Iterable Voice-Leading Schemas

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Abstract: *An iterable voice-leading schema combines a voice leading with a permutation that determines how the voice leading is to be reapplied. These structures model a wide range of repeating musical patterns from the Renaissance to the present day.*
Keywords: *Voice leading. Sequence. Round. Counterpoint. Orbifold. Geometry.*

This paper uses simple mathematics to analyze a not-so-simple collection of musical patterns in which a single voice leading is repeatedly reapplied. The resulting collection encompasses classical sequences as a special case, while also including a wider range of phenomena familiar from other musical styles. Mastering these repeatable patterns is an important component of contrapuntal expertise.

The mathematical background is adapted from previous work ([3],[4],[5],[1]): *pitches* are points in a line $\mathbb{R}$, with integer-valued pitches being *scale tones*; a scale here acts as a both a coordinate system and a metric whose unit is the *scale step*. *Pitch classes* are points in the circle $\mathbb{R}/c$ with c the (integral) size of the octave. A *path in pitch class space* is an ordered pair (p, r) with p a pitch class and r a real number indicating how that pitch moves; this lifts to a directed line-segment $p \to p{+}r$ in the pitch space $\mathbb{R}$. In this context, paths in pitch-class space can be understood either as points in the circle's tangent space or as homotopy classes of paths in the circle itself. Paths can be related by transposition or inversion, with $\mathbf{T}_x((p,r)) \equiv (\mathbf{T}_x(p), r)$ and $\mathbf{I}_x((p,r)) \equiv (\mathbf{I}_x(p), \text{-}r)$. A *voice leading* is a multiset of paths in pitch-class space, determining how the notes of one chord move to those of another; these are described colloquially by phrases such as "C major moves to F major by keeping the root fixed, moving E up by semitone to F, and G up by two semitones to A." A *transpositional voice leading* is one in which every path has the same real number—moving all its notes in the same direction by the same number of scale steps.

A voice leading $\mathcal{V} = A \to B$ defines a *transpositional voice-leading schema* $\boldsymbol{\mathcal{V}}$ that can be uniquely applied to any transposition of the initial chord, so long as it contains no pitch-class duplications and is not transpositionally symmetrical: $\boldsymbol{\mathcal{V}}(\mathbf{T}_x(A)) \equiv \mathbf{T}_x(\mathcal{V})$. When B is transpositionally related to A we can therefore reapply the voice-leading schema $\boldsymbol{\mathcal{V}}$ in a chain, generating a repeating musical pattern that sends each note n cycling through chordal elements:

$$n, \mathbf{T}_x(\varphi(n)), \mathbf{T}_{2x}(\varphi^2(n)), \mathbf{T}_{3x}(\varphi^3(n)), ..., \mathbf{T}_{ix}(n)) \tag{1}$$

with i the order of the permutation, so that $\varphi^i(n) = n$. When B is a transposition of A, then the permutation φ is uniquely determined by the voice leading (so long as both chords are suitably nonredundant). In the general case, where A is symmetrical or B is not related to A, we have to supply the permutation φ explicitly. (Geometrically, the permutation φ contains information about the path along which the vector $\mathcal{V}$ is parallel-transported from point A to B.) We therefore define an *iterable voice leading schema* $\boldsymbol{\mathcal{V}}_{\mathbf{I}}$ as a pair $(\mathcal{V},\varphi)$ with $\mathcal{V}$ a voice-leading and φ a permutation acting on the musical voices, allowing us to iterate the schema in analogy to (1).

Figure 1 shows iterated voice-leading schemas spanning more than four centuries. The *period* of a schema is the minimum number p such that the voice leading connecting chord 1 to chord $1 + p$ moves every voice by the same interval modulo the size of the scale; this is shown by the brackets on Figure 1. The *wraparound voice leading* connects the chord at the start of one period to the chord at the start of the next: a sequence is *transpositional* if this voice leading is transpositional (Figure 1b-d); if not, we have a *contrary-motion sequence* where the relative distance of voices changes by one or more octaves with each period (Figures 1a and 2).

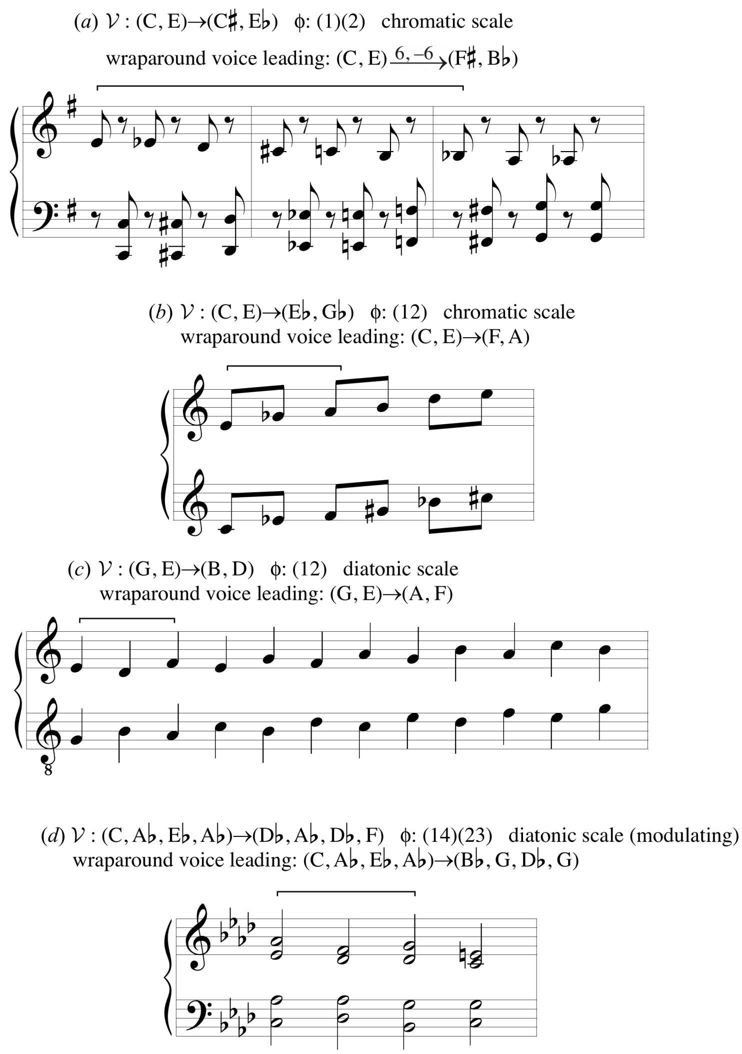

Figure 1: *. Iterated voice-leading schemas. (a) Beethoven Op. 90, I, mm. 105–107; (b) a central intervallic pattern in Stravinsky's* Firebird*; (c) a passage from the Sanctus of Josquin's Mass* L'Ami Baudichon*, mm. 14ff.; and (d) a reduction of the descending-fifth sequence in the development section of the first movement of Beethoven's Op. 2 no. 1. The cyclic notation (12) indicates that the music of voice 1 in the first chord passes to voice 2 in the second (counting from bottom to top), with the music of voice 2 passing to voice 1.*

Such sequences generally produce canons, with the nature of φ determining the structure of the canonic voices. When φ has a single cycle, each voice articulates the same pattern of intervals, forming a single canon as in Figure 1b-d. When φ has two cycles, repeated applications produce a *double canon* with two distinct groups of canonically related voices, as in Figures 1d and 2; more generally an n-cycle permutation produces an n-fold canon. (In the limiting case, where the schema uses n distinct cycles to connect n-voice chords, each voice progresses along its own interval independent of the others.) In Renaissance music, iterated voice-leading schemas tend to link adjacent chords (Figure 3); in classical music, they frequently connect nonadjacent sonorities (Figure 4).

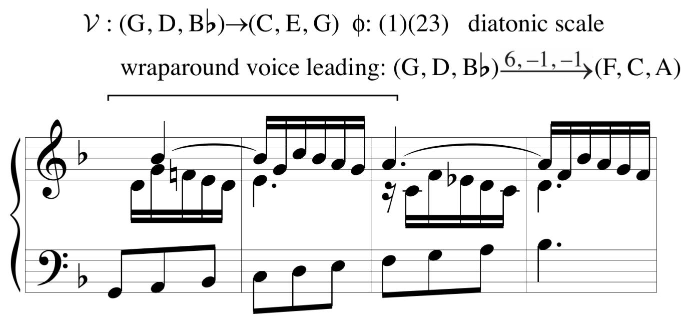

Figure 2: *A contrary-motion sequence in the first F-major fugue from the* Well-Tempered Clavier, *mm. 56ff.*

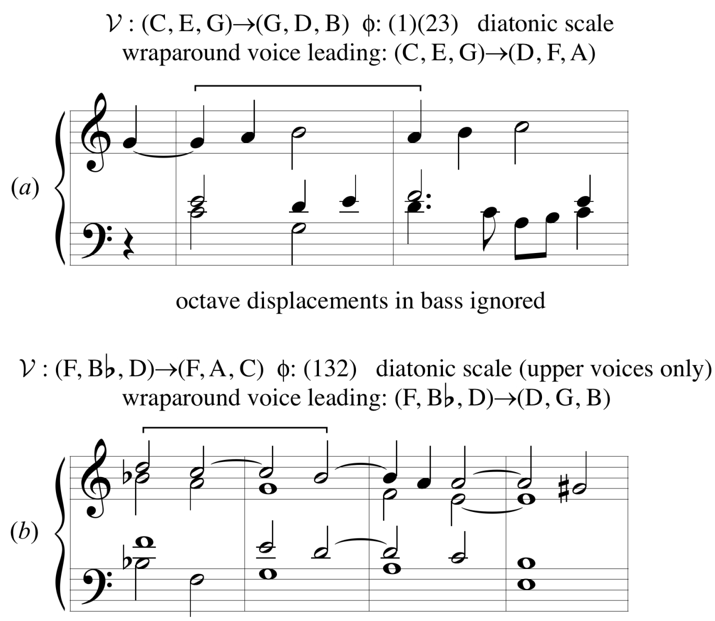

Figure 3: *Iterated voice leadings in (a) the Sanctus of Palestrina's Mass* Ave Regina Coelorum, *m. 19ff. and (b) the Sanctus of Palestrina's Mass* Spem in Alium, *mm. 93ff., presenting six successive ascending fifths in a row.*

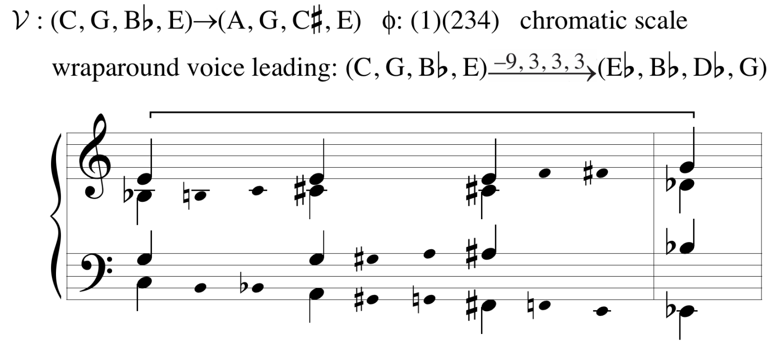

Figure 4: *The "Omnibus sequence," a common Romantic contrary-motion pattern [6].*

Many familiar musical patterns can be analyzed using this framework. In some cases, these structures have traditional music-theoretical names: for instance, a *round* is an iterable voice-leading schema whose generating voice leading is of the form $A \rightarrow A$, connecting a chord to itself (Figure 5). Similarly, previous theorists have explored *wedges* generated by the combination of a nontranspositional voice leading $\mathcal{V}$ with trivial permutation φ, so that all voices move along their own individual paths (e.g. Figure 1a, [2, p. 124ff.]). Sequences are *canonic* when φ is nontrivial (as in all but one of the preceding examples), and *noncanonic* otherwise (Figure 1a, Figure 6 below). A final possibility is a variable sequence in which either the transposition or the permutation changes over the course of the sequence: for instance, in Figure 6 V^6_5–I progressions descend by three thirds and one second, returning to their initial position after four units rather than seven; here the voice leading from C to F is *individually T-related* to the previous voice leadings [4], with the second chord being one step too high. More remarkable is Figure 7, where Bach changes the permutation while preserving the voice leading.

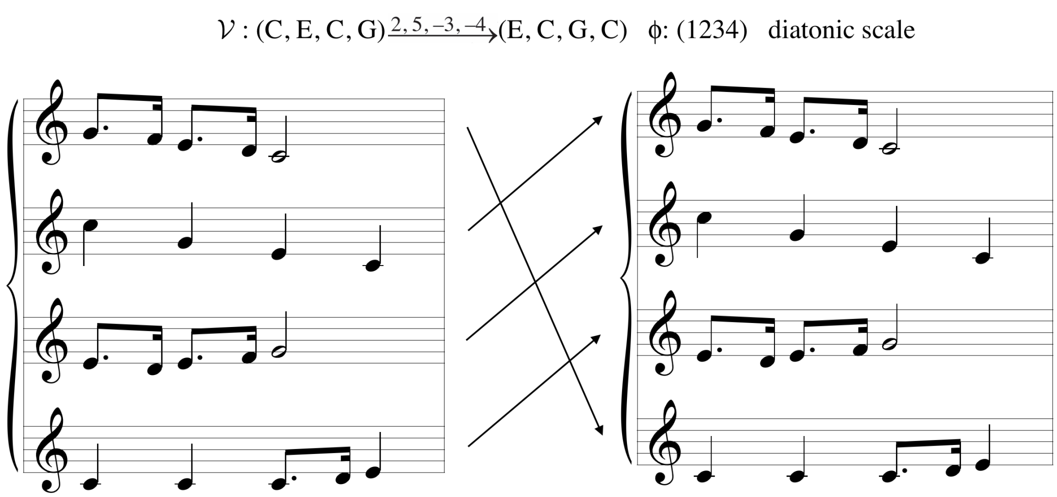

Figure 5: *The round "Row, row, row your boat."*

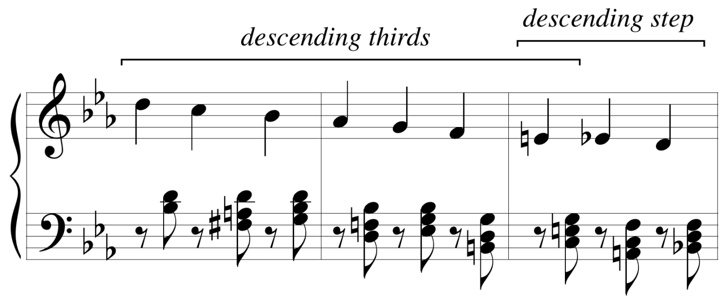

Figure 6: *A variable sequence in Beethoven's Op. 31 no. 3, I, mm. 68–70.*

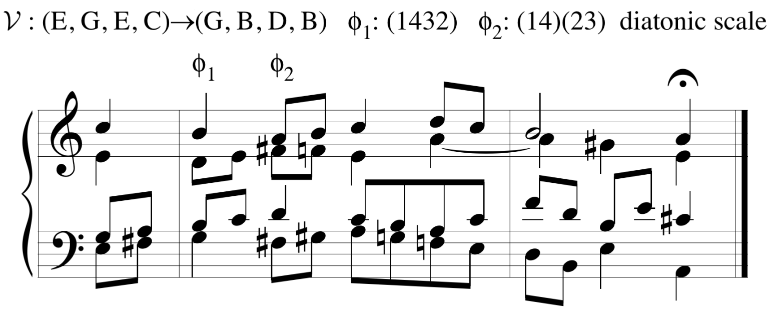

Figure 7: *The final phrase of Bach's chorale "Ach lieben Christen, seid getrost" (BWV 256, Riemenschneider 31).*

All of which is fairly clear when set out in abstract, mathematical form. However, I can testify that even an analytically minded musician can spend a lifetime working with iterated voice-leading patterns without clearly understanding their general structure.

References

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