enter key. The results are shown indexed from 1 to $n-1$ elements. One can use the form nPm, where n is an integer and m the number of parts (also hitting Enter key for the result be evaluated). In this case only partitions with m parts will be depicted. The resultant format is similar to the one depicted in J-Syncker's display in Figure. Actually, this one generates the output of the **factorization** (or multiplicative partition). To achieve that result, click the FAT button (or type letter "f") instead of P. Likewise, use the format nFm to obtain only multiplicative partitions of n with m parts. (The output showed in the display of J-Syncker in Figure 5 was obtained after entering 30F + Enter. To show only results with two values 30F2 + Enter would suffice.)

PER button **permutes** the elements of a list previously entered.

The two buttons with circular arrows (or left and right arrow keys) rotate a list in both directions. The two buttons with straight arrows (or shift+Left/Right arrow keys) shift a list in both directions. The basic unit is the step of shifting. (This operation resembles the correlated bitwise operation.)

The asterisk button performs the **product** operation. Its factors can be integers and/or lists of integers. Press the equal button to evaluate the result.

The caret button (or typing the caret character) is used to perform **exponentiations**. A list of integers can be passed as the operand in this operation. (Pressing equal button is also needed.)

The so-called **rhythmic complement** is performed by hitting the RhC button (or pressing the minus character). Performing it again over the result gives back the original.

Click **Play** button (or hit space bar) to listen the results of the various edits and operations in loop. (The same procedure can be used to **pause** and resume playing.) **Stop** button (or Esc key) stops the loop.

Other functionalities are also found[10] in the J-Syncker menu bar. The drop down menus are described as follows, according to Figure 6, in clockwise direction.

From the File menu, it is possible to save results as MIDI or text formats. MIDI files can be opened in other applications for editing. The text file format were conceived to report the musical arguments used in some operations. (This functionality is still in need of bug fixes.) Besides MIDI, it is possible to open some results automatically in the music notation software MuseScore [16],

[10]GUI buttons not described are related to operations not covered here. Synchronization of lists of durations, pitches, meter, voices and pitch contour deserve a specific report, thus lying beyond the scope of this paper. As an introduction to the subject see [17].

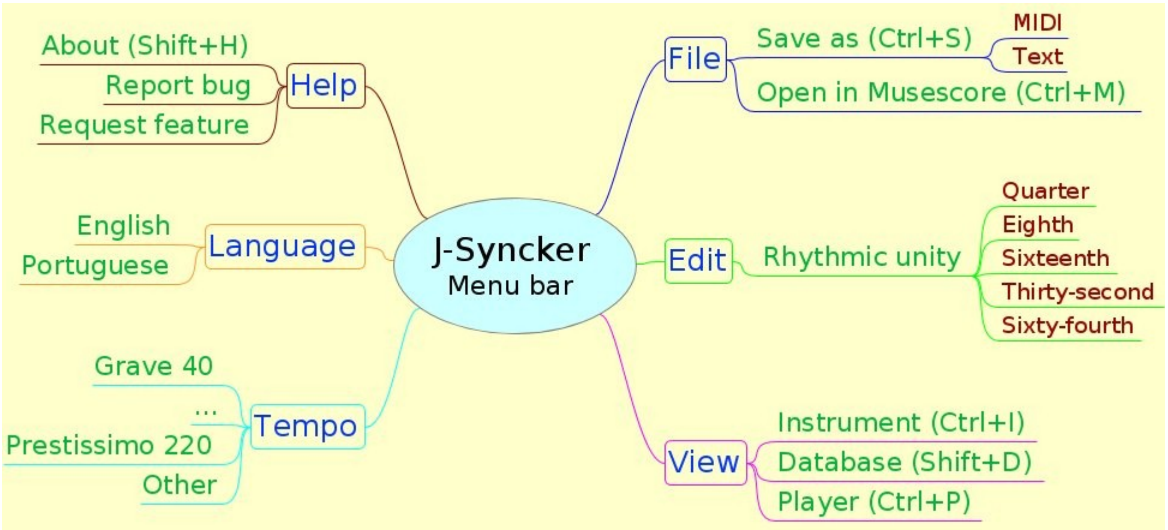

Figure 6: *J-Syncker's menu bar elements.*

once it is installed in the machine. (Currently, this is only possible in the Linux platform, due to MuseScore version updates.)

From the Edit menu, it is possible to set the basic rhythmic unit. That is, a number one in a list could be represented by one of the following rhythmic notes: quarter, eighth, sixteenth, thirty-second or sixty-fourth.

From the View menu, it is possible to change the MIDI program name, as a away to set a new timbre from a synthetic bank of instruments (or just Ctrl+I). The Database of rhythm attributes used in complexity measurement is made available also from this menu (or Shift+D). The player functionality was intended to be implemented in future versions.

From the Tempo menu, it is possible to set the speed of playback. One can choose from preset options (from *Grave to Prestissmo*) or enter a number through the "Other" option at the end of the list of tempos.

From the Language menu, it is possible to translate GUI elements between Portuguese and English.

From the Help menu, it is possible to retrieve information about the software (Shift+H), report a bug or request a feature.

In the next sections, we will bring back some of the previous discussion topics for further comments and lastly point to perspectives and future works.

VI. Discussion

The term "musical quality" was used here as an alternative to music element and music parameter. Element is too generic, meaning a constitutive part of a collection, function or system. Parameter has been used in a variety of definitions (even in the sense of quantity). Etymologically, it came from Greek, being an articulation of two words, *para* (beside) and *metron* (measure). Thus, a reference to a given meter. [7] It is defined in multiple ways such as "a rule or limit", "an arbitrary constant", "an independent variable" and "a set of physical properties". [9] The latter definition is the closer to its use in Music Theory. With that in mind, the term was put aside in order to avoid ambiguity and lack of clarity. Another reason to choose "musical quality" was to combine, compare or cooperate with the term "musical quantity". The aim was to bring forth the relation of these two terms, regarding manifestations of a musical stimulus, i. e., quality and quantity.

In Music Theory literature, numbers are commonly used to represent pitch related qualities (such as [6] and [20], among others). Still other authors expanded to other quality branches (such as [15], [10] and [1]).

Once quantitative variations described by musical qualities are numerically represented, they can be subjected to a variety of mathematical resources. The values plotted from a numeric list can also be seen as contours, bringing the underlying variation to an imagery media, helping prominently visual subjects.

It is important to highlight that this approach can be applied to any musical quality (durations, pitches, timbres, intensities and their combinations). It encourages students to face early on the elements assembled in a music composition. Such a comprehensive overview certainly brings benefits to "multi-quality" writing. (It may contribute to overcome the habit of presenting a score with only pitches and durations written.)

Typical techniques from the field of Music Composition [19] can be "re-presented" by means of operations on lists. This can be useful in aiding the process of obtaining variety out of the same musical seed.

Once established, such a paradigm makes clear the desire for an interface between Music and Mathematics. It can be used theoretically first in the form of a model. Later, it can be used technologically, in the form of a computer application.

The type of specification conducted here helps guide developers in the designing phases of a software project. J-Syncker is one example. Thus, a process for dealing with the articulation of music composition and mathematical modeling may constitute steps such as, formalizing, pseudocoding, coding and testing.

Listen and choose from computed results. This allows focus on the appreciation of computed results. The speed allows the composer to select from a large quantity of outputs.

That being said, one thing still needs to be considered: technologies come and go. As we have seen, they fall out of fashion, are discontinued or even became proprietary. Above all, what is worthy of attention is to focus on the paradigm. It is the paradigm that has a bigger chance to survive.

VII. Future Work

This paper was intended for students and scholars, as well as people interested in the intersection of Music Composition and Mathematics. By doing this, it also takes a step towards increasing the collaboration among such individuals.

In a broad perspective, we intend to continue developing pedagogical content in order to design a knowledge base (be it in a form of book, site or the like). Additionally, courses (and even curricula) based on this content is a desirable consequence. It will be possible also to adapt this approach to teaching existing courses (such as Scorewriting, Arranging, Analysis and so on.)

More specifically, the J-Syncker project has some known bugs to be fixed. It succeeded as a proof of concept and is being used as a central tool in musical courses at the School of Music at UFRN.

Pd-Syncker project needs the porting from Pd-Extended (discontinued) to Pd-Vanilla to be finished. The website needs to be translated to English. Coding externals in the C language also will bring benefits from the point of view of robustness.

A LibSyncker project has being designed to implement numerous operations on lists in a systematic way. We intend to update the lineage of both Synckers. This will demand a solution to work in a clever and broad way, to work with lists with various musical qualities, further advancing the concept of music synchronization.

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Contour Algorithms Review

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***Abstract**: In this paper, we present some problems of two Music Contour Relations Theory operations algorithms: the Refinement of Contour Reduction Algorithm, which was developed by Rob Schultz, and the Equivalence Contour Class Prime Form algorithm, which was developed by Elizabeth Marvin and Paul Laprade. We also propose two alternative algorithms to solve these problems.*

***Keywords**: Music Contour Theory, Reduction Algorithm, Algorithm, Equivalent contour classes.*

I. Introduction

Contour is the shape or format of an object and can be either bi- or multidimensional. In music, a contour is an abstraction of elements such as pitch, chord density, duration, timbre, or intensity. A melodic contour, for instance, is a map of pitches in time.

Musical contour is "a set of points in one sequential dimension ordered by any other sequential dimension" [10]. The study of contour is important because, as is the case with motifs and pitch class sets, it can contribute to musical coherence.

The Musical Contour Relations Theory provides concepts and operations with which to establish contour identity and similarity measures for analytical and compositional purposes. This theory has supported the analysis of works by composers such as W. A. Mozart [1], Luigi Dallapicolla [6], Arnold Schoenberg [4, 10], Anton Webern [3], Olivier Messiaen [12], Elliott Carter, Pierre Boulez, Iannis Xenakis, Alois Haba, Milton Babbitt, Harrison Birtwistle, and Igor Stravinsky [2], as well as video game soundtracks [8].

In this paper, we present some problems of two Contour Theory operations algorithms: the Refinement of Contour Reduction Algorithm [13] and the Equivalence Contour Class Prime Form algorithm [6, 7]. We also propose two alternative algorithms for solving them.

II. Contour Theory basic concepts and operations

A complete presentation of Contour Theory concepts and operations is outside the scope or this paper[1]. However, an understanding of contour space, contour segment, comparison matrix, class equivalence, prime form, and contour reduction is necessary to follow this paper's premise.

[1]See Beard [1], Bor [2], and Sampaio [11] for further information concerning Music Contour Theory.

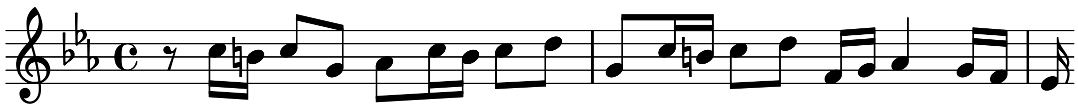


Figure 1: *Fugue 2 subject, The Well-Tempered Clavier by J.S. Bach*

Table 1: *Comparison matrix of the contour M < 5 4 5 2 3 >*

	5	4	5	2	3
5	0	-	0	-	-
4	+	0	+	-	-
5	0	-	0	-	-
2	+	+	+	0	+
3	+	+	+	-	0

Contour space[2] is a musical space composed of the contour elements (or contour points [CPs][3]). The contour space of pitches, for instance, is composed of all pitches represented as integers from low to high and numbered from 0 to $n-1$ [9], where n denotes cardinality. For instance, the contour space of pitches in the melody shown in Figure 1 is $S = \{0, 1, 2, 3, 4, 5, 6\}$, representing the pitches *Eb*, *F*, *G*, *Ab*, *B*, *C*, and *D*.

A contour segment, or, more simply, a contour, is an ordered set of contour points. For instance, the contour of the first five notes in Figure 1 is M < 5 4 5 2 3 >. Normalization (or translation) is an operation that simplifies the contour representation by re-enumerating from 0 to $n-1$. The motive M < 5 4 5 2 3 > normalization is < 3 2 3 0 1 >.

The relation of any pair of CPs is ternary: one CP is lower, equal to, or higher than the other. The contour can be analyzed and represented in a linear or combinatorial way. The linear contour regards only the relations between adjacent CPs; the combinatorial contour considers relations between adjacent and non-adjacent CPs. For instance, in the contour M, the relation between 5 and 4 is lower ($-$), whereas it is higher ($+$) between 2 and 3.

The relations among the contour points are represented in a comparison matrix (cf. Table 1).

Two contours are equivalent if they share the same comparison matrix. The set of equivalent contours forms a contour class. This class is represented by an equivalent class prime form[4]. According to Marvin and Laprade citeMarvin1987, the retrograded, inverted, and retrograded-inverted versions of a contour belong to its equivalence class. The prime form of this class can be calculated with a simple algorithm (cf. Section IV).

Any contour can be reduced to a prime form through the Contour Reduction Algorithm [10]. This algorithm, like the

Schenkerian reductive techniques,

> sets up hierarchical levels of pitch salience. High and low peaks are selected: "passing notes" and "inner voices" are pruned. That is, the algorithm provides a criterion for associating nonadjacent notes by picking out those that are presumably most obvious to the ear [10].

Robert Morris [10] proposed 25 basic contour prime forms to which any contour could be reduced. Rob Schultz [13] proposed a refinement to this algorithm because some contours such as

[2]See Lewin [5] for further information regarding musical spaces.

[3]A contour point is also known as a contour pitch, or c-pitch.

[4]Do not confuse this with prime contours, as proposed by Robert Morris [10].

A < 2 1 3 0 > are not reducible to one of the basic primes by the Morris algorithm.

The equivalence classes and prime contours are useful for checking the identity of the contours for analytical and compositional purposes. For instance, the contours A < 0 1 3 2 > and B < 1 0 2 3 > belong to the same class, as represented by their prime form < 0 1 3 2 >. The B contour is obtained by inverting and retrograding the A contour. As another example of this, the contours C < 0 2 4 1 3 > and D < 0 3 2 1 > have a common prime: P < 0 2 1 >, as obtained by the Refined Contour Reduction Algorithm.

III. Refined Contour Reduction Algorithm (by Schultz)

The Refined Contour Reduction Algorithm [13] (cf. Algorithm 1) removes intermediary CPs and preserves salient ones with the support of an auxiliary internal structure called the *max-/min-list*. This algorithm has a preliminary component (Steps 1 to 5), as well as a recurrent one (Steps 6 to 17).

CONTOUR-REDUCTION-ALGORITHM(C), where C is a contour. Let variable N:

Step 0: Set N to 0.

Step 1: Flag all maxima in C upwards; call the resulting set the max-list.

Step 2: Flag all minima in C downwards; call the resulting set the min-list.

Step 3: If all c-pitches are flagged, go to step 6.

Step 4: Delete all non-flagged c-pitches in C.

Step 5: N is incremented by 1 (i.e., N becomes N + 1).

Step 6: Flag all maxima in the max-list upward. For any string of equal and adjacent maxima in the max-list, flag all of them, unless: (1) one c-pitch in the string is the first or last c-pitch of C, then flag only it; or (2) both the first and last c-pitches of C are in the string, then flag (only) both the first and last c-pitches of C.

Step 7: Flag all minima in min-list downward. For any string of equal and adjacent minima in the min-list, flag all of them, unless: (1) one c-pitch in the string is the first or last c-pitch of C, then flag only it; or (2) both the first and last c-pitches of C are in the string, then flag (only) both the first and last c-pitches of C.

Step 8: For any string of equal and adjacent maxima in the max-list in which no minima intervene, remove the flag from all but (any) one c-pitch in the string.

Step 9: For any string of equal and adjacent minima in the min-list in which no maxima intervene, remove the flag from all but (any) one c-pitch in the strings.

Step 10: If all c-pitches are flagged, and no more than one c-pitch repetition in the max-list and min-list (combined) exists, not including the first and last c-pitches of C, proceed directly to step 17.

Step 11: If more than one c-pitch repetition in the max-list and/or min-list (combined) exists, not including the first and last c-pitches of C, remove the flags on all repeated c-pitches except those closest to the first and last c- pitches of C.

Step 12: If both flagged c-pitches remaining from step 11 are members of the max-list, flag any one (and only one) former member of the min-list whose flag was removed in step 11; if both c-pitches are members of the min-list, flag any one (and only one) former member of the max-list whose flag was removed in step 11.

Step 13: Delete all non-flagged c-pitches in C.

Step 14: If N $\neq$ 0, N is incremented by 1 (i.e., N becomes N + 1).

Step 15: If N = 0, N is incremented by 2 (i.e., N becomes N + 2).

Step 16: Go to step 6.

Step 17: End. N is the "depth" of the original contour C.

Algorithm 1 Refined Contour Reduction Algorithm (by Schultz)

A *maximum* point exists if, in a triad of CPs, the middle CP is equal to or greater than its neighbors. By default, the first and last CPs in a sequence are also maxima. The maxima-list is calculated in a linear manner with Algorithm 2 (in pseudocode). This algorithm returns a max-list from a given sequence of CPs. The *minima* and min-list are calculated in an analogous way. For instance, the maxima- and minima-list of the contour S < 1 0 3 2 > are $M = [1, 3, 2]$, and $m = [1, 0, 2]$.

Algorithm 2 *Maxima* list algorithm

MAX-LIST(S), where S is a sequence of CPs, represented as $S = \{S_0, S_1, ..., S_{n-2}, S_{n-1}\}$ and n is its cardinality.

```
Let max-list
Add S_0 to max-list
for i from 0 to n − 2 do
   if (S_{i+1} ≥ S_i) ∧ (S_{i+1} ≥ S_{i+2}) then
      Add S_i to max-list
   end if
end for
Add S_{n−1} to max-list
return max-list
```

In the Reduction Algorithm, the CPs are recurrently flagged as *maxima* and *minima*, and the CPs that are not flagged are removed. Each loop iteration increases the algorithm depth value that represents the complexity of the contour reduction. The loop finishes when all CPs are flagged.

In the first part, the contour is used as the sequence of CPs for max- and min-list calculus. In the second part, the max- and min-lists themselves are used for the calculus.

As an illustration, consider the reduction of the contour A[5] < 1 3 0 2 0 2 0 2 0 2 0 3 1 > following Algorithm 1.

Step 0. $N = 0$.
Step 1. *Maxima* flag: $\{A_0, A_1, A_3, A_5, A_7, A_9, A_{11}, A_{12}\}$ (Figure 2a).
Step 2. *Minima* flag: $\{A_0, A_2, A_4, A_6, A_8, A_{10}, A_{12}\}$ (Figure 2b).
Step 3. Conditional: all flagged, jump to Step 6.
Step 6. First part: Flag *maxima* from max-list: $\{A_0, A_1, A_5, A_7, A_{11}, A_{12}\}$ (Figure 2c).

Second part: The repeated adjacent *maxima* sequence ($\{A_5, A_7\}$) does not involve either the first or the last CP. Thus, both are flagged.

Step 7. First part: Flag *minima* from min-list: $\{A_0, A_2, A_4, A_6, A_8, A_{10}, A_{12}\}$ (Figure 2c).

Second part: The adjacent *minima* sequence ($\{A_2, A_4, A_6, A_8, A_{10}\}$) does not involve either the first or the last CP. Thus, both are flagged.

Step 8. The repeated adjacent *maxima* sequence ($\{A_5, A_7\}$) contains an intervening *minima* (A_6). Thus, the flag is kept.

Step 9. The repeated adjacent *minima* sequence ($\{A_2, A_4, A_6, A_8, A_{10}\}$) contains a slice with two intervening *maxima* ($\{A_4, A_6, A_8\}$) and two other slices without it ($\{A_2, A_4\}$ and $\{A_8, A_{10}\}$). The slices without the intervening *maxima* have the flag removed. The resulting min-list is $\{A_0, A_4, A_6, A_8, A_{12}\}$ (Figure 2d).

[5] In this analysis, we are representing the *maxima* and *minima* using small triangles above and below the CPs, with the letters M (for *maxima*) and m (for *minima*) and the contour as a sequence $A = \{A_0, A_1, ..., A_{11}, A_{12}\}$.

Step 10. There are CPs not flagged ($\{A_2, A_3, A_9, A_{10}\}$), and the combined *maxima* and *minima* repeat ($\{A_4, A_5, A_6, A_7\}$). Go to Step 11.

Step 11. Remove the flags from the combined CPs, keeping the CPs near the beginning and end of the contour (Figure 2e).

Step 12. The CPs flagged in Step 11 ($\{A_4, A_7\}$) belong to the max- and min-lists and remain unchanged.

Step 13. Remove non-flagged CPs ($\{A_2, A_3, A_5, A_6, A_9, A_{10}\}$) (Figure 2f).

Step 14. $N = 0$. Maintain the CPs as being unchanged.

Step 15. $N = 0$. Increment of 2 units: $N = 2$.

Step 16. Return to Step 6.

The second iteration begins with the CPs, *maxima* and *minima* flags illustrated in the figure 3a[6]

Step 6. First part: Flag *maxima* from max-list: ($\{A_0, A_1, A_{11}, A_{12}\}$) (Figure 3b).

Second part: The repeated adjacent *maxima* sequence ($\{A_1, A_{11}\}$) does not involve either the first or the last CP. Thus, all *maxima* are flagged.

Step 7. First part: Flag minima from min-list: ($\{A_0, A_4, A_8, A_{12}\}$).

Second part: The repeated adjacent *minima* sequence ($\{A_4, A_8\}$) does not involve either the first or the last CP. Thus, all *minima* are flagged.

Step 8. The adjacent repeated *maxima* sequence ($\{A_1, A_{11}\}$) has two *minima* intervene ($\{A_4, A_8\}$).

Step 9. The adjacent repeated *minima* sequence ($\{A_4, A_8\}$) has not yet had the *maxima* intervene. The flag of the one repeated *minima* (A_8) is removed (Figure 3c).

Step 10. There are CPs that are not flagged ($\{A_7, A_8\}$). Go to Step 11.

Step 11. There is no combined CP repetition.

Step 12. There is no combined CP repetition.

Step 13. Remove non-flagged CPs ($\{A_7, A_8\}$) (Figure 3d).

Step 14. $N = 2$. Increment of 1 unit: $N = 3$

Step 15. $N = 3$. Maintain the contour as being unchanged.

Step 16. Return to Step 6.

The third iteration begins with the CPs, *maxima* and *minima* flags illustrated in the figure 4: $\{A_0, A_1, A_{11}, A_{12}\}$ and $\{A_0, A_4, A_{12}\}$.

Step 6. First part: Flag *maxima* from max-list: ($\{A_0, A_1, A_{11}, A_{12}\}$) (Figure 4).

Second part: The repeated adjacent *maxima* sequence ($\{A_1, A_{11}\}$) does not involve either the first or the last CP. Thus, all *maxima* are flagged.

Step 7. First part: Flag *minima* from min-list: ($\{A_0, A_4, A_{12}\}$) (Figure 4).

Second part: There are no repeated adjacent *minima* sequences.

Step 8. The repeated adjacent *maxima* sequence contains an intervening *minima.* Thus, the flag is kept.

Step 9. There are no adjacent repeated *minima* sequences.

Step 10. All CPs are flagged, and there are no combined *maxima* and *minima* repetitions. Jump to Step 17.

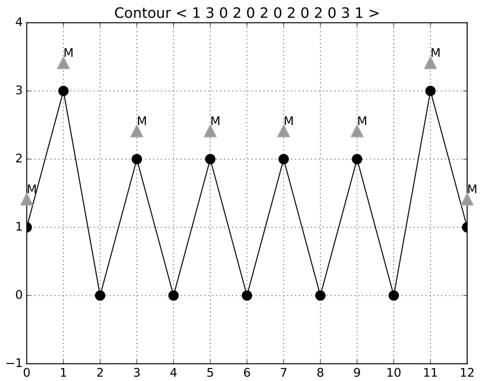
Step 17. The contour A has a depth of 3, is reduced to < 1 3 0 3 1 >, and is normalized to < 1 2 0 2 1 >.

i. Reduction algorithm review

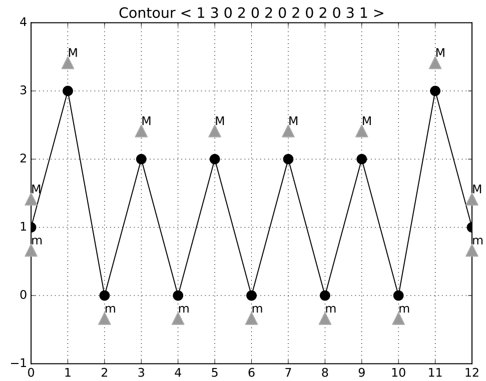
There are two problems in this algorithm:

1. Combined adjacent *maxima* and *minima* repetition is in Step 10.

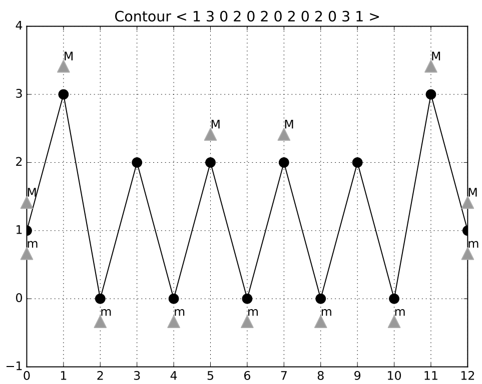
[6]For simplification reasons, we keep the original sequence index.



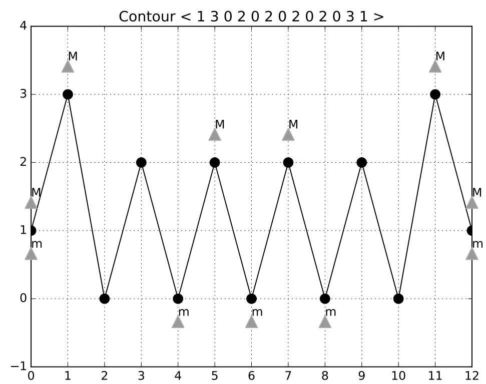
(a) *Step 1*



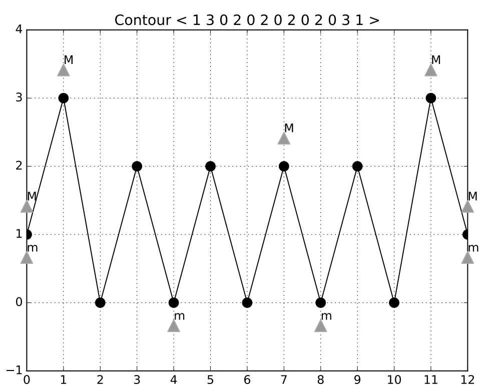
(b) *Step 2*



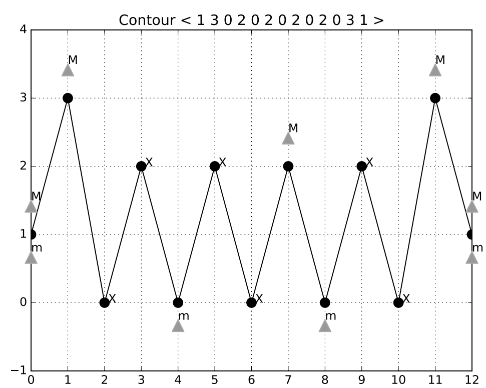
(c) *Steps 6–8*



(d) *Steps 9 and 10*

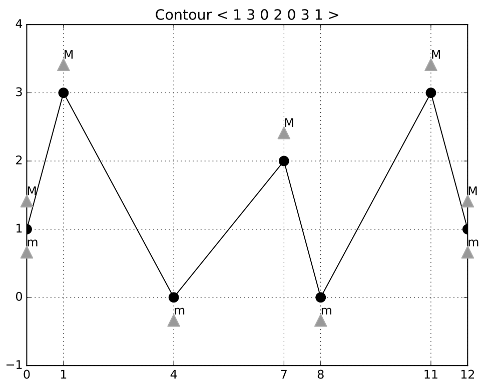


(e) *Steps 11 and 12*

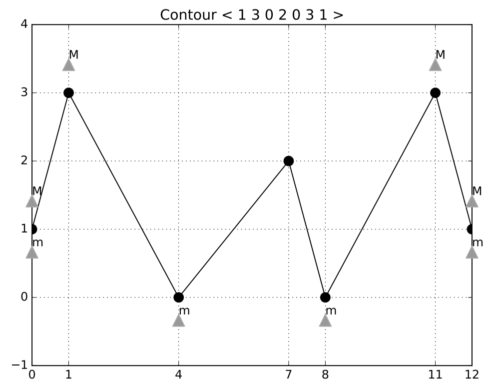


(f) *Steps 13–16*

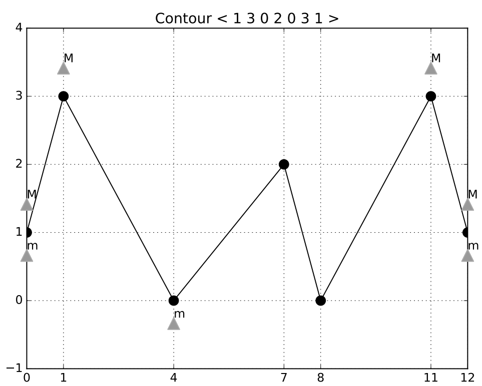
Figure 2: *Reduction algorithm flags in the first iteration*



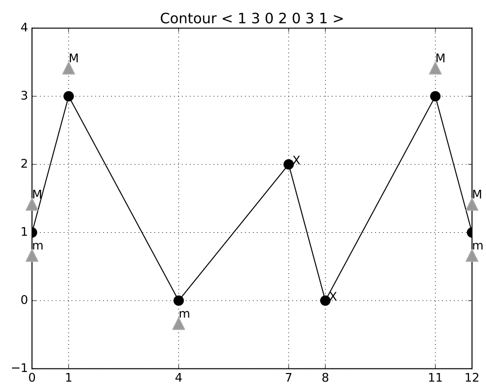
(a) *Initial state*



(b) *Steps 6–8*



(c) *Steps 9–12*



(d) *Steps 13–16*

Figure 3: *Reduction algorithm flags in the second iteration*

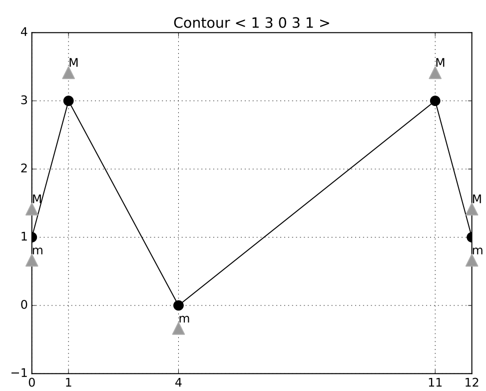


Figure 4: *Reduction algorithm flags in the third iteration*

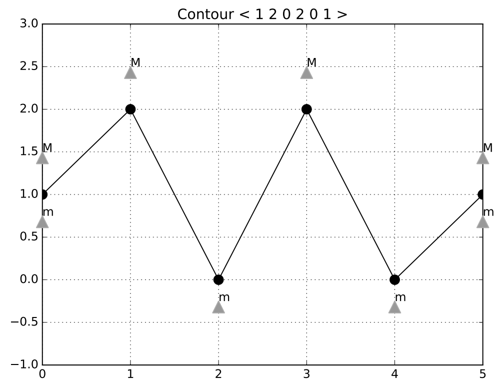


Figure 5: *Combined adjacent maxima and minima repetitions*

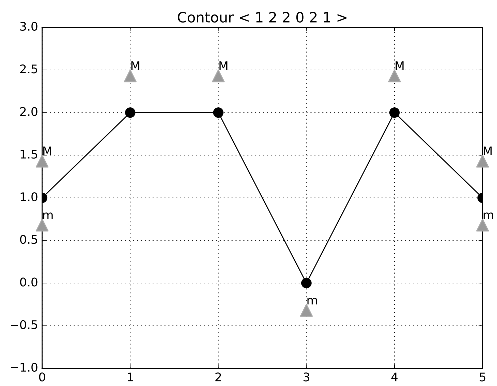


Figure 6: *Contour < 1 2 2 0 2 1 > and flagged maxima and minima.*

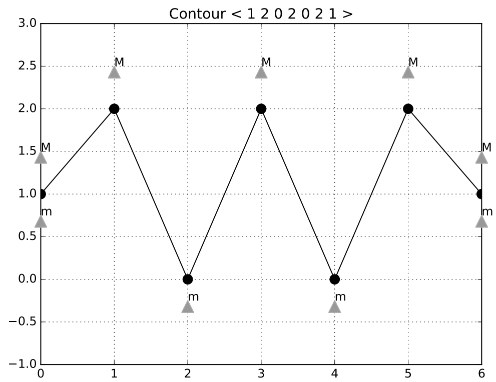
2. There is indefiniteness in the repeated *maxima* and *minima* sequences of Steps 8 and 9.

Step 10 contains two conditions with two possible results: continue with Step 11 or jump to the algorithm's end at Step 17. In the second condition, the expression "no more than one" leads to an error because prime contours cannot have combined *maxima* and *minima* repetitions, and this expression allows for at least one. Thus, the algorithm must not end while these repetitions are in the contour. For instance, the contour A < 1 2 0 2 0 1 > (Figure 5) has all CPs flagged and only one max-min combined repetition of CPs—2 and 0. Thus, in Step 10, the algorithm ends, and the repetition is not removed.

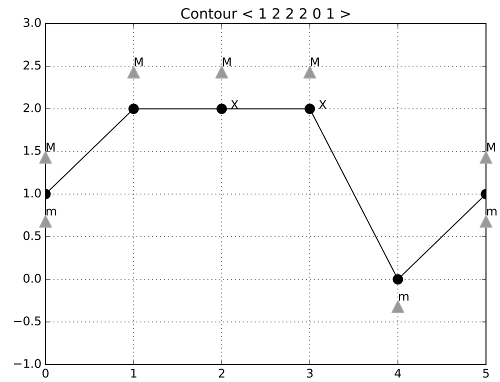
To fix this problem, this step should be rewritten as "and there are no repeated combined CPs in terms of the *maxima* and *minima*."

The expression "string of equal and adjacent *maxima*" in Step 8 is not precise concerning the length of the string. For instance, the contour A < 1 2 2 0 2 1 > (Figure 6) has three repeated *maxima*. There are two possible interpretations of "string" in this example: the unique sequence with three repeated *maxima* $\{A_1, A_2, A_4\}$, or each of the two sequences of repeated *maxima* $\{A_1, A_2\}$ and $\{A_2, A_4\}$.

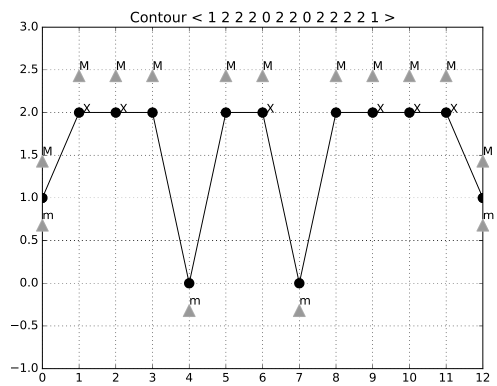
Considering the unique sequence $\{A_1, A_2, A_4\}$ as the string, there is an intervening *minima*



(a) *with minima between the maxima*



(b) *without minima between the maxima*



(c) *with and without minima between the maxima*

Figure 7: *Contour with and without minima between maxima*

(A_3), and we should move to Step 9. Considering each pair of repeated *maxima* as a string, there is no intervening *minima* in the first pair, and we should remove the flag from one of these *maxima*.

In the first case, with a unique string, the algorithm ends with the unchanged contour < 1 2 2 0 2 1 >, and with the repeated *maxima*. In the second case, with two strings, one of the *maxima* is removed, and the algorithm finishes with the correct prime contour < 1 2 0 2 1 >.

There are three particular situations for the equal and adjacent *maxima* in maxima-list:

1. with *minima* between the *maxima* (Figure 7a).
2. without *minima* between the *maxima* (Figure 7b).
3. with slices with and without *minima* between *maxima* in the same string (Figure 7c).

In the first situation, all the *maxima* flags must be kept; in the second, only the first *maxima* must keep it flag; and in the third situation, in the slice without *minima* between the *maxima*, these *maxima* must be unflag, and, in the slices with *minima* between *maxima*, the *maxima* adjacent to the intervening *minima* must be kept. In this third situation, there are cases where two equal adjacent *maxima* are between two *minima*. In these cases, the second *maxima* must be unflag (See C_6 in the figure 7c).

There is a similar problem in the analogous Step 9, related to *maxima* between equal adjacent *minima*.

We propose a new version of the algorithm with these questions reviewed (Algorithm 3).

CONTOUR-REDUCTION-ALGORITHM-REVIEW(C), where C is a contour.

Let variable N.

First part

Step 0: Set N to 0.

Step 1: Flag all *maxima* in C upwards, and call the resulting sequence the max-list; flag all *minima* in C downwards, and call the resulting sequence the min-list. All modification in a max- or min-list will result in a new max-/min- list.

Step 2: If all CPs are flagged, go to Step 4.

Step 3: Delete all non-flagged CPs in C, and increment N by 1 (i.e., N becomes $N + 1$).

Second part

Step 4: Flag all *maxima* in the max-list upward, and flag all *minima* in the min-list downward.

Step 5: If there are no adjacent repeated *maxima* in max-list and *minima* in min-list, go to Step 10.

Step 6: For any string of equal and adjacent *maxima* in the max-list:

if First and last CP are present in the string **then**
  Flag them.
else if First or last CP are present in the string **then**
  Flag it.
else
  Flag all *maxima* in the string.
end if

Step 7: For any string of equal and adjacent *minima* in the max-list:

if First and last CP are present in the string **then**
  Flag them.
else if First or last CP are present in the string **then**
  Flag it.
else
  Flag all *minima* in the string.
end if

Step 8: For any string of equal and adjacent *maxima* in max-list:

if There is no *minimum* between the *maxima* **then**
  Flag the first *maxima* and unflag the others.
else if There are *minima* between all the *maxima* **then**
  Flag all the *maxima*.
else
  Flag the *maxima* that are adjacent to the *minima* and unflag the others.
  if There is two adjacent *maxima* between two *minima* **then**
    Unflag the second *maxima*.
  end if
end if

Step 9: For any string of two equal and adjacent *minima* in the min-list:

if There is no *maximum* between the *minima* **then**
Flag the first *minima* and unflag the others.
else if There are *maxima* between all the *minima* **then**
Flag all the *minima*.
else
Flag the *minima* that are adjacent to the *maxima* and unflag the others.
if There is two adjacent *minima* between two *maxima* **then**
Unflag the second *minima*.
end if
end if

Step 10: If there is no CP repetition in the max-list and min-list (combined), not including the first and last CPs of C, go to Step 14.

Step 11: Remove the flags on all repeated CPs except those closest to the first and last CP of C.

Step 12: If both flagged CPs remaining from Step 11 are members of the max-list, flag any one (but only one) former member of the min-list whose flag was removed in Step 11.

Step 13: If both flagged CP remaining from Step 11 are members of the min-list, flag any one (but only one) former member of the max-list whose flag was removed in Step 11.

Step 14: If all CP are flagged, go to Step 17.

Step 15: Delete all non-flagged CP in C and, if $N = 0$, increment N by 1 (i.e., N becomes $N + 1$); otherwise, increment N by 2 (i.e., N becomes $N + 2$).

Step 16: Go to Step 4.

Step 17: End. N is the "depth" of the original contour C.

Algorithm 3 Reviewed Contour Reduction Algorithm

IV. Equivalence Contour Class Prime Form

The equivalence contour class prime form is obtained with the algorithm proposed by Marvin and Laprade [7] (Algorithm 4).

Algorithm 4 Equivalence Contour Class Prime Form Algorithm

EQUIVALENCE-CONTOUR-CLASS-PRIME-FORM(cseg), where cseg is contour:
Step 1: If necessary, translate the cseg so its content consists of integers from 0 to $(n - 1)$,
Step 2: If $(n - 1)$ minus the last c-pitch is less than the first c-pitch, invert the cseg,
Step 3: If the last c-pitch is less than the first c-pitch, retrograde the cseg.

The main feature of this prime form is the compactness in the contour beginning. This feature is inspired by the Post-Tonal Theory's set class prime form. The Contour Interval Succession[7] is a useful tool for how compact the beginning of the contour is. For instance, the contour A < 0 2 1 3 4 > belongs to the same class of its retrograde, inverted, and retrograded/inverted versions: R(A) < 4 3 1 2 0 >, I(A) < 4 2 3 1 0 > and RI(A) < 0 1 3 2 4 >. Table 2 contains the interval successions of these contours. The RI (A) version starts with the lowest positive value of the Contour Interval Succession.

Marvin and Laprade's algorithm has some conditions to invert and/or retrograde the contour to find the prime form of its equivalent class.

[7] A sequence with the differences (or contour intervals) between adjacent CPs [4].

Table 2: *Contour Interval Succession of contour A < 0 2 1 3 4 >, R(A), I(A) and RI(A)*

Contour	Contour Interval Sucession
A < 0 2 1 3 4 >	$[2, -1, 2, 1]$
R(A) < 4 3 1 2 0 >	$[-1, -2, 1, -2]$
I(A) < 4 2 3 1 0 >	$[-2, 1, -2, -1]$
RI(A) < 0 1 3 2 4 >	$[1, 2, -1, 2]$

i. Algorithm review

This algorithm, however, fails in 28 of 235 equivalent classes according to the Marvin and Laprade table [7] (cf. prime form fails in Table 3). Each of these classes has two prime forms. For instance, both contours A < 0 1 3 2 4 > and RI(A) < 0 2 1 3 4> belong to the *cseg-class* 5-3, but according to the Marvin/Laprade algorithm, each one has a particular prime form. They are not changed by the algorithm steps.

Let the processing of the two contours A and RI(A) occur while comparing each step:

Step 1: Both contour A and RI(A) are normalized. Maintain the contour as being unchanged.

Step 2: Both contours have the same cardinality ($N = 5$), and the same last CP (4). Therefore, the condition $(5 - 1) - 4 < 0$ is false. Maintain both contours as being unchanged.

Step 3: Both contours have the same first (0) and last CP (4). Thus, the condition $4 < 0$ is false. Maintain both contours as being unchanged.

Hence, this algorithm fails with some classes. To fix this problem, we propose a new algorithm that is tested using the Marvin/Laprade table (Algorithm 5).

Algorithm 5 Equivalence Contour Class Prime form Algorithm Reviewed

EQUIVALENCE-CONTOUR-CLASS-PRIME-FORM-REVIEW(cseg), where cseg is the contour:
Let the array *arr*:
Step 1: Normalize *cseg*.
Step 2: Get the result of inversion (I), retrogression (R), and retrogression-inversion (RI) and add them to *arr*.
Step 3: Sort the array *arr*.
Step 4: The prime form is the first contour of the array *arr*.

V. Conclusions

In this paper, we revealed problems with the Refined Contour Reduction and the Equivalence Class Prime Form algorithms that lead them to fail with some contour inputs. We demonstrated how and why these algorithms fail, as well as how to fix the issues, and we proposed alternative algorithms.

The problems with the Refined Contour Reduction Algorithm raised here do not allow for an output of the correct reduction of some of the contours with combined maxima-minima like < 1 2 0 2 0 1 >, and with strings of repeated maxima (or minima) with intervening minima (or maxima), such as < 1 2 2 0 2 1 >. However, a review of the string definition and a small change in the algorithm's tenth step fixed these problems.

The problem with the Equivalent Class Prime Form Algorithm led to it returning two prime forms for the same class in 28 of the 235 Marvin/Laprade classes, as shown in the table showing the same [7]. The alternative algorithm proposed in this paper fixed this problem.

Table 3: *Equivalent Class with Two Prime Forms*

Cseg class	FP correta	FP incorreta
5-3	< 0 1 3 2 4 >	< 0 2 1 3 4 >
5-8	< 0 2 3 1 4 >	< 0 3 1 2 4 >
5-25	< 1 0 4 2 3 >	< 1 2 0 4 3 >
5-27	< 1 2 4 0 3 >	< 1 4 0 2 3 >
6-3	< 0 1 2 4 3 5 >	< 0 2 1 3 4 5 >
6-9	< 0 1 3 4 2 5 >	< 0 3 1 2 4 5 >
6-13	< 0 1 4 2 3 5 >	< 0 2 3 1 4 5 >
6-15	< 0 1 4 3 2 5 >	< 0 3 2 1 4 5 >
6-31	< 0 2 3 4 1 5 >	< 0 4 1 2 3 5 >
6-37	< 0 2 4 3 1 5 >	< 0 4 2 1 3 5 >
6-53	< 0 3 2 4 1 5 >	< 0 4 1 3 2 5 >
6-59	< 0 3 4 2 1 5 >	< 0 4 3 1 2 5 >
6-115	< 1 0 2 5 3 4 >	< 1 2 0 3 5 4 >
6-119	< 1 0 3 5 2 4 >	< 1 3 0 2 5 4 >
6-125	< 1 0 5 2 3 4 >	< 1 2 3 0 5 4 >
6-127	< 1 0 5 3 2 4 >	< 1 3 2 0 5 4 >
6-134	< 1 2 3 5 0 4 >	< 1 5 0 2 3 4 >
6-139	< 1 2 5 3 0 4 >	< 1 5 2 0 3 4 >
6-144	< 1 3 2 5 0 4 >	< 1 5 0 3 2 4 >
6-149	< 1 3 5 2 0 4 >	< 1 5 3 0 2 4 >
6-178	< 2 0 1 5 4 3 >	< 2 1 0 4 5 3 >
6-180	< 2 0 4 5 1 3 >	< 2 4 0 1 5 3 >
6-181	< 2 0 5 1 4 3 >	< 2 1 4 0 5 3 >
6-182	< 2 0 5 4 1 3 >	< 2 4 1 0 5 3 >
6-184	< 2 1 4 5 0 3 >	< 2 5 0 1 4 3 >
6-186	< 2 1 5 4 0 3 >	< 2 5 1 0 4 3 >
6-188	< 2 4 1 5 0 3 >	< 2 5 0 4 1 3 >
6-190	< 2 4 5 1 0 3 >	< 2 5 4 0 1 3 >

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Sound Shizuku Composition: a Computer-Aided Composition System for Extended Music Techniques

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Abstract

Abstract: *We discuss in this paper a new environment for computer aid musical composition which is designed to create works centered on the creative use of instrumental extended techniques. The process is anchored on computational techniques to retrieve musical information via audio descriptors. We developed an analytical process, based on the extraction of spectral characteristics of a Sound DataBase (SDB), and on supporting the compositional planning as follows: relate statistical measures to the spectral behavior of specific execution modes of various instruments contained in the SDB. The result of the process is a palette of possibilities that assists the composer decisions regarding to the desired orchestration to be applied in a musical piece. The paper presents then the motivation and context to develop the environment, describes and characterizes the audio descriptors that have been studied, presents the computer system architecture and discusses the results obtained with Sound Shizuku.*

Keywords: *Composition. Computer-Aided Orchestration. Audio Descriptors. Extended Techniques. Interdisciplinary Music Computation.*

I. Introduction

Among the contemporary music compositional techniques, some of them can touch upon the control factors related to musical timbre [1] and significantly alter the spectral characteristics of each single note heard. It could be compared to a palette of color where mixed extended instrumental techniques produce new shades and, finally, create new orchestral sounds. In line with the use of timbre as a potential space for composition, there is an increasingly concern with

[1]The issues related to the term 'tone' as used in this paper, exceeds the definition by 'exclusion to' that timbre is an identification of property and distinction, whose sound sources have the same intensity and pitch. We attribute the term 'timbre' a spectral morphological identity, as discussed by Smalley [37] .

getting more refined and particular timbre results, both for the compositional planning and for the instrumental/vocal realization. The idea of timbre as a 'metadimension' [14, p. 45] shows the interest to consider it not as a simple 'color' but as a potential space for integration of other musical features and thus become the central focus of the composition. From the artistic point of view, the timbre is a concept linked to the *modus operandi* of musical language, concurrent to aesthetics and musical form. Nevertheless there is still a fundamental issue centered on the difficulty of relating to a 'musician qualitative intuition' on timbre with a 'quantitative assessment' of possible categories of measures and objective analysis of the musical timbre behavior [5, p. 162].

This paper establishes a dialogue between the study of musical timbre, as poetic and musical approach, to a scientific point of view. More specifically, we work with recent studies on music information retrieval based on spectral content that are the inner microstructures of musical timbre and therefore might help the development of a more refined and conscious compositional planning. This view has its origins in the pioneering research of Hermann von Helmholtz whose treatise related timbre to the presence and the magnitude of spectral components with respect to its fundamental component [15]. This study provided important subsidies to timbre analysis focused on the spectral characteristics of the sound [26, 31, 13]. Other researches from Berger [4] and form Wedin and Goude [41] pointed to a correlation between the accuracy of timbre recognition with the attack and decay time of the sound source. As for Pierre Schaeffer, the timbre of a sound is perceived by the variation of its spectral behavior and its evolution in time [28]. Schaeffer was the pioneer by separating the physical phenomenon of the sound of his own perception phenomenological.

Based on these concepts we present a man-machine interaction methodology that connects computer aid sound analysis with the symbolic notation of a music score. We conducted a study on musical information retrieval via low-level audio descriptors that are centered on feature extraction of sound frequency spectrum. In this sense, using audio features as composition architectural tools, two approaches to aid the compositional planning were developed: *a)* extract from sound frequency spectra specific features *b)* relate them to modes for the extended instrumental techniques, including transcription to symbolic music information and music orchestration. For this goal we have developed an virtual analytical environment that recommends orchestral sonorities called *Sound Shizuku Composition – SSC*. To present this environment and its compositional implications, in Section II we discuss the main stages of the sound analysis and music orchestration assisted by computer. On Section III practical results are briefly discussed. Finally, we conclude our article in Section IV discussing forthcoming projects.

II. Architecture of the Methodology

The scope of the computer-assisted music orchestration system presented here is to apply audio descriptors to provide a pallet of contrasting timbre variations. The goal is to produce a refined blending of sounds derived from set of extended techniques. Therefore the creative process relates sounds, described by audio descriptors, and instrumental settings to transcriptions of these relations into a music score. Finally, the transcriptions improve the original compositional planning in face of the computer aid orchestration. The first step developed here was to build 'Sound Mixtures' that can be defined as computer simulations to generate audio files that will expand possibilities of instrumental mixes. Sound Mixtures, are generated by superimposing modes of playing, articulations and various extended instrumental techniques storage as audio samples in the Sound DataBase (SDB).

Secondly the mixtures are analyzed with audio descriptors in order to extract their related spectral features. Section i presents the audio descriptors used to process that extraction. Figure 1 is the general outline of the proposed methodology in our research into computer-assisted music orchestration.

i. Audio Descriptors Technical Definitions

In this section i we discuss the use of audio descriptors to provide sound analysis capability to the music orchestration system. We introduce only audio descriptors that were studied in our research. The scientific knowledge area on this subject is called Music Information Retrieval or simply MIR [6, 7, 29, 38]. Studies on MIR use mathematical functions, supported by statistical measurements and psychoacoustic models to proceed the so-called audio features extraction. According to [22, p. 01] the methodology to describe the characteristics of a sound signal have been proposed by the scientific community to recognize patterns of speech and musical instrument classification. These procedures are also significant tools on the context of musical composition and orchestration. Several methods for analyzing the spectral content of digitized audio signals are performed by Short-Time Fourier Transform or STFT, which is defined as follows by Sheh and Ellis [30, p. 02]:

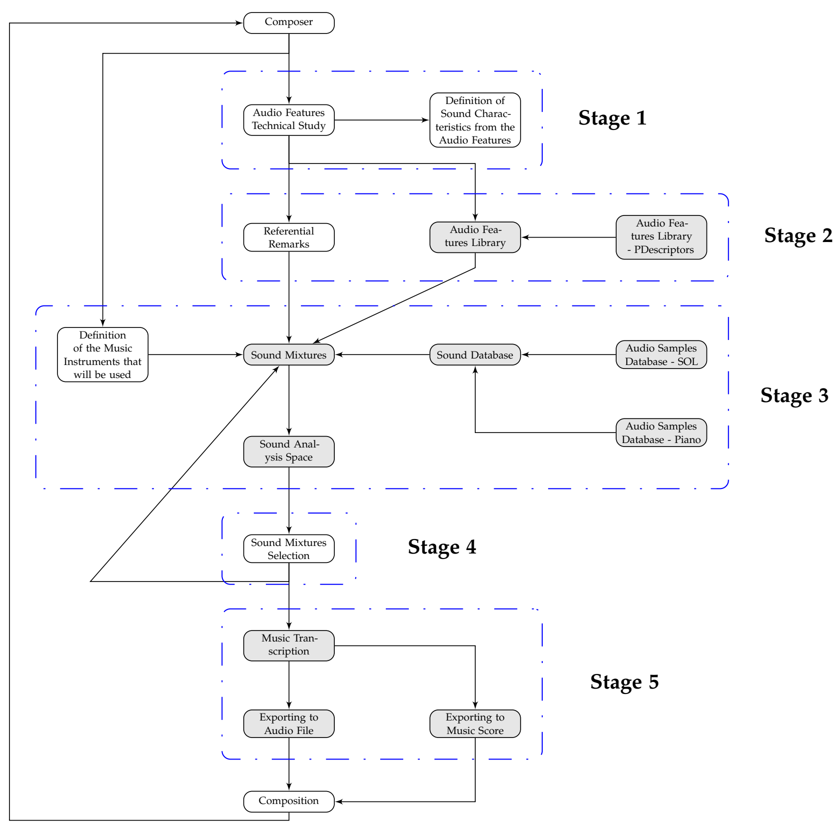


Figure 1: *General scheme of the computer-assisted music orchestration* Sound Shizuku Composition - SSC. *The gray blocks represent the computational flow data for sound analysis. The white blocks represent user interaction with the system itself. The dotted and dashed blocks comprise the tasks of each of the five steps of our methodology architecture.*

$$\mathbf{STFT}_{[\mathbf{k,n}]} = \sum_{m=0}^{N-1} x[n-m]w[m]e^{-j2\pi km/n} \tag{1}$$

Table 1: *Summary of the audio features.*

Feature	Definition	Sound Correlative	Possible Application
Spectral Centroid	Center of the Mass	Brightness/Opacity	Sound Detection Centroidariation
Spectral Standart Deviation	Spectral Bandwidth	Sound Mass	Spectral Bands Equalization
Spectral Skewness	Asymmetry or Obliquity	Hot and Rounded/Bright and Penetrating	Detection of percussion instruments
Spectral Kurtosis	Flattening of the Distribution	Noise	Transient Detection
Spectral Flux	Time Attack	Attack	Detection of Sound Events
Spectral Flatness	Ratio of Geometric Mean with Arithmetic Mean	Noise/Tone	Noise Removal
Spectral Irregularity	Difference Magnitude Spectrum	Velvety and Smooth/Rough and Ribbed	Spectral Band Equalization
Spectral Roll-Off	Spectral Slope Envelope	Roughness	Mastering Voice and Music
Odd-to Even Ratio	Quotient of the Magnitude of the Spectral Components	Nasal/Soft	Detection of Musical Intensities
RMS Energy	Root Mean Square of the Energy	Strong/Weak	Detection of Sound Intensities
Loudness	Auditory Sensation of Sound Intensity	Strong/Weak	Sound Intensity Perception
Zero-Crossing Rate	Signal Changes in Time	Noise	Sound Noise Detection
Spectral Decreasing	Energy Spectrum	Percussion Sounds	Detection of Percussive Sounds
Temporal Centroid	Temporal Center of the Mass	Percussion Sounds	Detection of Percussive Sounds
Spectral Chroma	Spectrum Analysis by Musical Pitches	Tonality	Harmony Identification

where k indexes the frequency axis with $0 \leq k \leq N-1$, n is the short-time window center, and $w[m]$ is an N-point Hanning window.

From the widespread view in the area of MIR, audio descriptors are tools for sound analysis and most of them are represent by one-dimensional curves. As pointed out by Rimoldi [27, p. 01], the audio features are useful tools for a taxonomy of features related to the spectral content of the analyzed sound signal even though with their reductionist characteristics in relation to the analyzed object. Such features can be correlated and not necessarily equivalent with subjective attributes of the perception of the sound signal, such as 'brightness', 'opacity', 'roughness', 'noisiness', 'softness', among others.

To our research we use a set of fifteen audio features: Spectral Centroid [39, pp. 460-461], Spectral Standard Deviation [9, 27], Spectral Skewness [9], Spectral Kurtosis [1], Spectral Flux [22, 24], Spectral Flatness [8, p. 01], Spectral Irregularity [16], Spectral Roll-Off [19, p. 47], Odd-to Even Ratio [22], RMS Energy [17, p. 113], Loudness [42, 10, 20, 40, 25], Zero-Crossing Rate [24, 21], Spectral Decreasing [18], Temporal Centroid [23] and Spectral Croma [11, 12]. Such statistical measures estimate particular characteristics of a digital audio signal. As already pointed out, audio descriptors are powerful tools for the creation of a taxonomy of spectral characteristics. This taxonomy can be correlated but not necessarily equivalent to the subjective attributes of the human perception. Table 1 summarizes the main highlighted points for the audio descriptors. In it, we summarized the presentation of the features with their possible applications.

ii. Sound DataBase - SDB

The audio samples used to generate Sound Mixtures belong to two databases compiled by Ballet *et. al* [2] and Barbancho *et. al* [3]. Such samples have durations between five to seven seconds in .aiff audio format. In Ballet research called Studio OnLine or SOL the repository of instrumental sonorities relates to 'some aspects of the sound of contemporary instrumental music' [2, p. 124]. In total, the SOL database has 16 musical instruments such as accordion, tuba, bassoon, clarinet, trumpet, contrabass, alto saxophone, flute, guitar, harp, horn, oboe, trombone, violin, viola and cello. The collection of samples includes some extended instrumental techniques.

The database belonging to Barbancho [3], focuses on piano sounds. The research covers an extensive study on piano sounds, from a single note to a whole chord with up to ten simultaneous notes. There are several recordings of the piano in different registrations, intensities in *staccato* and *ordinary* playing techniques with the presence or absence of the damper pedal. In both databases, there are three different musical dynamics: *pianissimo* or *p*, *mezzo-forte* or *mf* and *fortissimo* or *ff*. In the current version of our research we chose to use the piano audio samples playing only the one single note. The current version of our database (SDB) has an approximate size of 30 GB[2]. Following Section iii describes the main steps that established the construction the sound analysis and orchestration environment, named as *Sound Shizuku Composition*

iii. Sound Analysis Environment - *Sound Shizuku Composition - SSC*

The *Sound Shizuku Composition* or *SSC* [3] was built in modules that provide orchestration cues using the SDB, described in Section i. *SSC* was developed in Pure Data (PD) using a library of audio descriptors developed by Monteiro [21] at the Interdisciplinary Nucleus for Sound Studies (NICS). Next Subsections discuss each of the modules and all the other computational routines that was also implemented in Pure Data. There are seven modules as follows:

- Module 1 - Selection of musical instruments and the desired instrumental techniques
- Module 2 - Define orchestration blending to be evaluated by audio features
- Module 3 - Calculation of orchestration algorithm of sonorities
- Module 4 - Selection of the audio descriptors
- Module 5 - Analysis of sonorities via audio descriptors
- Module 6 - Interaction and choice of sound mixtures arranged in the GUI visual cues
- Module 7 - Selection of output formats of sound mixtures in audio format and musical score transcription

iii.1 Module 1 - Selection of musical instruments and instrumental techniques

In the first stage the composer defines the desired musical instrumentation from a total 16 choices of musical instruments. Choices of instrument are repeated in such way that a selection of an instrument is followed by the choice of an instrumental techniques. The current version of *SSC* does not allow selection of the same instrument, that is, the system enables only one flute, one clarinet, one trumpet, one tuba etc. Figure 2 illustrates the Step 1.

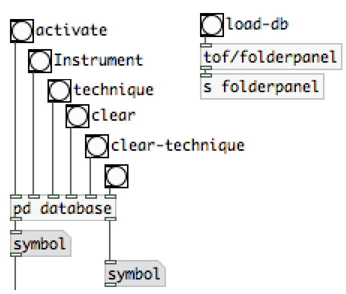


Figure 2: *Figure of Module 1. To startup the system is necessary to load the database using the* load-db *in the upper right corner.*

iii.2 Module 2 - Define orchestration blending

In this module, the composer is able to restrict the amount of Sound Mixtures (SM) to reduce computer calculation when search and analyse mixtures. We also implemented a restriction

[2]Because of its size, we can not attach the sound database. It is suggested to contact the author to get the current version of the sound database. email: mieysimurra@gmail.com

[3]The term *Shizuku* is Japanese for water drop.

algorithm for searching orchestration solutions based on the presence of a pitch profile using the Spectral Chroma, audio descriptors. This procedure ensures that the SM are restricted to a certain pitch or at least to the presence of a specific musical time. It is possible to use the pitch profile to calculate a percentual pitch presence. The algorithm calculates the presence in the range $[0, ..., 1]$. When presence is 100 %, the search algorithm process the orchestral indication with the greatest pitch influence. If the user do not indicate the presence of pitch the search algorithm performs the selection of the SM randomly. This second possibility was accomplished with the use of the function *urn*, in Pure Data. Next, Figure 3 illustrates the module 2 showing the quantity of orchestral blending, given specific pitch and its percentage of presence.

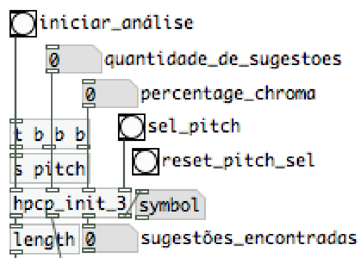


Figure 3: *Figure of module 2 of the* SSC *system*

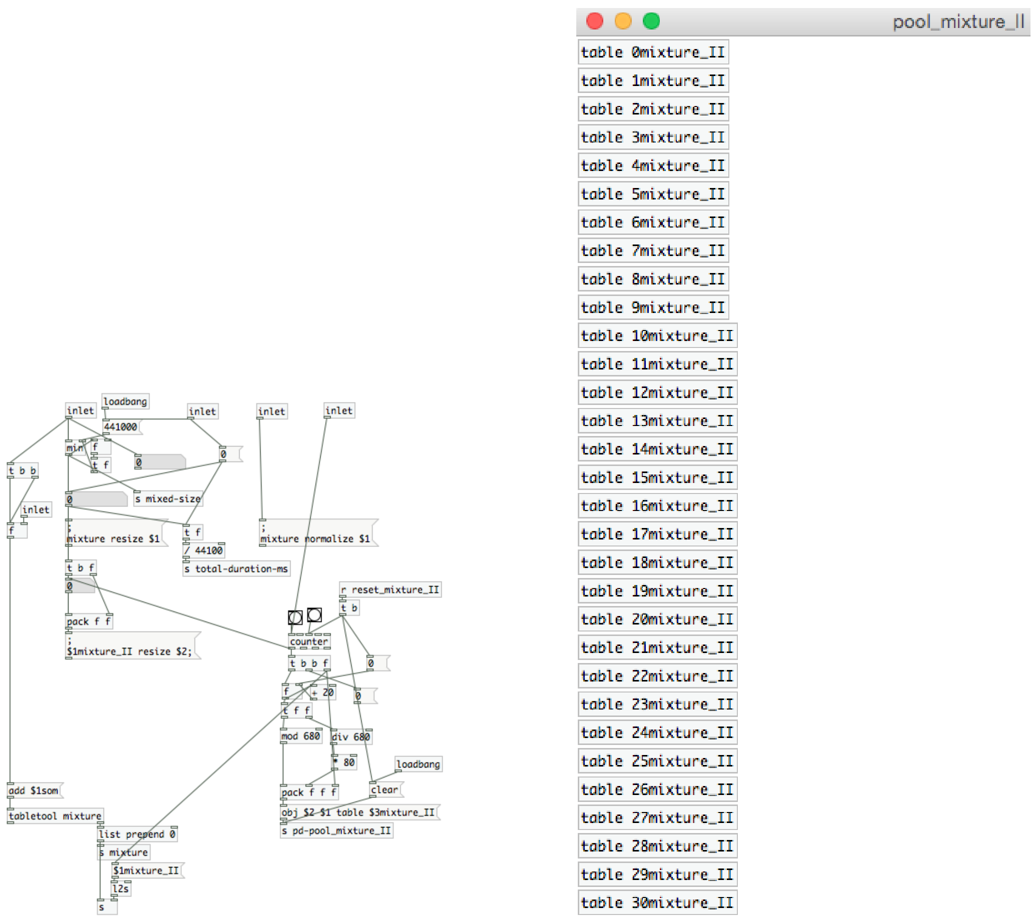
iii.3 Module 3 - Orchestration

The orchestration step uses the pitch presence, defined in the previous section, to perform overlays of audio files from the sound database (SDB). This routine is performed using the object *tabletool*, from *TimbreID* library, developed by William Brent [4]. Each audio file is edited so that the overlapping is performed on files with the same length. For this, we use the object *min*, from Pure Data (PD), which identifies the smallest window of the data collected. The overlays are rendered and stored in tables that will be used to extract the audio features. Figure 4a presents the overlay algorithm of audio samples defined by the Module 1. The Figure 4b, represents the audio samples *corpus*.

iii.4 Module 4 - Selecting Audio Features

After establishing the *corpus* of sound mixtures, sound analysis is conducted. In total, it is used a set of four pairs of features which are arranged in a two dimensional space, a coloured graphic display. As discussed in Section i, audio descriptors project the retrieved information of the sound spectrum to one-dimensional curves. However, as discussed in the Introduction, timbre is a perceptual feature which has several parametric dimensions. In order to help the composer to expand the analysis scope on specific sonic characteristics, a set of four pairs of audio descriptors is present in a graphic display. This tool enabled a refined detailing of various sound characteristics. Section i then presents the available audio features in the current stage of our system. Figure 5 illustrates the selection of the four pairs of audio features. Indications 'x' and 'y', below each feature represent their disposal in the operating interface of Sound Mixtures performed by Module 3.

[4]For more information about the *TimbreID*, see: <http://williambrent.conflations.com/pages/research.html.>



(a) (b)

Figure 4: *Figure for the audio samples overlays. In Figure 4a, the process is performed by the object* tabletool*, from the* TimbreID *library. Figure 4b, the overlays are stored in the* corpus *named* mixture_II*. The* corpus *will be analyzed by audio features in Module 4.*

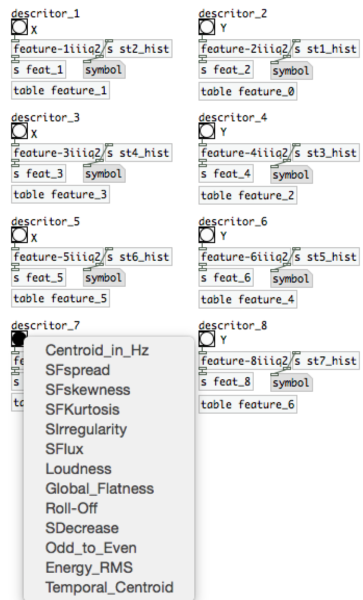


Figure 5: *Figure of the Module 4 in which the user can select the set of four pairs of audio features that will analyze the sound mixtures.*

iii.5 Module 5 - Sound Mixtures Analysis

In Module 5, the system performs the sound mixtures analysis via audio descriptors. The features are based on the suitable *PDescriptors* library developed at NICS/UNICAMP [21]. It calculates the mean of the extracted values of each audio feature. These means are accumulated in a list of data to be arranged in a space for exploration and analysis. Figure 6a represents one of the four pairs of the features chosen in module 4. In this *patch* the data analysis are collected. The mean of the data are stored in sub-module *pd accum-symbol*. These means are arranged in the space of operation which will be described in Module 6.

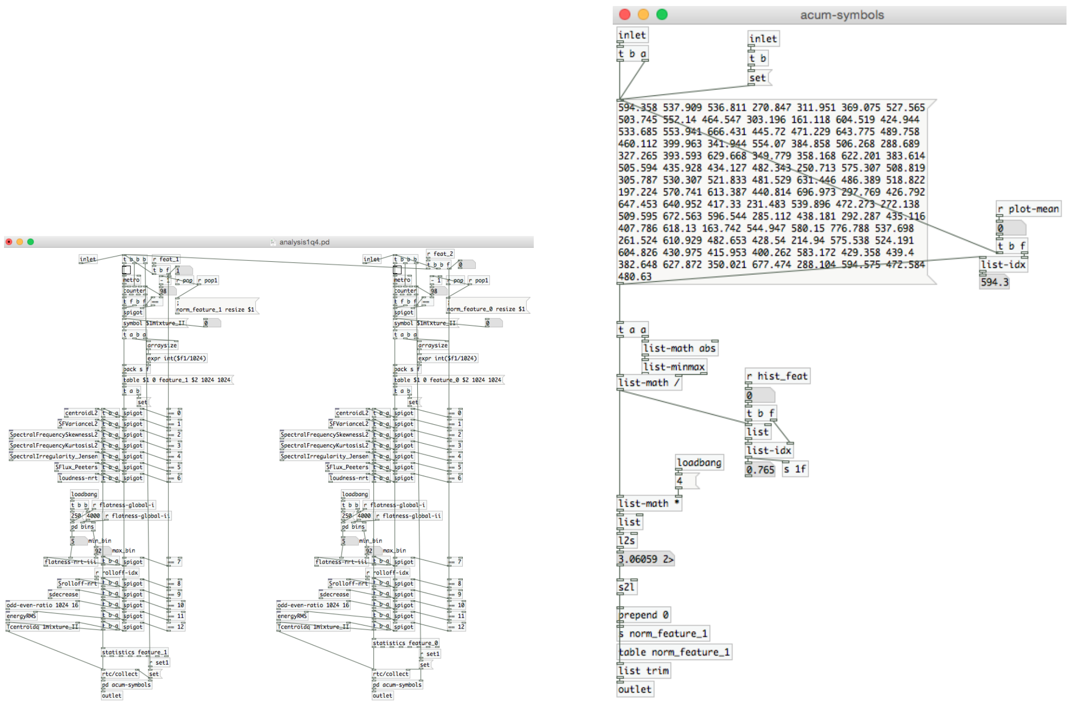
iii.6 Module 6 - Creation of the Sound Mixtures Space Exploration

this module, we have implemented a graphical user interface for the interaction, exploitation and selection of the sound mixtures. It was used sound mixtures using the GEM (*Graphics Environment for Multimedia*) library. The graphical *SSC* interface enables the visualization of four pairs of audio features and allows up to listening to the sound mixtures arranged on the GUI. Figure 7 presents the patch of the sound mixtures search and the four bi-dimensional graphic visualization. The first space is represented by yellow dots. The second space is represented by green dots. The third space is represented by the purple dots. Finally, the fourth space is represented by red dots.

iii.7 Module 7 - Selection of Sound Mixtures and Transcriptions

Module 7 controls the system output formats and there are two specific formats: a) audio file *.aif* and b) music score that is performed by an external PD object called *notes* developed by *Waverly Labs*, at *New York University - NYU*[5]. According to the description of *notes* the external object for Pure Data was conceived as an aid for computer assisted composition (CAC), generative music,

[5]For more information, visit: `http://nyu-waverlylabs.org/notes/`.



(a) (b)

Figure 6: *Figure of the Module 5 which performs the analysis of Sound Mixtures. The collected data is extracted by the audio features (Figure 6a). Figure 6b is the sub-module that calculates the mean of the collected data.*

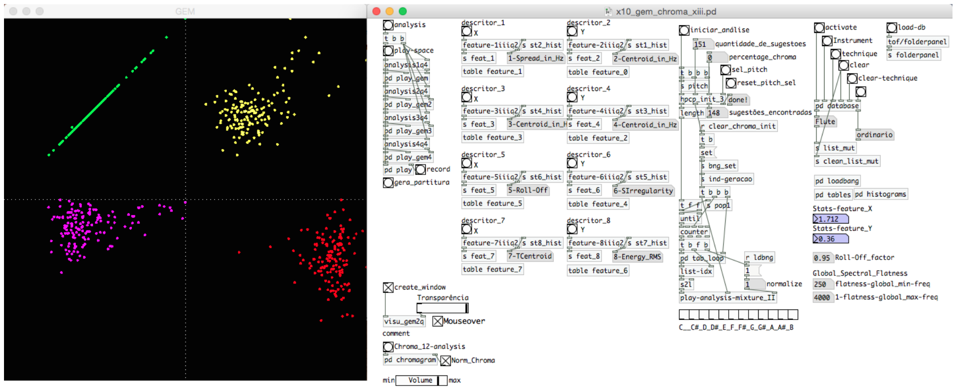


Figure 7: *Illustration of the Module 6 in which the user can interact with the sound mixtures in the exploitation space. In this module the user can hear the sound mixtures.*

and other places where symbolic music notation might be useful. This object interprets the data collected from the PD environment and converts them into a musical graphical notation in *Lilypond* format. The symbolic data that will be converted into musical notation must be configured in the particular syntax of the *Lilypond*. The module that interprets the symbolic data on musical transcription is called the *score-ssc1.pd*. Moreover, this module was not designed to produce final scores although this is conceivable. The composer often goes to lilypond and edit, copy, combine and modify scores in various ways.

There is the object *inst* that receives data such as 'musical instrument', 'musical pitch', 'dynamic' and 'instrumental technique'. Each musical instrument has its own object *inst*. In general, the algorithm receives a message with musical symbolic data and the object *inst* sends each information for its specific sub-module. The sub-module interprets the specific data and converts it in the *Lilypond* syntax. The next step creates a single message with all the information that will be interpreted by the *notes*. The diagram in Figure 8 summarizes all the steps of the musical information.

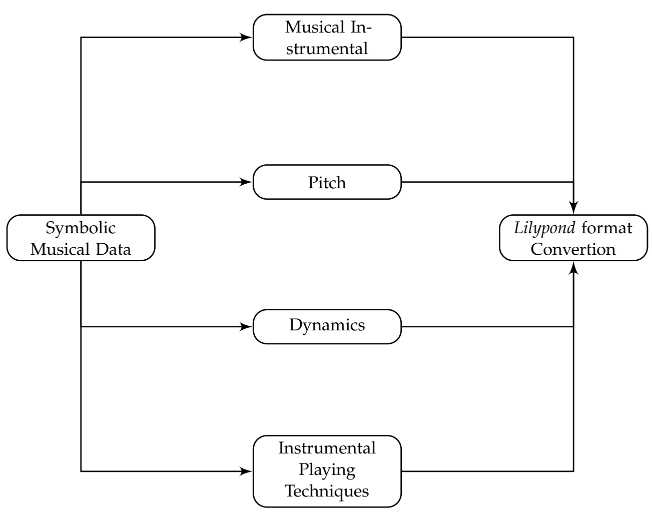
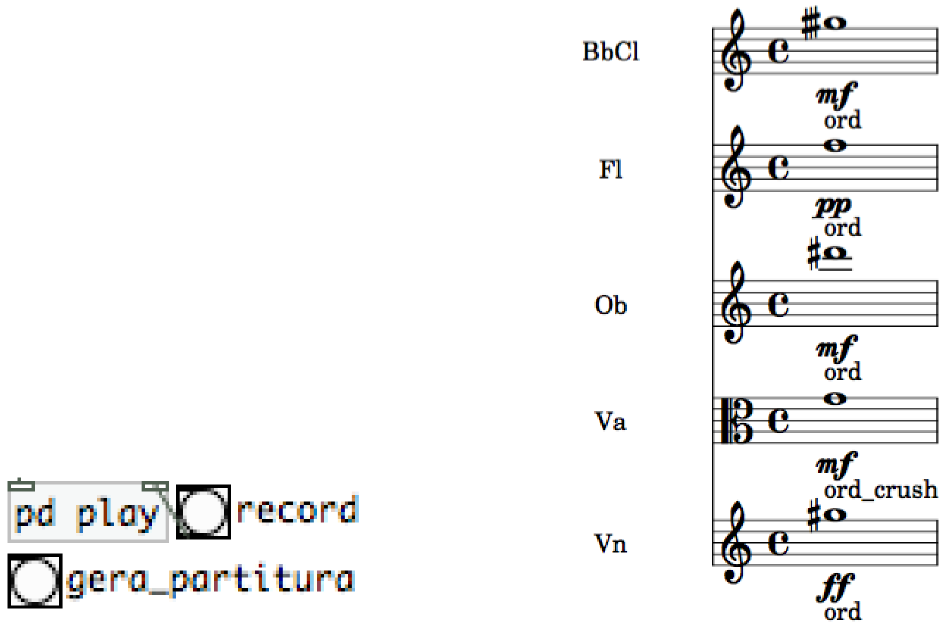


Figure 8: *Diagram Blocks for the musical information, convertion in* Lilypond *format.*



(a) (b)

Figure 9: *Figure of Module 7, which selects the sound mixture and stores it in audio format and in musical notation format in* lilypond*. Figure 9a is the* patch *interaction with the module 7. Figure 9b illustrates the score of a given sound mixture.*

In the current version of *SSC*, there is no temporal information for the orchestral sonorities. Each interaction will produce only an orchestral setting with previously established duration. Figure 9a illustrates the *patch* to store sound mixtures, in *.aif* format or in music sheet format, in *lilypond*. Figure 9b, represents an example of the score of a sound mixture.

iii.8 *Sound Shizuku Compostion - SSC* General Architecture

In the *SSC* system the orchestral possibilities result from the interaction of the analysis of audio descriptors with their potential semantic correlates. Timbre has several perceptual characteristics that may be intrinsically associated or orthogonally different. The sound analysis tools describe certain aspects that can highlight one or more specific characteristics related to the subjective attributes of timbre perception. Figure 10 illustrates the general outline for the orchestration computer-aided orchestration architecture.

III. Practical Applications

The system for supporting the compositional planning presented here focus on how musical orchestration connects two distinct universes *a)* instrumental extended techniques and *b)* computational tools to analyze and statistically describe the spectral content of the material generated by these techniques. Therefore, we developed a method to help the composer to relate: *a)* the high-level descriptions or symbolic data, called 'sonority' with *b)* the specific modes of extended playing techniques. Next we present three compositional that was created with the system, briefly.

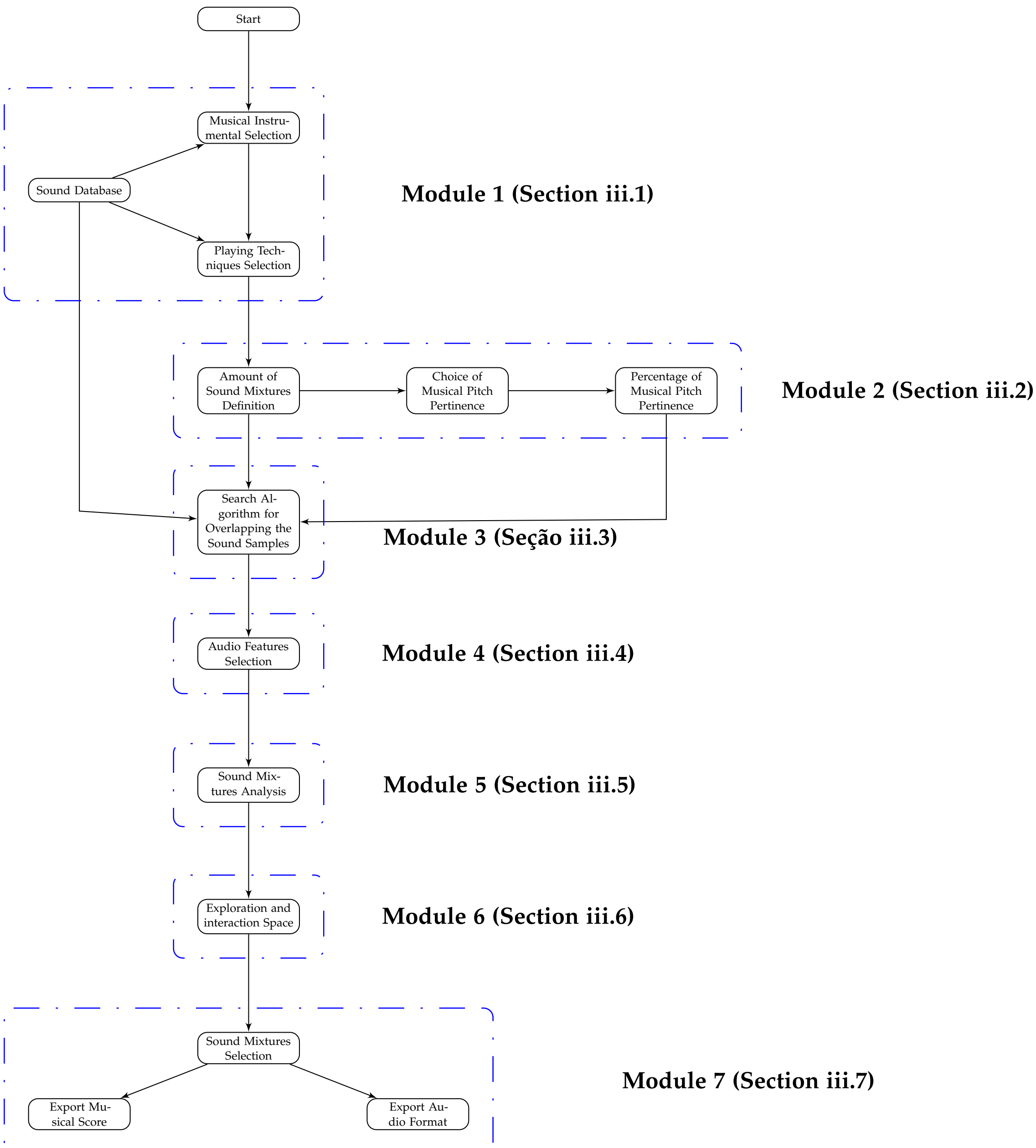


Figure 10: Sound Shizuku Composition - SSC *General Architecture.*

In the particular case of *Lana Tai*, the methodology expounded on the construction of 'Sound Mixtures', as discussed in Section II, which were anchored in two audio features: spectral chroma and spectral centroid. In *Lana Tai* the audio features were related to two contrasting ideas: *a)* opacity and *b)* brightness. The main ideas about the compositional planning can be found in [35, 33, 34].

The work *The oil, the moon and the river* was anchored in three audio features: Loudness, Spectral Irregularity and Spectral Chroma. The compositional planning consisted of contrasting sonorities called velvety and rough and the variation of their intensity in different dynamic levels. According to the analysis from the spectral irregularity, we find that the different playing techniques alter the timbre perception of each analyzed sound. Instrumental techniques which are characterized by instrumental noise insertion tend to relate to rough and ridged sonorities. Conversely, for velvety and smooth sounds we used certain instrumental techniques to result in clean and clear sound like *whistle tones*, on flutes. In the analysis using Spectral Chroma, we find the polarization of musical pitches in which we have established the basis of the melodic structure of the work. Published works for the analysis of the composition can be found in [36].

Finally in *Labori Ruinae* we used audio descriptors to produce gradual timbre transformations. Such analysis was anchored in a vector consisting of a set of six audio features. The formal structure of the work relates to the spectral transformation of five pairs of sonorities. Each sonority has been described by a vector with six audio descriptors. We interpolated each pair of sound from its degree of dissimilarity, in ascending order. We began to work with the pair of sonorities with lower dissimilarity index. Consequently, the work ends with the pair of the higher rate.

IV. Final Considerations

This article discussed a system to work as a new strategy on composition and orchestration within the vast domain of sounds produced by extended playing techniques. The research enabled the formal dialogue between analysis, audio descriptors with the conceptual, aesthetic and subjectiveness providing to the composer a tool to be applied into the process of musical composition. We presented the general architecture of the computer system and how aid to orchestration is done. In this architecture, we introduced five stages concerning to the creative process: *a)* defines the timbre characteristics to be exploited through the audio features. This step will define the aspects and timbre characteristics which will be worked compositionally; *b)* establishes the remarks within the space of characteristics, known as 'Referential Remarks'; *c)* conducts experiments in instrumental mixtures, known as 'Sound Mixtures', via orchestration of audio samples of several playing techniques. These configurations were built from a sound database of various instrumental playing techniques; *d)* defines the orchestral settings weighted by the particular preferences of the composer. This procedure ensures the effective participation of the composer in the final result of his own musical compositional; finally *e)* stores the sound mixture selected by the composer in musical notation and in audio format.

We introduced the audio descriptors used in our analysis with a computer environment. In total there are fifteen audio descriptors available and our perspective is associated with *SSC* focuses on improving and refining the algorithm analysis and the overlapping audio samples using techniques and tools of computer music and other computer models.

Moreover, we intend to publish other results obtained with the current version of *SSC* and also further advance the stage of the system. One of our goals is to expand the sound database by adding more audio samples. Another issue that we will address is to study correlations between orchestral sonorities and text descriptions of timbre characteristics with the affective/emotional states that may be induced or evoked by them.

V. Acknowledgments

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Discrete and Combinatorial Mathematics, Geometry and Mathematics of Continuous Functions Used in Some of my Compositional Projects

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Abstract: *This paper intends to demonstrate the different ways many of my compositional projects used mathematical tools, from the pre-compositional stage through a final product done with sound synthesis. These tools are of diverse nature, depending on the theoretical needs of the problem faced. In some cases, the project employed discrete and combinatorial mathematics. In other cases, geometry was a useful tool to visualize rhythmic manipulations. Irrational numbers were the basis of a non-conventional tuning proposition. Continuous functions, like "sine", are at the core of digital sound synthesis and, in a particular project, served to the design of a digital filter.*

Keywords: *Algorithmic composition. Inharmonic tuning. Fractal diminution. Set similarity. Cyclic rhythm. Digital filter.*

I. Introduction

Many music compositions of the 20th Century have benefited a great deal from Mathematical, Physical or Technologica l knowledge and many continue to do so nowadays. We will concentrate the approach of this paper in the contribution of three different branches of Mathematics: Discrete and Combinatorial Mathematics, Euclidian Geometry and all kinds of Continuous Functions Mathematics.

We may distinguish two main general compositional approaches used in this period. The first one prolongs the validity of the very ancient idea of reducing the complexity of the musical phenomenon to a symbolic representation called the "note" which embraces some of the predominant characteristics of sound to human perception: pitch, duration, dynamics and timbre. This approach allowed the development of musical notation. It still represents, to most composers, their daily tool for music conception and representation. The second approach, that had only subsidiary relevance until the 19th Century, depends on the possibility of dealing with the internal characteristics of the sound. Some, as [7], say that is music composed with the sound itself. Some, as Landy [8], call it "organized sound" and do not even defend that we need to call them "music" anymore. Of course, we are talking of sound products in which the author intentionally explores the internal qualities of the sound evolving in time. Therefore, they belong to the realm of sound design, electronic, concrete, acousmatic or electroacoustic music, or whatever other name is used to identify music to which the concept of "note" is, at most, of secondary importance.

The mathematical tools of discrete and combinatorial mathematics, and geometry, apply mostly to music that continues to use traditional notation, while the mathematics of continuous functions holds the conceptual basis for music that, besides employing technological means to generate sound, treats the sound from inside out.

I might use compositions of most of the established composer and the major names of the 20th Century to demonstrate my point but I choose to use my own compositions in order to state my personal view of how important I consider the influence of mathematical thinking in my compositional trajectory. The selection of cases intends to illustrate the use of different mathematical tools, notwithstanding that more than one may have contributed to develop each particular compositional project.

II. Pitch numerical representation allowing a processual form

It may seem a problem of nostalgic self-indulgence to resort to one of my first attempts in music composition to illustrate how the elementary idea of representing the chromatic scale with numbers emerged to me. Indeed, the circumstance around this report is what makes it interesting. It was the year of 1970 and I was seventeen years old. I had just started to attend college classes and one of the required freshman courses was "Introduction to Computer Programming". The instructor taught us the Fortran 1.0 computer language. We had to punch cards and stay on line to run our codes on the only IBM mainframe computer available in the school, a device that filled a large room. We could not enter the room, only glimpse through a door window. Besides the scheduled homework, we were supposed to come up with real world problems that a computer program might solve. The teacher used to say that the computer was a solution in search of problems. I was already interested in music composition, following whatever reached me of the European avant-garde music. This means that I had some information about basic concepts of dodecaphonic and aleatory music.

During that year, among other projects, I devised the idea of composing automatically a short piece of atonal music with the aid of a computer program. The name of resulting piece of music was *Three Episodes for piano*. Its definitive version dates 1974. The first problem I had to face in that project was how to represent the notes of the chromatic scale with numbers. My first attempt was to assign ten pitches to the numbers 1 to 10, substituting 0 for 10 to deal only with single digits. Therefore my numeric code was: $1 = C$, $2 = C\sharp$, $3 = D$, $4 = D\sharp$, 5= E, (...) through $9 = G\sharp$ and $0 = A$. As I was missing numeric representations for $A\sharp$ and B, I circumvented the problem with a systematic rotation of the numeric correlation assignment to include all the pitches.

I keep to this day a print out of the output, but unfortunately, the code itself was lost. It reads like that:

4	1	7	8	3	0	9	5	2	6
5	8	5	1	3	9	4	7	8	0
3	3	6	4	2	3	1	5	8	5
6	9	0	6	5	4	6	3	3	8

etc..

The idea of the program was to calculate the numbers of each new line adding the adjacent numbers of the previous line. For instance, the second line is based on the first: $4 + 1 = 5$,

$1+7=8$, until the last, which should turn around to the beginning and retrieve the first element, $6+4=10$, however making $10=0$. Insofar, when the sum exceeded 10, the program kept only the last digit. For instance, $9+5=14$, but 14 was replaced by 4.

Soon I realized that I did not need to restrain myself to single digits. I could operate the same reduction used for numbers above 10, using the concept of base 12. For a new programming attempt, I chose a more practical correlation that begun with the association $C=0$, as practiced nowadays. Therefore, a second program using the principle of mode 12, processing a new sequence of twelve numbers without repetition, yielded the following result:

10	4	1	7	8	3	0	9	5	2	6	11
2	5	8	3	11	3	9	2	7	8	5	9
7	1	11	2	2	0	11	9	3	1	2	11
8	0	1	4	2	11	8	0	4	3	1	6
etc.......											

These attempts of 1970 precede the publication of Forte's pioneer book on musical set theory [4]. Took me almost two decades to acknowledge the development of a set theory of music in other part of the world. For sure, I had assumed that it might be happening, so intuitive the approach seemed to me. The only problem was that, at that time, before the internet, information reached Brazil much slower than today.

The more interesting aspect of that first attempt was how it allowed the generation of pitch data by numeric manipulation. One cannot add pitches, unless numbers replace them. The purpose was to build a machine that makes music using a process that only stops when it reaches a certain condition, for instance, the completion of one hundred loop cycles. At that time, I was only vaguely aware of the concept of "music as process" and the major trend it represented. Still years later, when critics commented the first performance of the piece, referring to it as piece of serial music, I thought they were mistaken because I did not follow the rules of serial music. For me, then, serial music was dodecaphonic music with its principles extended to others parameters. I realized that the pitch generation of that piece was somehow unpredictable and therefore closer to stochastic music. One thing particularly pleased me: the process allowed pitch repetition, a negative imperative to Schoenberg. All I knew at that time, concerning serial music, followed the teachings of Krenek (1940). We knew very little details about the explosion of the series promoted by Boulez and Stockhausen techniques, but the ear guided me to obtain similar results, although the technique used was somehow original.

Figure 1: *Coelho de Souza's* Three Episodes for piano*, mov. 3, m. 14-16.*

Even though Figure 1 does not show an analysis of the pitch generation, it illustrates the style of the music produced by the numerical process above described.

As a corollary to this line of reasoning, we may question what would be the meaning of negative numbers in this context. In fact, the model is consistent because the chromatic scale supports a symmetrical reflection:

...	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
...	G	$A\flat$	A	$B\flat$	B	C	$C\sharp$	D	$D\sharp$	E	F	$F\sharp$	G

On the other hand, what happens with actual pitch frequencies? For instance, consider $A0 = 27.50Hz$ as the low A in the scale above. Calculating the descending pitches according to the tempered tuning, we obtain:

$$\begin{aligned}
A(0) &= \text{27,50 Hz} \\
A\flat(0) &= \text{25,96 Hz} \\
G(0) &= \text{24,50 Hz} \\
G\flat(0) &= \text{23,12 Hz} \\
F(0) &= \text{21,82 Hz} \\
E(0) &= \text{20,60 Hz} \\
E\flat(0) &= \text{19,44 Hz} \\
D(0) &= \text{18,35 Hz} \\
D\flat(0) &= \text{17,32 Hz} \\
C(-1) &= \text{16,35 Hz} \\
B(-1) &= \text{15,43 Hz} \\
B\flat(-1) &= \text{14,56 Hz} \\
A(-1) &= \text{13,74 Hz}
\end{aligned}$$

Therefore, the pitches plunge into a sub-sonic frequency realm, asymptotically tending to zero. There are no negative frequencies and even if we forcefully assign a negative value to the frequency of a pitch, in physical terms this will be the same sound of the equivalent positive frequency with a 180 degrees inverted phase. Therefore, this physical reality impairs the dualistic principle used by Hugo Riemann to justify his Theory of Functional Harmony because its postulate requires the existence of an inverted harmonic series. He missed that we can draw pitches in a linear scale that supports negative numeric values, but these pitches map frequencies into a logarithm curve that asymptotically approaches zero, never assuming negative values or any symmetrical shape.

III. Tuning with the Golden Section

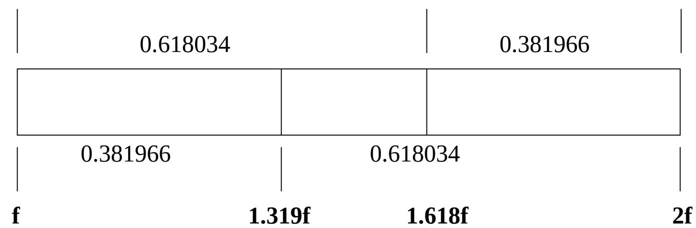
From my first compositional project, I jump now to my most recent project that is an opera named *The Machine of Pascal in Pernaguá*. For this project, I have also rescued from my memory the generative operations described above and used them to compose some of the scenes. This is also a reason to mention the procedure in this report. Unfortunately, I could not find a short good example of the method taken from the opera therefore I resorted to that old but clearer example.

Going now to the point, one of the main scenes of the drama depicts an hypothetical ability of Pascal's machine to produce music. The story is set in the 18th Century but, obviously, the Pascal's machine is a metaphor of today's computer. The music associated with the machine has characteristics that go beyond human motor control or limits of instrumental performance. One of such aspects is the microtonal tuning used by the computer-generated music of this scene. It implements a scale division inspired by the golden section.

The purpose of this tuning model is to obtain inharmonic relations between pitches. As the basic generative proportion is an irrational number, we do not expect to obtain any of the harmonic relations, known since Pythagoras, based on integer numbers.

$$\varphi = (1 + 5^{1/2})/2 = 1.6180339887\ldots \tag{1}$$

In this construction, we also applied two principles very fond to mathematical reasoning: the principle of symmetry and the principle of self-similarity. Applying symmetry, reflecting the division around the middle point, we obtain a first step of the division:



The following steps are recursive applications of the golden ratio, dividing each remaining segment into three parts that replicate self-similarly the scheme above. For a matter of clarity, we present the results in a vertical table instead of horizontally, as above.

The first column of Table 1 shows a linear division of the octave applying a nested golden ratio proportion. The next column shows the linear increments: adding the values of the first and second columns, we obtain the next line of the first column. However, we know that the human hearing is not linear, but logarithmic. The next column shows a similar division of the octave with logarithmic scaling. The fourth column shows the logarithmic increments: multiplying the values of the third and fourth columns, we obtain the next line of the third column. The fifth column depicts the tempered division of the octave. Of course, there is no perfect equivalence with, neither the first, nor the third column, but we emphasize in bold italic that the values of the tempered fourth and fifth degrees are very close of those in the golden rate division column. The small discrepancy of values is not only a matter of accuracy. We tried to proof a mathematical equivalence and performed a more precise evaluation of the results too. We find out that values are indeed not equal, but only a coincidence up to a certain degree of precision. However, we cannot perceive the difference between these pitches because they are within the JND (just noticeable difference) limit.

The marks on sixth column show the nesting branches. Centered "x"s indicate the first division step. A left positioned "x" shows the second branch e right positioned "x" the third interaction. The last column applies the results to the interval A2-A3. Notice that we obtain a microtonal scale divided in 25 intervals. There are three kinds of intervals that on the second column are identified by the increments 0.022. 0.034 e 0.056. Notice that the sum of the first two equals the third. Therefore. for practical purposes. we might reduce the division to 21 intervals of only two sizes.

These frequencies can be considered the fundamental of a complex note but also harmonic partials of inharmonic sounds. They can disperse in many octaves or concentrate in clusters. A digital synthesis program can implement any kind of pitch combination and their relative weight. That is what we have done in the above-mentioned opera scene.

Table 1

linear proportion	**linear increment**	**logarithmic proportion**	**logarithmic increment**	**tempered d = 2 1/12**	**fractal nesting**	**pitches for A2 = 110.0**
1.000 f	+ 0.056	**1.000 f**	x 1.0396	**1.000 f**	x	**110.0 Hz**
1.056 f	+ 0.034	1.040 f	x 1.0238		x	**114.4 Hz**
1.090 f	+ 0.056	1.064 f	x 1.0396	1.059 f	x	**117.0 Hz**
1.146 f	+ 0.034	1.106 f	x 1.0238		x	**121.7 Hz**
1.180 f	+ 0.022	1.113 f	x 1.0154	1.122 f	x	**122.4 Hz***
1.202 f	+ 0.034	1.150 f	x 1.0238		x	**126.5 Hz**
1.236 f	+ 0.056	1.178 f	x 1.0396	1.189 f	x	**129.6 Hz**
1.292 f	+ 0.034	1.224 f	x 1.0238		x	**134.6 Hz**
1.326 f	+ 0.056	1.253 f	x 1.0396	1.260 f	x	**137.8 Hz**
1.382 f	+ 0.034	1.303 f	x 1.0238		x	**143.3 Hz**
1.416 f	+ 0.022	***1.334 f***	x 1.0154	***1.335 f***	x	**146.7 Hz**
1.438 f	+ 0.034	1.355 f	x 1.0238		x	**149.1 Hz***
1.472 f	+ 0.056	1.387 f	x 1.0396	1.414 f	x	**152.6 Hz**
1.528 f	+ 0.034	1.442 f	x 1.0238		x	**158.6 Hz**
1.562 f	+ 0.022	1.476 f	x 1.0154		x	**162.4 Hz***
1.584 f	+ 0.034	***1.499 f***	x 1.0238	***1.498 f***	x	**164.9 Hz**
1.618 f	+ 0.056	1.534 f	x 1.0396		x	**168.7 Hz**
1.674 f	+ 0.034	1.595 f	x 1.0238	1.587 f	x	**175.5 Hz**
1.708 f	+ 0.056	1.633 f	x 1.0396		x	**179.6 Hz**
1.764 f	+ 0.034	1.698 f	x 1.0238	1.681 f	x	**186.8 Hz**
1.798 f	+ 0.022	1.738 f	x 1.0154		x	**191.2 Hz***
1.820 f	+ 0.034	1.765 f	x 1.0238	1.782 f	x	**194.2 Hz**
1.854 f	+ 0.056	1.807 f	x 1.0396		x	**198.8 Hz**
1.910 f	+ 0.034	1.879 f	x 1.0238	1.888 f	x	**206.7 Hz**
1.944 f	+ 0.056	1.923 f	x 1.0396		x	**211.5 Hz**
2.000 f	-	**2.000 f**	-	**2.000 f**	x	**220.0 Hz**

IV. A process of fractal rhythmic diminution

The above mentioned section of the opera also uses another mathematical procedure that departs from a rhythmic motive based on seven beats, irregularly divided with four kinds of durations: a half note, two quarter notes, a dotted quarter note and three eight notes, assembled in a sequence that produces syncopation, as displayed in the first line of Figure 2.

Figure 2: *Fractal rhythmic diminution (Coelho de Souza's The Machine of Pascal in Pernaguá).*

Each line represents a new level of self-similar diminution of the original line. In the second level, within the duration of a half note, a tuplet of seven eight notes (replacing four notes) reproduces the rhythmic proportions of the original measure. Pitches are the same seven pitches but on a different permutation. The process continues towards three more levels of diminution, two of them displayed in Figure 2. The last one, not shown in Figure 2, replaces the three eight notes by a 7:8 diminution. This musical process was inspired by a visual graphic, proposed by Peano. to generate the design of a snowflake, as we can see in Figure 3. Although not identical, these two processes of fractal self-similar diminution exhibit certain common features.

The main differences results from the asymmetrical internal structure of the first musical level, which induces, by self-similarity, a quite chaotic rhythm, as the process continues through the other levels. The snowflake design, on the other hand, appears to be much more regular. In fact, we listen to that fragment of music almost as a random sequence of events, like a Brownian movement, although, in a deeper level of perception, we realize that there is a coherent structure resulting from the multi-level rhythmic self-replication.

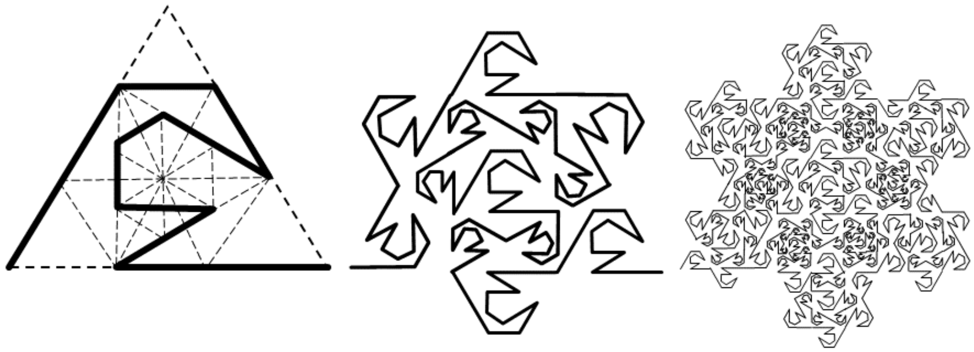


Figure 3: *Snowflake curve depicted as a process of fractal diminution ([11, p. 190]).*

V. A particular set property

The principle of *permutation* is another tool of combinatorial mathematics that I have used since 1984 when I composed *Rébus* for piano solo. Applying it to pitches yields trivial results but applying it to intervals allows us to study an interesting similarity relation that is not part of the standard Set Theory proposed by Allen Forte [4]. Forte devised a property related to the interval content of sets that he calls the *Z relation*. This property measures the similarity of two sets based on the identity of their interval vector, which counts the number of occurrence of each interval between the pitches of the set. Forte was able find some pieces in which the segmented sets exhibit the Z relation, but they are very likely to have occurred by chance, not intentionally by the composer.

The relation that I have proposed is different from the Z relation, as I demonstrated in an article [2]. It starts with the CORD vector, proposed by Soderberg [10], that lists the intervals of a set class. Going a step further, I have proposed a PCORD set that rearranges the intervals of the CORD vector to normal order, or actually, without any additional transformation, to its prime form.

This proposition differs from Forte's Z relation because it aims to be not only an analytical tool, but also a generative model. Based in a single PCORD, we can generate sets of different set classes. These sets have a second degree of structural similarity although to standard set theory they seem to be unrelated.

If we segment this music grouping the pitches of each measure in the treble clef and the bass clef. and reduce these sets to their prime form. the result will be:

Measures $1-2$:
Treble clef: $\{C\sharp.F\sharp.D.A\sharp.G\sharp\} \rightarrow$ set class (01468)
Bass clef: $\{B.C.D\sharp.F.A\} \rightarrow$ set class (02368)
Measures 3-4:
Treble clef: $\{B\flat.A.G.D\flat.E\flat\} \rightarrow$ set class (02368)
Bass clef: $\{E.G\sharp.F\sharp.C.B\} \rightarrow$ set class (01468)
Measures 5-6:
Treble clef: $\{D.F.C.A\flat.E\flat\} \rightarrow$ set class (02358)
Bass clef: $\{G.A.C\sharp.E.F\sharp\} \rightarrow$ set class (02358)

Figure 4: *Fragment (reduced) from Coelho de Souza's Concerto for Percussion (w. in progress).*

At a first approach. it seems that three different unrelated set classes have been used in these six measures of the piece: (01468), (02368) and (02358). If we calculate the CORD vector of these sets classes. reorder them to their prime form to calculate PCORD form. we obtain:

(01468) $\rightarrow$ [[1322]] $\rightarrow$ ((1223))
(02368) $\rightarrow$ [[2132]] $\rightarrow$ ((1223))
(02358) $\rightarrow$ [[2123]] $\rightarrow$ ((1223))

Therefore. we realize that this entire passage has been generated from a single PCORD. namely ((1223)). In the above mentioned article [2]. we listed all the set classes that are related by PCORD similarity. for each cardinality. In Table 2 we reproduce only the list of cardinality 5 because the sets used in the music of Figure 4 all the sets as based on five pitches. As expected in the column of PCORD ((1223)) we find the three set classes used in that fragment: 5-30 (01468), 5-25 (02358) and 5-28 (02368).

VI. Geometric representation of rhythm

The music of Figure 4 allow us to approach another mathematical tool that can be used to generate or analyze music: the depiction of rhythm by geometry means. This is particularly efficient when the rhythm presents cyclic features. That is the case of the music of Figure 4. We may calculate the rhythms based on the smallest division value. in this case sixteenths:

Measures 1-2 (right hand): 3 - 3 - 3 - 3 - 4
Measures 3-4 (left hand): 2 - 3 - 3 - 3 - 4 - 1
Measures 5-6 (right hand): 1 - 3 - 3 - 4 - 3 - 2

Table 2: *List of PCORD x Set Classes with Cardinality 5*

PCORD	((1111))	((1112))	((1113))	((1114))	((1122))	((1123))
Set Classes	5-1 (01234)	5-2 (01235) 5-3 (01245)	5-4 (01236) 5-6 (01256)	5-5 (01237) 5-7 (01267)	5-9 (01246) 5-10 (01346) 5-Z12 (01356) 5-8 (02346)	5-Z36 (01247) 5-14 (01257) 5-16 (01347) 5-19 (01367) 5-Z18 (01457) 5-11 (02347)

((1124)	((1133))	((1222))	((1223))	((1233))	((2222))	((2223))
5-13 (01248) 5-15 (01268) 5-Z17 (01348) 5-20 (01568)	5-Z38 (01258) 5-21 (01458) 5-22 (01478) 5-Z37 (03458)	5-24 (01357) 5-23 (02357)	5-27 (01358) 5-29 (01368) 5-30 (01468) 5-25 (02358) 5-28 (02368) 5-26 (02458)	5-31 (01369) 5-32 (01469)	5-33 (02468)	5-34 (02469) 5-35 (02479)

Although a certain degree of similarity in these rhythmic patterns induce us to suspect the existence of some hidden consistency. the numerical strings do not allow an immediate realization of some intentional process. On the other hand. it is clear to the ear that there are cyclic rhythmic patterns driving the discourse. When we represent these three rhythmic patterns in a circle. assigning striking points to sixteen possible positions, we obtain a much clearer visualization of the displacement procedure (Figure 5)

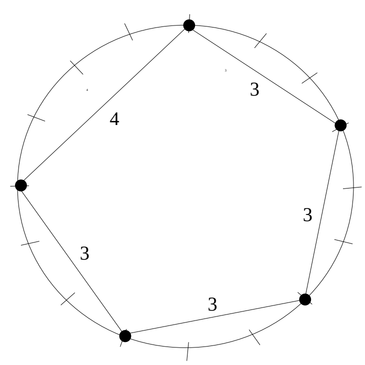


Figure 5: *Rhythm pattern of measures 1-2.*

This representation allows us to realize that there is only one rhythmic pattern altogether in the passage. Actually a single pattern is rotated at $+45°$ and $-115°$. as shown in Figures 6 and 7. so we have it starting at a different point of 16 points grid cycle at each two bars. A similar linear representation is possible and in that instance. instead of rotation. the procedure to consider is displacement. Another instance of geometric representation is the well know method of representing the twelve pitches in a circle. as the hours in a clock. This is a standard procedure of the set class theory. found in any textbook on the subject. Although we have used this kind of representation to illustrate principles of symmetry in my own compositions. we chose to

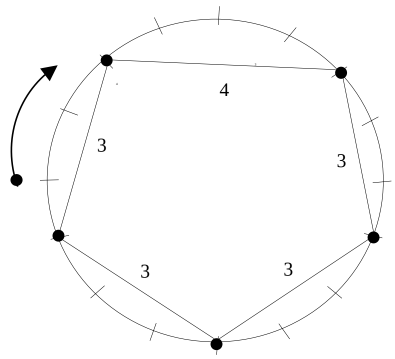


Figure 6: *Rhythm pattern of measures 3-4.*

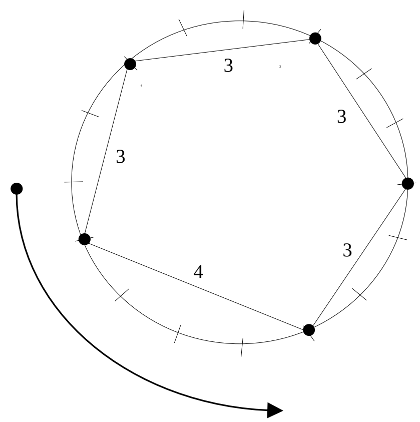


Figure 7: *Rhythm pattern of measures 5-6.*

illustrate the use of geometry as a tool for music composition. using cyclic rhythms because it is a less known problem. although some recent releases like [12] are quickly becoming popular. Another reason is that for rhythm the number of points represented in the circle is variable. and not a fixed clock face. We might also bring about the subject of geometry representation used in the neo-Riemannian theory. especially the Riemannian Tonnetz (see [5]). besides other achievements like those proposed by [13]. I did not show these devices because I have not used them in my music. In this paper. I voluntarily limited myself to tools that I have employed in my own compositions to demonstrate how mathematical tools can be helpful for establishing a pre-compositional background for building a personal style.

VII. A mathematical filter used to transform a sound signal

So far. we have examined cases where the mathematical tools described belong to chapters of discrete and combinatorial mathematics (see [6]) or geometry. I have promised to approach also examples of the mathematics of continuous functions. Indeed. I might collect a bunch of examples from my compositions and. fortunately. the mathematical foundation of them certainly belong in one of the two volumes of the all-encompassing book on the subject written by Gareth Loy [9].

Nevertheless. since I am trying to demonstrate that my fellow composers can easily understand these mathematical tools. I will restrain myself to only one example because the mathematics involved in the continuous functions usually depends on the knowledge of calculus and other high-level mathematics.

The following *Csound* program. besides the usual opcodes offered as presets by the program. uses a transformation of the wave signal applying directly to it a sine function multiplication. We highlight that line of the code with boldface. As we know sine waves are continuous functions. and according to Fourier's theorem. we can analyze any sound wave as a sum of harmonic sine waves. if we found the appropriate variables. In this case. however. the sine wave works as a kind of mathematical filter.

This *Csound* experiment is based on a standard two-stack frequency modulation design. but with the above mentioned transformation. we tried in this project. to perform a filtering that somehow works like the waveshaping technique. however done with a brute-force mathematical function instead of the usual tables. Most of the *Csound* opcodes are based in continuous functions but the composer does not have to deal directly with them. For more information on computer music synthesis see [3] and on *Csound* programming see [1].

```
;"EXPERIMENT 1.orc"
;instrument with time variable timbre

sr = 44100
kr = 4410
ksmps = 10
nchnls = 2

instr 1
  idur = p3
  iamp = p4
  ipitcar = p5
  iratefreq = p6
  iindex = p7

;transient for the attack
  ifrtr = cpspch(ipitcar)
  ikftr = .975
  imfrtr = ikftr*ifrtr
  kamptr1 expon iamp.idur.0.1
  kamptr2 oscili kamptr1.imfrtr.2.0
  aout1 oscili kamptr2.ifrtr.2.0

;filter units
  kamp = 1
  ifrcar = cpspch(ipitcar)
  ifrmod = ifrcar*iratefreq
  idev = iindex*ifrmod
  kamp2 linen kamp..50*idur.idur..50*idur
  amod oscili idev.ifrmod.1.0
  afreq = (ifrcar)+(amod)
```

```
  aosc oscili kamp2.afreq.1.0
  afilt1 = sin(aosc*3.14159/2) ;ïČ§ sine filter
  again = ((1/(aosc+1.5))-1.2)*1.25
  kvar line 0.idur.1
  aout2 = ((afilt1*kvar)+(again*(1-kvar)))*2*iamp
  kenv linen 1..05*idur.idur..05*idur
  aout2 = aout2*kenv

;triangle
  ktrian linen 0.5*iamp..01*idur.idur..95*idur
  aout3 oscili ktrian.ifcar.2
  aout4 = (aout1+aout2+aout3)*.75
  outs aout4.  aout4
  endin

;test score "EXPERIMENT 1.score"
  f01 0 512 10 1
  f02 0 512 10 1 0 .1111 0 .04 0 .0204 0 .01234 0 .00826

;a sequence of notes with harmonic relations of modulation

;instr start dur amp pitch fm/fc I=d/fm
  i01 0  6 10000 8.09 1   3
  i01 3  6 10000 8.03 2   3
  i01 6  6 10000 7.11 2   4
  i01 9  6 10000 7.05 1.5 3
  i01 12 6 10000 8.06 1.5 4

;a sequence of notes with inharmonic relations of modulation
;instr start dur amp pitch fm/fc I=d/fm
  i01 0  1 10000 8.09 1.54  3.04
  i01 2  1 10000 7.11 1.55  4.05
  i01 4  1 10000 8.03 2.56  3.06
  i01 6  1 10000 7.06 2.57  4.07
  i01 8  1 10000 7.09 1.25  5.08
  i01 10 1 10000 8.01 1.414 3.14
  e
```

VIII. Conclusions

We tried to demonstrate. describing a varied set of examples assembled from my own composition projects. how mathematical tools can be useful to shape a style. from the pre-compositional stages through a final sound synthesis stage. In some cases. these tools use discrete mathematics. for instance. in set theory that employs note representation by numbers. or in combinatorial mathematics applied to algorithmic composition based in set manipulation. We can also resort to irrational numbers to implement non-conventional tunings other than tempered or traditional tunings based on integer proportions.

On the other hand. the universe of continuous functions is at the base of human hearing. as far as logarithmic functions explain the perception of pitch and sound dynamics. However. when we jump into the world of direct manipulation of the sound. the mathematics of Fourier transforms. Hilbert transform. convolution. filters (simple. FIR. IIR or Z transf). resonance. acoustic systems modeling (with finite differential equations). and also techniques of sound synthesis (like AM. FM. vocal synthesis. physical modeling. etc). dynamic spectra (Gabor. short time Fourier transform). sound vocoder. and so on. are matters in which the deep understanding of their mathematical foundations enhance the use of their capabilities.

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