The State of the Art: New Directions in Music and Mathematics

Robert W. Peck
Louisiana State University
rpeck@lsu.edu

Abstract: *This article is adapted from the author's 2018 keynote lecture at the Third National Congress of Music and Mathematics in Rio de Janeiro. It discusses the history of the fields of mathematical and computational musicology, as well as the formation of the* Journal of Mathematics and Music, *which the author co-founded in 2007. Further, it identifies recent trends in the fields of mathematical and computational musicology through an examination of the journal's special issues.*

Keywords: *Mathematical Musicology. Computational Musicology.*

The application of mathematics to music involves many areas of work: theories of music, music analysis, composition, performance; indeed, all (or, at least, nearly all) musical activities can be informed by mathematical thought. Such application endeavors to understand what is rational in musical practice, be it in the creation or appreciation of music. However, the questions that inform our conjectures and theorems proceed from a logic that is unique to music itself. In addition to reason, music as an applied mathematical art challenges us to discover a language for the expression of musical concepts and structures. Not only must it be able to describe the physical, acoustic properties of music, but also its deeply psychological aspects. It seeks to situate all of these viewpoints into a coherent formal description.

Over the centuries, this language has drawn from many branches of mathematics. The earliest extant work in our field incorporated techniques of what is now considered number theory in the study of musical intervals (however, at the time, it was called arithmetic). In the sixth century B.C., the Pythagoreans were among the first mathematical music theorists, through their belief that all things are either numbers or endowed with the properties of numbers. Musical intervals held a special position for them in this connection: they were numbers made audible. This tradition was passed through the neo-Pythagoreans into the Roman world during the first century A.D. It was the Roman Boethius who, around the year 500, essentially translated Nichomachus's *Manual of Harmonics* from Greek into Latin in his treatise *De institutione musica*, ensuring its preservation into the Dark Ages in Europe following the collapse of the ancient world.

The study of musical intervals thus became one of the seven liberal arts. Following the study of grammar, rhetoric, and logic—the trivium—students in the Middle Ages learned about musical intervals as one of the four disciplines in the medieval quadrivium: arithmetic, geometry, music, and astronomy. And, just as astronomy was viewed as geometry in motion, music was considered arithmetic in motion. Music, along with its six sister studies, was prerequisite to the study of philosophy; as an applied mathematical art, it was considered an essential component of an education.

With the advent of the modern era, this work continued in the writings of such scholars as Kepler and Galileo (as well as Galileo's father, Vincenzo), largely in their descriptions of Harmonia

Mundi (or the Music of the Spheres). Through their efforts, mathematical applications in music became increasingly sophisticated. The work of René Descartes, particularly his *Compendium musicae*, served as a bridge between the ancient and modern worlds in terms of mathematical musical thought. It was during this time that the emphasis turned from the mere numerical properties of intervals to the acoustical properties of musical sounds and to rational description of the perception of these sounds. To that end, questions concerning which sounds we perceive as consonances or dissonances came to occupy a more central position, marking a shift from the belief that such things as musical intervals are themselves endowed with these properties, to an attitude that it is our experience of them that defines their qualities.

The formalism of these new ideas, representative of the Enlightenment in Europe, required new mathematical tools. Composers such as Jean-Phillipe Rameau applied these contemporary scientific ideas to their musical theories. Scientists such as Gottfried Leibniz (the founder of calculus) and Leonhard Euler (who published over 10,000 pages of research in all areas of mathematical thought) also applied their own original ideas to music. For example, answering the question of consonance versus dissonance now required measuring tools. Where does a particular interval fall on a continuous spectrum between pure consonance and pure dissonance? How do the acoustical properties of the instrument making the sound affect our perception of consonance or dissonance? What of the environment through which the sound travels? How does the human ear function in this regard? An interval incorporates two tones—what if we want to examine chords with more than two notes? This line of inquiry continued well into the nineteenth century in the work of Hermann von Helmholtz on "psychophysics," and its testing of the limits of human perception.

With the twentieth century came even greater strides in the application of mathematics to music. Arnold Schoenberg's development of the "Method of Composing with Twelve Tones" in the early part of the century inspired composers and music theorists alike for the next several decades with an essentially algebraic, group-theoretical approach to composition and analysis, based on permutations of pitch classes and row-order positions. (Granted, Schoenberg himself was not concerned with this mathematical basis. It was later that composer-theorists such as Milton Babbitt discussed Schoenberg's technique in these terms.) What is significant here is that the mathematics served not only to describe properties of musical events, such as whether they are consonant or dissonant, but to generate music itself as a compositional method. This had the effect of moving mathematical applications in music out of a merely descriptive role to a structural one. However, Schoenberg's method was not the only mathematically inspired technique from around this time: Josef Hauer arrived at a similar compositional method, using tropes (or unordered, complementary hexachords). Joseph Schillinger developed his own mathematical method of musical composition, which taught composers such as George Gershwin ways of organizing musical forms.

Returning to Babbitt, however, we observe a line of inquiry that leads directly into the mathematical music theories of today. Babbitt—as we observed previously—was the first to analyze the group-theoretical basis for Schoenberg's 12-tone method in the 1950s. The subsequent generation of composers and music theorists continued Babbitt's work, applying algebraic techniques not only to ordered rows (of pitch classes or attack points, etc.), but also to unordered sets. In the 1960s, Howard Hanson began the categorization of the power set of the aggregate, the set of all subsets of the chromatic scale. This work was furthered by Allen Forte, in his seminal work from 1973, *The Structure of Atonal Music* [17], which introduced the notion of pitch-class sets, interval vectors, set classes, the Z relation, etc., to a wide audience of music theorists. Meanwhile, John Clough was applying similar mathematical techniques to the study of the diatonic system. In addition to music theorists, composers were also publishing research along these lines: John

Rahn in his *Basic Atonal Theory*, and Robert Morris in his several significant articles and his book *Composition with Pitch Classes*.

In the 1980s and 1990s, the application of group theory to music also found a home in the study of historical music theories. David Lewin's work in transformational theory [32] included examples of generalized interval systems that derived from nineteenth-century and early twentieth-century models of tonal relations by Hugo Riemann, Arthur von Oettingen, Moritz Hauptmann, and others. Through this "neo-Riemannian theory," the mathematical analysis of earlier, tonal music came to occupy a prominent position, tantamount to the analysis of contemporary compositions. This work was furthered greatly though the efforts of Richard Cohn. The defining characteristic of much of the neo-Riemannian analysis, Cohn's notion of "parsimonious voice leading," engendered further investigation into efficient voice leading among not just consonant triads, but among and between other sonority types, leading ultimately to the geometric, orbifold models of voice leading developed by Clifton Callender, Ian Quinn, and Dmitri Tymoczko in the first decade of the twenty-first century.

Simultaneous to these developments in North America, the formation of another mathematical music theory took place in Europe. However, whereas much of the work in the Americas was conducted by musicians who had a keen interest in mathematics, the European practitioners were mostly mathematicians who were also highly skilled in music. Heading this movement was the Swiss mathematician and jazz pianist Guerino Mazzola. His teaching and research reached several other important figures in other European countries as well, including Thomas Noll in Germany, Moreno Andreatta in France, Domenico Vicinanza in Italy, Francisco Gomez in Spain, Anja Volk in the Netherlands, Christina Anagnostopoulou in Greece, and many others.

The recent mathematical music theories on both sides of the Atlantic Ocean have incorporated increasingly sophisticated mathematical models. In addition to the group-theoretical work that began in the middle twentieth century, composers and music theorists have applied many additional branches of mathematics to the study and creation of music: including, but not limited to, category theory, knot and braid theory, algebraic topology, combinatorics, topos theory, combinatorial word theory, module theory, homotopy theory, homology theory, graph theory, algebraic geometry, mathematical physics, machine learning, and computation. In fact, even ideas from mathematics education have had a bearing on musical studies.

Because of the plethora of concepts and techniques that have been introduced into our field, it became necessary to establish a central forum for the sharing of these ideas. That was largely the impetus behind the founding of the Journal of Mathematics and Music, and its associated Society for Mathematics and Computation in Music. In the next part of this article, I will describe briefly the process that led to their formation.

In January of 2001, the American Mathematical Society (AMS) and the Mathematical Association of America held their national meetings (as part of the annual Joint Mathematics Meetings) in New Orleans, Louisiana—a city that is 120 kilometers from my home. My wife's mother, Judith Baxter, and stepfather, Stephen Smith, are professional mathematicians, and they planned to attend that meeting. Because I had been working in the field of mathematical music theory, I thought it would be opportune for me to present some of my recent research at that meeting, so I submitted presentation proposals to both societies, which were accepted by the respective program committees. That experience was very positive, and I received interesting feedback from a number of mathematicians who were interested in music, and who were in attendance at the meeting. Consequently, when the American Mathematical Society announced that they would be holding a sectional meeting in Baton Rouge, Louisiana—my own city—in 2003, I planned to submit another proposal. Judith Baxter suggested that, because of the interest generated at the previous meeting, rather than a single presentation, she and I propose a whole special session devoted to mathematics and music. The proposal was accepted, and the first AMS special session on Mathematical Techniques in Musical Analysis was held in March of that year. It featured talks by twenty mathematicians and music theorists and composers, including Thomas Noll from Germany, and was attended by several mathematicians who were at the conference.

Based on the success of that special session at a sectional meeting, I proposed another special session at the next national meeting, in Phoenix, Arizona, in January of 2004. This session was somewhat larger, and generated a great deal of interest at the meeting. The session incorporated talks by several European mathematicians. One of the presenters at this session, Richard Cohn, suggested an organizational meeting during the conference to discuss the idea of an international journal devoted to the mathematical study of music. The idea was received with tremendous enthusiasm, and a working group was formed to begin the process of bringing this idea to life. That working group consisted of Thomas Noll and Moreno Andreatta from Europe, and Norman Carey, Ian Quinn, and myself from the Americas.

Another special session on Mathematical Techniques in Musical Analysis was accepted for the AMS sectional meeting in fall of 2004 in Evanston, Illinois. The working committee reconvened at that session, along with several members of the growing community of interested mathematicians and musicians. Another meeting was held to continue the work of creating the journal. Topics discussed at that meeting were the name of journal, possible publishers, and the election of the first editors-in-chief. It was decided that there would be two editors: one mathematician and one musician, and one from each side of the Atlantic. The first two editors were decided to be Thomas Noll and myself.

During the years 2005 and 2006, the editors, along with the working committee, continued the necessary planning for the journal. After much debate, it was decided that the publisher would be Taylor & Francis of London, England. Further, we had to choose how many issues we would publish per year, the approximate page count for issues and how many articles they should include, and related questions. We decided to publish the journal in both print and online versions. We also issued the initial call for submissions. Among the most significant of our plans for the journal was to devote one special issue per year to a topic of current research interest. We would invite experts in these areas to guest edit the special issues, allowing them to solicit papers or to invite authors to contribute articles.

Taylor & Francis, asked us if there was a related society whose members might receive a reduced subscription rate to the journal. At that time, there were various organizations for mathematicians (such as the AMS) and other organizations for musicians (such as the Society for Music Theory), but none dedicated solely to the interdisciplinary study of music and mathematics. So, we founded at that time the Society for Mathematics and Computation in Music. From the beginning, it was intended to be an international undertaking, and to hold its biennial meetings alternately on opposite sides of the Atlantic.

The Journal of Mathematics and Music launched in January 2007. The first issue, Volume 1, Number 1, included a "Welcome" by the editors [44]; and research articles by John Rahn [49], Guerino Mazzola and Moreno Andreatta [36], and Jack Douthett and Richard Kranz [15]. The first meeting of the Society for Mathematics and Computation in Music followed in May of that year. It was held at the State Institute for Music Research in Berlin, Germany.

In July 2007, we published Volume 1, Number 2—the first special issue, devoted to "The Legacy of John Clough in Mathematical Music Theory." The guest editor for this issue was David Clampitt (a former student of Clough's). In addition to Clampitt's guest editorial [12], it included articles by Norman Carey ("Coherence and sameness in well-formed and pairwise well-formed scales") [8], Julian Hook ("Enharmonic systems: A theory of key signatures, enharmonic equivalence and diatonicism") [21], and Thomas Noll ("Musical intervals and special linear transformations")

[42]. John Clough's work has held much relevance for mathematical research in music theory since the late 1950s. His continued research into the diatonic system served as the basis for much of modern scale theory. It was a natural choice to devote this first special issue to his legacy. This issue investigated several mathematical-music topics that grew out of Clough's pioneering work, including generic and specific interval measures; Myhill's property (the condition in which intervals among the members of a scale come in two specific sizes), the property called "cardinality equals variety," which generalizes Myhill's property to subsets of a scale for any size, and which was one of the early examples of a significant mathematical result's being discovered first in a musical context; contextual operations; the theory of well-formed scales; and musical applications of continued fractions; the Farey series; the Stern-Brocot tree; the special linear group $SL(2,\mathbb{Z})$; and aspects of combinatorial word theory.

The second special issue, Volume 2, Number 2, was published in July of 2008. It was guest edited by Elaine Chew, Alfred Cramer, and Christopher Raphael, and was devoted to the topic of computation in music research - the first of a few special issues that dealt with aspects of computation. It featured articles by Chantal Buteau and John Vipperman ("Representations of motivic spaces of a score in *OpenMusic*") [7], Leigh M. Smith and Henkjan Honingh ("Time-frequency representation of musical rhythm by continuous wavelets") [50], and Anja Volk ("Persistence and change: Local and global components of metre induction using Inner Metric Analysis") [53]. Of particular concern in this issue are the processes and methods of computing data that represent musical information. Such a line of inquiry has implications not only for music theory and analysis, but also composition and improvisation, performance, and cognition and music perception. For instance, aspects of motivic manipulation can be generated or analyzed by machine - often more efficiently or thoroughly than by the human hand. Another application of computation could be found in determining algorithms. One may use computation to deduce the key of a piece or section of music, or its meter (based on finding and relating weights of adjacent and non-adjacent periodicities). Such determinations can be compared to those of more traditional music-theoretic analysis, or to the empirical results of cognitive research, thereby opening a new space for investigation that occupies the intersection of humanistic and scientific approaches to music study. One interesting question raised by the guest editors of this issue concerns the bottom-up versus top-down methods that are possible in a computational study [11]. Does one start with the data and build a theory up from there, or does one begin with a central idea that drives the pursuit of finer levels of detailed inquiry? The articles in this issue incorporate different approaches in this regard, which the editors acknowledge and offer to the reader for comparison.

The third special issue appeared in July 2009. Volume 3, Number 2, was devoted to "Tiling Problems in Music," and was guest edited by Moreno Andreatta and Carlos Agon. This issue included the following three articles: "New perspectives on rhythmic canons and the spectral conjecture," by Emmanuel Amiot [1]; "Algorithms for translational tiling," by Mihail N. Kolountzakis and Máté Matolcsi [30]; and "Tiling the integers with aperiodic tiles," by Franck Jedrzejewski [27].

In their introductory editorial to the issue [5], Andreatta and Agon quote Leibniz's famous letter to Christian Goldbach from April 17, 1712: "Musica est exercitium arithmeticae occultum nescientis se numerare animi" (which translates to: "music is a hidden arithmetic exercise of the soul, which does not know that it is counting"). In the case of tiling structures in music, the editors pararphrase Leibniz to say: "In some cases, mathematics is an 'exercitium musicae'." They continue: "What characterizes a 'mathemusical' problem is the fact that settling the originally musical problem in an appropriate mathematical framework not only gives rise eventually to new mathematical results, but also paves the way to new musical constructions that would have been impossible to conceive without the process of 'mathematization'. It is this double movement, from music to mathematics and backwards, which makes a 'mathemusical' problem so intriguing to both mathematicians and musicians" [5, pp. 63-64]. In this case, the problem is construction of rhythmic tiling canons: These are special musical forms which consist of a rhythmic pattern that completely tiles the musical time axis by temporal translations. Think of it as a strict rhythmic canon in which every beat receives one and only one impulse. Such tiling canons are not a new concept. Indeed, they have been in use since the Ars Nova though more recent American Minimalism and beyond.

The creation and study of tilings has a long and distinguished history. It incorporates results from several branches of mathematics: geometry, certainly, but also group factorizations and direct sums, characters and the discrete Fourier transform, Minkowski's (number-theoretical) tessellation problem, Fermat's last theorem, the classification of groups (they cannot be factorized into a direct sum of subsets without at least one of the factors being periodic), cyclotomic polynomials, and the use of machine computation. The study of tilings also has connections to at least one open mathematical conjecture, the Fuglede or spectral conjecture: This conjecture deals with the relation between the spectral property of a domain in n-dimensional Euclidean space and its tiling character. It states that such a domain admits a spectrum if it tiles n copies of $\mathbb{R}$ by translation. Another topic discussed in the issue is the means of testing of mathematical theories for determining the number of isomorphism classes of rhythmic tiling canons.

Of particular interest throughout this issue is the construction of Vuza canons, so-named after Dan Vuza, who discovered them, and who first published on them in the journal *Persepectives of New Music* in 1991 [55]. In fact, the mathematical problem in Vuza's *Perspective of New Music* article was rediscovered several years later by mathematicians. Vuza canons are one example of tiling canons that can be generated by means of cyclotomic polynomials. Other musical applications of tiling canons include homometric structures, such as the well-known Z-related pairs of sets classes in traditional pitch-class set theory; Boulez's chord multiplication, such as he used in his composition *Le marteau sans maître*; and transpositional combination.

In July 2010, Volume 4, Number 2 appeared—another special issue devoted to computation, titled: "Computational Music Analysis: Can computational music analysis be both musical and computational?" The guest editors for this issue were Christina Anagnostopoulou and Chantal Buteau. All of the articles in this issue incorporate analyses of the same piece of music: Brahms's String Quartet No. 1 in C Minor, Op. 51, No. 1. The articles are by Darrell Conklin ("Distinctive patterns in the first movement of Brahms' String Quartet in C minor") [13], Atte Tenkanen ("Tonal trends and α-motif in the first movement of Brahms' String Quartet op. 51 nr. 1") [51], and Philippe Cathé ("Harmonic vectors and stylistic analysis: a computer-aided analysis of the first movement of Brahms' String Quartet Op. 51-1") [10].

The guest editors take as a point of departure a quotation from Ian Bent: "music analysis is the means of answering the question: 'How does it work?"' [6, p. 5]. It relies on the comparison of data, and, via comparison, it discovers the structural elements of a piece of music and the specific functions of these elements. To conduct such an analysis requires a neutral level, which does not necessarily take into account the composer's intentions (the poetic level) or the listener's cognitive mechanisms, intuitions, aesthetic judgements, emotions, or reactions (the aesthesic level). In this regard, certain analytical methods, such as those of Jean-Jacques Nattiez, are cricitized: a human analysis may reflect the analyst's own perceptions, and would thus not be neutral. However, computational analysis can get closer to this neutrality. However, the human analyst cannot be eliminated altogether, when one seeks to combine objectivity and scientific rigor with the interpretative nature of music analysis. With this in mind, the computational side of music analysis has certain well-defined aims: to produce musicologically interesting results, to formulate a (neutral) analytical process, and to perform calculations that would have been difficult, tedious, or impossible by hand. Finally, an additional aim is to test computational methodologies.

Future work in computational music analysis should address various limitations and problems in the field:

1. The division that exists between analyses that start from various symbolic representations (like notation) and others from the audio signal. Often these two worlds do not meet.

2. The lack of emphasis on the representational issues, which nevertheless are crucial both for the formalization and the result aspects.

3. The plethora of approaches, with a distinct lack of comparisons and discussions between them—although the guest editors suggest that the formation of MIREX (the Music Information Retrieval Evaluation eXchange) is an important step towards this direction.

4. The lack of connections to the field of more traditional music analysis, a situation that has resulted in varying opinions from both sides represented in the special issue.

5. The means of musical evaluation of a system's results. For example, can the data alone reveal what a traditional motivic analysis proposes or the evolution of tonality in a piece that is evident in, say, a Schenkerian analysis?

In general, computational music analysis needs to address the delicate issue of the balance between computation, musicologically sound methodology, and the proper evaluation of results. Ultimately, the human factor is still crucial and necessary in any analytical approach, as is the inherent diversity in music analysis.

In 2011 (which coincided with Volume 5 of the journal), no special issue appeared: all three issues were devoted to contributed papers. However, Volume 6, Number 2, which appeared in July 2012, was another special issue, Mathematical and Computational Approaches to Music: Three Methodological Reflections, guest edited by Anja Volk and Aline Honingh. Unlike the previous special issues, this one followed closely from a workshop at the previous year's meeting of the Society for Mathematics and Computation in Music in Paris. In this workshop, three senior scholars in the field, Guerino Mazzola, Geraint A. Wiggins, and Alan Marsden presented their views on the current state of mathematical and computational analysis, and each was also invited to comment on the others' remarks. Essentially, the special issue reproduced this exchange, with position papers by all three authors: Guerino Mazzola ("Thinking music with precision, depth, and passion") [35], Geraint A. Wiggins ("Music, mind and mathematics: theory, reality and formality") [56], and Alan Marsden ("Counselling a better relationship between mathematics and musicology") [34], and responses from the others. The result is an interesting and challenging exchange of ideas.

In their introduction to the issue, the guest editors provide a background to the debate [54]. They cite Nicholas Cook, who states that we have been standing quite long at a moment of opportunity with respect to the relation between computational approaches and musicology, without reaching the full potential of the interdisciplinary enterprise. Mathematical music analysis seems too detailed for more traditional musicologists, and at the same time it is too vague for mathematical scientists. Nevertheless, it is a growing field, both in terms of computational and mathematical applications.

The first computational projects in music began in the 1960s: the idea was that machines could process more information than humans can process practically. More recently, computation has been a major tool in music theory and analysis, musical performance research, historical musicology, ethnomusicology, and cognitive musicology. The earliest known applications of mathematics to music were by the Pythagorean school, who associated intervals and numbers. Since that time, mathematics has been used to study aspects of tuning and temperament, consonance and dissonance, musical set theory, scale theory, transformational theory, musical topos theory, and various compositional models.

The articles focus on four main topics: what are the benefits of mathematical and computational music analysis? What are its failures? What are the challenges going forward? And what opportunities are available for interdisciplinary discourse among mathematical, computational, and musicological approaches to music research? With regard to these topics: in terms of benefits, What key contributions did mathematical and computational approaches bring to the field of music research according to your point of view? Each of the authors (with certain overlap) lists some specific benefits:

- key contributions to music technology, e.g., to mp3-format (Mazzola, Marsden)
- conceptual clarification of music theoretic concepts (Mazzola)
- mathematics allows to understand existing music from its positioning within a broader framework of possible musics (Marsden, Mazzola)
- mathematics affords explanations of the consequences of tuning (Marsden)
- generalizations in mathematics allow addressing the evolution of music (Mazzola)
- computation allows rigorous testing of evolutionary hypotheses (Wiggins)
- computation is a hard test of the efficiency and precision of conceptualization and operationalization of musical thought (Mazzola)
- computation allows creation of educational music tools (Marsden, Wiggins)

With regard to failures: What are examples of pitfalls that occurred within computational and/or mathematical approaches to music research in the past? What can we learn from them? Again, the authors list particular shortcomings of existing research in the field:

- it is a failure to study the musical products without the musical processes (Wiggins)
- it is a failure to treat music theory not as fundamentally perceptual (Wiggins)
- it is a failure to use information-theory approaches to music (Mazzola)
- it is a failure to neglect the sign-theoretical shape of musical phenomena in favor of purely formal aspects (Mazzola)
- it is a failure that mathematicians concentrate too much on scale theories; musicologists are more interested in musical pieces (Marsden, Mazzola)
- it is a failure to reject higher math as a means to understand complex musical structures on the grounds that everybody understands music in some sense (Mazzola)

Next, what challenges are we facing now within computational and/or mathematical approaches to music? What unexplored fields and questions have the potential to move our understanding of music forward with the help of computational and/or mathematical approaches? What steps need to be taken now and in the near future in order to fully unfold the potential of computational and/or mathematical approaches to music? The authors identify the following challenges that we are facing in computational and mathematical approaches to music:

- musicologists have not yet taken up the tools offered by mathematics and computation (Marsden)
- accounting for the 'messiness' of humans is a challenge for the preciseness in mathematics and computation (Marsden, Wiggins)
- classification of musical objects in mathematics alone is not enough; we need to apply them to musical pieces within computational experiments (Mazzola)
- to achieve a meta-theory of music, we need to rise above the study of particular examples of music tying all musical cultures together through time (Wiggins)

- we need to model musical behavior rather than merely music products (Wiggins)
- mathematicians need to consider more empirical, statistical work for a better connection between rationalism and empiricism (Marsden)

And, finally, how can we strengthen the connections among the three fields of mathematical, computational, and musicological approaches to music? Are there different ways of 'understanding' music in these three fields? In what context are the differences between the disciplines (mathematics, music research, computer science) a useful source for innovative research on music-related questions? When do the differences between the disciplines become a stumbling block for interdisciplinary research, and what needs to be done to overcome that? In the interdisciplinary discourse, we need to consider:

- humility is essential for interdisciplinary work (Marsden)
- honesty about what is within scope and what not is important (Marsden)
- using the fuzzy concept of 'musicality' as an argument to dismiss mathematical or computational models of music is not appropriate (Mazzola)
- achieving mathematical theories that are computational and can be tested by comparison with humans requires substantial interdisciplinarity (Wiggins)

In answering all these questions, the three authors discuss the use of models and the issue of abstraction; the precision of mathematics and computation versus the imprecision of humans; and, ask what kind of theory is music theory—what are the ultimate consequences for mathematical and computational approaches to music?

The special issue in 2013, Volume 7, Number 2, was devoted to the subject of "Mathematical Theories of Voice Leading." It followed from the John Clough Memorial Conference in 2013, which dealt with the same topic. The conference and the issue endeavored to seek a reconciliation of mathematical theories of voice leading by the authors who presented work at that conference, and whose articles appeared in the issue: Julian Hook ("Contemporary methods in mathematical music theory: a comparative case study") [23], Richard Plotkin and Jack Douthett ("Scalar context in musical models") [46], Dmitri Tymoczko ("Geometry and the quest for theoretical generality") [52], and Jason Yust ("Tonal prisms: iterated quantization in chromatic tonality and Ravel's 'Ondine"') [57].

In his introductory editorial, guest editor Marek Žabka notes that this issue marked a shift away from the algebraic basis of the mathematical music theory of the previous decades, particularly the transformational theories of Lewin. However, Lewin's notion of "abstract (musical) spaces as extratemporal universe[s] of quasi-spatial potentialities," where actual pieces of music are conceived as "human gestures that move through chronological time" continues to serve as a central theme in these theories of voice leading [33, p. 41]. Specifically, Hook's 'cross-type transformations,' Tymoczko's 'paths in pitch class space,' and Douthett's 'stroboscopic portraits' all consider musical motion as a form of gesture in an abstract space—although their specific means of describing these gestures differ significantly from Lewin's. Another point of commonality among the articles is their reliance on Clough and Douthett's theory of maximal evenness. This property remains a central issue in the types of voice-leading structures employed in much music, and suggests its universal significance.

The next special issue, Volume 8, Number 2, which was published in July 2014, represented a different direction: guest editors Jason Yust and Thomas M. Fiore put together an issue that does not engage directly with a research technique, but rather with "Pedagogies of Mathematical Music Theory." They invited six articles by scholars who have taught courses in the field: Jon Kochavi ("*Musica speculativa* for the twenty-first century: integrating mathematics and music in the liberal arts classroom") [29], Rachel Wells Hall ("Acoustics labs for a general education math and music course”) [20], James R. Hughes (“Creative experiences in an interdisciplinary Honors course on mathematics in music”) [25], Robert Peck (“Mathematical music theory pedagogy and the ’New Math”’) [45], Mariana Montiel and Francisco Gómez (“Music in the pedagogy of mathematics”) [39], and Thomas Noll (“Getting involved with mathematical music theory”) [43]. The first three of these articles describe specific courses, their goals, and pedagogical philosophies, provide samplings of their contents, and reflect on successes and challenges. The next three papers are essays of a more general nature about the teaching and learning of mathematical music theory, and its place in academic institutions and the public sphere at large.

The editors proceed from the adage that “a discipline defines itself by its pedagogy” [59]. In this connection, curricula provide a basis of shared knowledge, and a means to organize, classify, and categorize that knowledge. Further, it records and passes along the history of the discipline, defining along the way its classic results and its most significant contributors. In the case of mathematics and music theory, both fields require individually a great deal of specialized knowledge and skills to access basic research. When these subjects are combined in the interdisciplinary context of mathematical music theory, these challenges of accessibility are multiplied substantially. Several themes emerged across the articles in this issue. Among the most ubiquitous is that creative and discovery-based learning seems to be an effective method for the teaching of mathematical music theory. Certain challenges also reappear in several of the articles: these problems center primarily around the issue of accessibility and making mathematical music theory available to a general audience. Solutions to these problems include popularization, such as creating museum exhibits and other public institutional venues that promote the topic of music and mathematics.

The special issue from July 2015, Volume 9, Number 2, is another unique issue. It is the only special issue (indeed the only issue of the *Journal of Mathematics and Music*) that is devoted primarily to a single paper: Harald Fripertinger’s and Peter Lackner’s “Tone rows and tropes” [18]. Guest Editors Julian Hook and Robert Peck also invited responses by Robert Morris [41] and Andrew Mead [37], both composers of 12-tone music and leading experts in its theory.

The main article deals with a classification of the set of 479,001,600 12-tone rows. The authors complete the categorization of tone rows and tropes using Pólya’s Theorem, which derives from and essentially generalizes Burnside’s lemma on the number of orbits of a group action on a set. In this case, the set is the collection of all 12-tone rows, and the groups the authors employ involve the canonical operations of transposition, inversion, retrograde, rotation, and multiplication of pitch classes and order position numbers. Whereas classifying and counting musical objects have long been among music theorists’ typical endeavors, using sophisticated techniques from the field of combinatorics is rather novel. Julian Hook published an article that contains an application of Pólya’s Theorem to the universe of pitch-class sets in 2007 [22], but that is a significantly smaller set.

As the guest editors discuss [24], bringing such mathematical complexity to music also creates certain problems. The theorems and proofs of the mathematicians leave the musicians nonplussed; even those few who can follow the mathematical detail often find that it falls short in musical relevance or sensitivity, in historical or cultural reference, and in connections to the ways they have learned to think about music. The writings of the musicians, meanwhile, may be filled with revealing musical examples and analyses but frustrate the mathematicians via imprecise prose, mathematical terms and notations deployed in ways that are non-standard or outright wrong, and rambling commentary in the place of logical demonstration. The divide is both made evident and also mended to a certain degree in Morris’s and Mead’s responses to the main article. Both responders speak to earlier results in the world of 12-tone composition that anticipate many of the paper’s findings.

A special issue on "Machine Learning and Music Generation" followed in 2016: Volume 10, Number 2. It was guest edited by José M. Iñesta, Darrell Conklin, and Rafael Ramírez, and featured articles by Darrell Conklin ("Chord sequence generation with semiotic patterns") [14], Sergio Giraldo and Rafael Ramírez ("A machine learning approach to ornamentation modeling and synthesis in jazz guitar") [19], Phillip B. Kirlin and Jason Yust ("Analysis of analysis: Using machine learning to evaluate the importance of music parameters for Schenkerian analysis") [28], Katerina Kosta, Rafael Ramírez, Oscar F. Bandtlow, and Elaine Chew ("Mapping between dynamic markings and performed loudness: a machine learning approach") [31], and Pedro J. Ponce de León, José M. Iñesta, Jorge Calvo-Zaragoza, and David Rizo ("Data-based melody generation through multi-objective evolutionary computation") [47].

Music generation research has generally employed one of two strategies: first, knowledge-based methods that model style through explicitly formalized rules; and, second, data mining methods that apply machine learning to induce statistical models of musical style, training data, music representation, candidate generation, and evaluation, and modeling and generating music, including melody, chord sequences, ornamentation, and dynamics. In such machine learning, models are induced from either audio data or symbolic data. The guest editors identify a challenging issue in computational music generation, especially if fully automated: the determination of concrete musical events that cohere with certain more abstract or underlying structures [26]. Examples of these situations occur as metrical regularities or as both adjacent and distant repetitions.

In general, machine learning is incorporated in the generation of events and event sequences through statistical models. For instance, one might employ model-based prediction or sampling from a statistical model, or in evaluating candidate-event sequences or ranking generated sequences that are based on a statistical model. In studies of repetition in music, it develops a way to represent and generate repetition using statistical models. Further, it develops a pattern representation that explicitly describes relations between events using variables that are instantiated during generation. A semiotic pattern describes a formal language of sequences, and sequences satisfying the pattern are generated by sampling from a statistical model. The statistical model derives from a corpus and is an instance of the viewpoint-modeling approach used previously with success for prediction, classification, and music generation. Of particular interest are genetic algorithms. These provide an optimization technique that mimics the biological evolution of living beings and natural selection. A population of individuals, representing different possible solutions, are subjected to crossovers and mutations, and a selection stage decides which individuals might best solve the problem. Those individuals are allowed to procreate a new generation of, supposedly, better individuals until convergence.

The articles in this issue involve topics such as creating a machine-learning system that synthesizes a notated jazz melody, while adding ornamentations that are characteristic of a natural musical interpretation. Such a system might rely on a classifier that learns from a dataset of recorded performances in deciding which notes to ornament and which notes not to ornament. Another topic is experiments with learning probabilities for melodic reductions, such as in modeling Schenkerian analysis, directed at discovering the importance of different features that most closely match an accepted analysis. Another article examines the relationship between dynamic markings in music scores and performed loudness by applying machine-learning techniques to induce predictive models of loudness levels corresponding to dynamic markings, and to classify dynamic markings given loudness values.

2017 produced a double special issue: both Numbers 2-3 in Volume 11 dealt with the topic of "Perfect Balance and the Discrete Fourier Transform." This issue was edited by Thomas M. Fiore, and included two articles by Emmanuel Amiot ("Decompositions of nil sums of roots of unity. An adaptation of 'Sommes nulles de racines de l'unité"' and "The discrete Fourier transform of distributions") [2] and [3], Andrew J. Milne, David Bulger, and Steffen A. Herff ("Exploring the space of perfectly balanced rhythms and scales") [38], Norman Carey ("Perfect balance and circularly rich words") [9], and Jason Yust ("Harmonic qualities in Debussy's 'Les sons et les parfums tournent dans l'air du soir"') [58].

Fiore's editorial observes that applications of the discrete Fourier transform in music scholarship span decades, countries, and continents [16]. This issue develops the mathematical framework of the discrete Fourier transform for music theory, and it connects it to salient music-theoretical problems and questions. In particular, the authors in the issue examine the mathematical ramifications of perfectly balanced scales and rhythms via the discrete Fourier transform, and musically explore the smooth manifold they inhabit. They study perfectly balanced scales from a word-theoretic perspective, focusing on those perfectly balanced scales with circular palindromic "richness" and relatively few step "differences." The final article analyzes Debussy's prelude "Les sons et les parfums tournent dans l'air du soir" from a Fourier perspective, highlighting perfectly balanced pitch-class sets.

While it has not appeared in print yet, 2018's special issue examines "Combinatorics on Words." It is guest edited by Marc Chemillier, Christophe Reutenauer, and Srecko Brlek, and features articles by Jean-Paul Allouche, Tom Johnson, David Clampitt, Norman Carey, Thomas Noll, Marc Chemillier, Lorraine Ayad, and Solon Pissis. Exciting new work is being done in the application of combinatorial word theory to music, as several results in the mathematical field have been anticipated by the work of these music researchers.

* * *

The respective fields of music and mathematics are each tremendously rich, and they represent some of the greatest achievements of individuals, as well as society. Their interdisciplinary combination highlights aspects of each field that are not always in evidence. Such amalgamation also presents certain problems: particularly, we do not always have the language needed to describe these newfound facets. The work of the special issues of the *Journal of Mathematics and Music*, as well as of this conference, is to find new and successful ways of expressing and disseminating these ideas and thereby laying the groundwork for future studies in our field.

References

[1] Amiot, E. 2009. New Perspectives on Rhythmic Canons and the Spectral Conjecture. *Journal of Mathematics and Music*, v.3, n.2, pp. 71-84.

[2] Amiot, E. 2017. Decompositions of nil sums of roots of unity. An adaptation of "Sommes Nulles de Racines de l'Unité". *Journal of Mathematics and Music*, v.11, n.2, pp. 67-75.

[3] Amiot, E. 2017. The Discrete Fourier Transform of Distributions. *Journal of Mathematics and Music*, v.11, n.2, pp. 76-100.

[4] Anagnostopoulou, C., and Buteau, C. 2010. Can Computational Music Analysis be Both Musical and Computational? *Journal of Mathematics and Music*, v.4, n.2, pp. 75-83.

[5] Andreatta, M., and Agon, 2009. C. Special Issue: Tiling Problems in Music Guest Editors' Foreword. *Journal of Mathematics and Music*, v.3, n.2, pp. 63-70.

[6] Bent, I. 1987. *Analysis*. London: McMillan Press.

[7] Buteau, C., and Vipperman, J. 2008. Representations of Motivic Spaces of a Score in OpenMusic. *Journal of Mathematics and Music*, v.2, n.2, pp. 61-79.

[8] Carey, N. 2007. Coherence and Sameness in Well-formed and Pairwise Well-formed Scales. *Journal of Mathematics and Music*, v.1, n.2, pp. 79-98.

[9] Carey, N. 2017. Perfect Balance and Circularly Rich Words. *Journal of Mathematics and Music*, v.11, n.2, pp. 134-54.

[10] Cathé, P. 2010. Harmonic Vectors and Stylistic Analysis: a Computer-aided Analysis of the First Movement of Brahms' String Quartet Op. 51-1. *Journal of Mathematics and Music*, v.4, n.2, pp. 107-19.

[11] Chew, E., Cramer, A., and Raphael, C. 2008. Guest Editors' Foreword. *Journal of Mathematics and Music*, v.2, n.2, pp. 57-60.

[12] Clampitt, D. 2007. The Legacy of John Clough in Mathematical Music Theory. *Journal of Mathematics and Music*, v.1, n.2, pp. 73-78.

[13] Conklin, D. 2010. Distinctive Patterns in the First Movement of Brahms' String Quartet in C minor. *Journal of Mathematics and Music*, v.4, n.2, pp. 85-92.

[14] Conklin, D. 2016. Chord Sequence Generation with Semiotic Patterns. *Journal of Mathematics and Music*, v.10, n.2, pp. 92-106.

[15] Douthett, J., and Krantz, R. 2007. Continued Fractions, Best Measurements, and Musical Scales and Intervals. *Journal of Mathematics and Music*, v.1, n.1, pp. 47-70.

[16] Fiore, T.M. 2017. Introduction to the Special Issue on Perfect Balance and the Discrete Fourier Transform. *Journal of Mathematics and Music*, v.11, n.2, pp. 65-66.

[17] Forte, A. 1973. *The Structure of Atonal Music*. New Haven: Yale University Press.

[18] Fripertinger, H., and Lackner, P. 2015. Tone Rows and Tropes. *Journal of Mathematics and Music*, v.9, n.2, pp. 111-72.

[19] Giraldo, S., and Ram/'irez, R. 2016. A Machine Learning Approach to Ornamentation Modeling and Synthesis in Jazz Guitar. *Journal of Mathematics and Music*, v.10, n.2, pp. 107-26.

[20] Hall, R.W. 2014. Acoustics Labs for a General Education Math and Music Course. *Journal of Mathematics and Music*, v.8, n.2, pp. 125-30.

[21] Hook, J. 2007. Enharmonic systems: A Theory of Key Signatures, Enharmonic Equivalence and Diatonicism. *Journal of Mathematics and Music*, v.1, n.2, pp. 99-120.

[22] Hook, J. 2007. Why Are There Twenty-Nine Tetrachords? A Tutorial on Combinatorics and Enumeration in Music Theory. *Music Theory Online*, v. 13, n. 4.

[23] Hook, J. 2013. Contemporary Methods in Mathematical Music Theory: a comparative case study. *Journal of Mathematics and Music*, v.7, n.2, pp. 89-102.

[24] Hook, J., and Peck, R. 2015. Introduction to the Special Issue on Tone Rows and Tropes. *Journal of Mathematics and Music*, v.9, n.2, pp. 109-10.

[25] Hughes, J.R. 2014. Creative Experiences in an Interdisciplinary Honors Course on Mathematics in Music. *Journal of Mathematics* and Music, v.8, n.2, pp. 131-43.

[26] Iñesta, J.M., Conklin, D., and Ramírez, R. 2016. Machine Learning and Music Generation. *Journal of Mathematics and Music*, v.10, n.2, pp. 87-91.

[27] Jedrzejewski, F. 2009. Tiling the Integers with Aperiodic Tiles. *Journal of Mathematics and Music*, v.3, n.2, pp. 99-115.

[28] Kirlin, P.B., and Yust, J. 2016. Analysis of Analysis: Using Machine Learning to Evaluate the Importance of Music Parameters for Schenkerian Analysis. *Journal of Mathematics and Music*, v.10, n.2, pp. 127-48.

[29] Kochavi, J. 2014. *Musica Speculativa* for the Twenty-first Century: Integrating Mathematics and Music in the Liberal Arts Classroom. *Journal of Mathematics and Music*, v.8, n.2, pp. 117-23.

[30] Kolountzakis, M.N. and Matolcsi, M. 2009. Algorithms for Translational Tiling. *Journal of Mathematics and Music*, v.3, n.2, pp. 85-97.

[31] Kosta, K., Ramírez, R., Bandtlow, O.F., and Chew, E. 2016. Mapping between Dynamic Markings and Performed Loudness: a Machine Learning Approach. *Journal of Mathematics and Music*, v.10, n.2, pp. 149-72.

[32] Lewin, D. 1987. *Generalized Musical Intervals and Transformations*. New Haven: Yale University Press.

[33] Lewin D. 1993. *Musical Form and Transformation*. New Haven: Yale University Press.

[34] Marsden, A. 2012. Counselling a Better Relationship between Mathematics and Musicology. *Journal of Mathematics and Music*, v.6, n.2, pp. 145-53.

[35] Mazzola, M. 2012. Thinking Music with Precision, Depth, and Passion. *Journal of Mathematics and Music*, v.6, n.2, pp. 83-91.

[36] Mazzola, G., and Andreatta, M. 2007. Diagrams, Gestures and Formulae in Music. *Journal of Mathematics and Music*, v.1, n.1, pp. 23-46.

[37] Mead, A. 2015. Remarks on "Tone Rows and Tropes" by Harald Fripertinger and Peter Lackner. *Journal of Mathematics and Music*, v.9, n.2, pp. 173-78.

[38] Milne, A.J., Bulger, D., and Herff, S.A. 2017. Exploring the Space of Perfectly Balanced Rhythms and Scales. *Journal of Mathematics and Music*, v.11, n.2, pp. 101-33.

[39] Montiel, M., and G/'omez, F. 2014. Music in the Pedagogy of Mathematics. *Journal of Mathematics and Music*, v.8, n.2, pp. 151-66.

[40] Morris, R. 1987. *Composition with Pitch Classes*. New Haven: Yale University Press.

[41] Morris, R. 2015. Review of "Tone Rows and Tropes" by Harald Fripertinger and Peter Lackner. *Journal of Mathematics and Music*, v.9, n.2, pp. 179-95.

[42] Noll, T. 2007. Musical Intervals and Special Linear Transformations. *Journal of Mathematics and Music*, v.1, n.2, pp. 121-37.

[43] Noll, T. 2014. Getting Involved with Mathematical Music Theory. *Journal of Mathematics and Music*, v.8, n.2, pp. 167-82.

[44] Noll, T., and Peck, R. 2007. Welcome. *Journal of Mathematics and Music*, v.1, n.1, pp. 1-6.

[45] Peck, R.W. 2014. Mathematical Music Theory Pedagogy and the "New Math". *Journal of Mathematics and Music*, v.8, n.2, pp. 145-50.

[46] Plotkin, P., and Douthett, J. 2013. Scalar Context in Musical Models. *Journal of Mathematics and Music*, v.7, n.2, pp. 103-25.

[47] Ponce de León, P.J., Iñesta, J., Calvo-Zaragoza, J., and Rizo, D. 2016. Data-based Melody Generation through Multi-objective Evolutionary Computation. *Journal of Mathematics and Music*, v.10, n.2, pp. 173-92.

[48] Rahn, J. 1980. *Basic Atonal Theory*. New York: Longman.

[49] Rahn, J. 2007. Cool tools: Polysemic and Non-commutative Nets, Subchain Decompositions and Cross-projecting Pre-orders, Object-graphs, Chain-hom-sets and Chain-label-hom-sets, Forgetful Functors, Free Categories of a Net, and Ghosts. *Journal of Mathematics and Music*, v.1, n.1, pp. 7-22.

[50] Smith, L.M., and Honingh, H. 2008. Time-frequency Representation of Musical Rhythm by Continuous Wavelets. *Journal of Mathematics and Music*, v.2, n.2, pp. 81-97.

[51] Tenkanen, A. 2010. Tonal Trends and A-motif in the First Movement of Brahms' String Quartet op. 51 nr. 1. *Journal of Mathematics and Music*, v.4, n.2, pp. 93-106.

[52] Tymoczko, D. 2013. Geometry and the Quest for Theoretical Generality. *Journal of Mathematics and Music*, v.7, n.2, pp. 127-44.

[53] Volk (née Fleischer), A. 2008. Persistence and Change: Local and Global Components of Metre Induction Using Inner Metric Analysis. *Journal of Mathematics and Music*, v.2, n.2, pp. 99-115.

[54] Volk, A., and Honingh, A. 2012. Mathematical and Computational Approaches to Music: Challenges in an Interdisciplinary Enterprise. *Journal of Mathematics and Music*, v.6, n.2, pp. 73-81.

[55] Vuza, D.T. 1991. Supplementary Sets and Regular Complementary Unending Canons (Part One). *Perspectives of New Music*, v. 29, n. 2., pp. 22-49.

[56] Wiggins, G.A. 2012. Music, Mind and Mathematics: Theory, Reality and Formality. *Journal of Mathematics and Music*, v.6, n.2, pp. 111-23.

[57] Yust, J. 2013. Tonal prisms: Iterated Quantization in Chromatic Tonality and Ravel's“"Ondine.” *Journal of Mathematics and Music*, v.7, n.2, pp. 145-65.

[58] Yust, J. 2017. Harmonic Qualities in Debussy's “Les sons et les parfums tournent dans l'air du soir.” *Journal of Mathematics and Music*, v.11, n.2, pp. 155-73.

[59] Yust, J., and Fiore, T.M. 2014. Introduction to the Special Issue on Pedagogies of Mathematical Music Theory. *Journal of Mathematics and Music*, v.8, n.2, pp. 113-16.

[60] Žabka, M. 2013. Toward the Reconciliation of Mathematical Theories of Voice Leading. *Journal of Mathematics and Music*, v.7, n.2, pp. 81-88.