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Foreword

IT is a great honor for us to present the second number of 2018's volume of *MusMat – Brazilian Journal of Music and Mathematics*. This issue opens with the keynote lecture given by **Robert Peck** at the Third National Congress of Music and Mathematics in Rio de Janeiro—*The State of the Art: New Directions in Music and Mathematics*. As the title indicates, it is a very enlightening survey on the most recent trends in the convergent fields of music and mathematics. The lecture also highlights the formation of the *Journal of Mathematics and Music*, co-founded by the author in 2007. **Ciro Visconti** proposes a new approach for the representation of any contextual inversion operations between members of any set class. **Liduino Pitombeira** discusses the theoretical bases of the analytical-compositional methodology called Systemic Modeling, using Debussy's *Prélude No.1* as a case study. **Marco Sampaio** proposes two new algorithms for melodic contour similarity that can be used with small and large contours. **Robert Morris**' article addresses two important issues in Pitch-Class Set Theory: Z-Related Hexachords explained by Transpositional Combination and the Complement Union Property.

Carlos Almada
December 2018

The State of the Art: New Directions in Music and Mathematics

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Abstract: *This article is adapted from the author's 2018 keynote lecture at the Third National Congress of Music and Mathematics in Rio de Janeiro. It discusses the history of the fields of mathematical and computational musicology, as well as the formation of the* Journal of Mathematics and Music, *which the author co-founded in 2007. Further, it identifies recent trends in the fields of mathematical and computational musicology through an examination of the journal's special issues.*

Keywords: *Mathematical Musicology. Computational Musicology.*

The application of mathematics to music involves many areas of work: theories of music, music analysis, composition, performance; indeed, all (or, at least, nearly all) musical activities can be informed by mathematical thought. Such application endeavors to understand what is rational in musical practice, be it in the creation or appreciation of music. However, the questions that inform our conjectures and theorems proceed from a logic that is unique to music itself. In addition to reason, music as an applied mathematical art challenges us to discover a language for the expression of musical concepts and structures. Not only must it be able to describe the physical, acoustic properties of music, but also its deeply psychological aspects. It seeks to situate all of these viewpoints into a coherent formal description.

Over the centuries, this language has drawn from many branches of mathematics. The earliest extant work in our field incorporated techniques of what is now considered number theory in the study of musical intervals (however, at the time, it was called arithmetic). In the sixth century B.C., the Pythagoreans were among the first mathematical music theorists, through their belief that all things are either numbers or endowed with the properties of numbers. Musical intervals held a special position for them in this connection: they were numbers made audible. This tradition was passed through the neo-Pythagoreans into the Roman world during the first century A.D. It was the Roman Boethius who, around the year 500, essentially translated Nichomachus's *Manual of Harmonics* from Greek into Latin in his treatise *De institutione musica*, ensuring its preservation into the Dark Ages in Europe following the collapse of the ancient world.

The study of musical intervals thus became one of the seven liberal arts. Following the study of grammar, rhetoric, and logic—the trivium—students in the Middle Ages learned about musical intervals as one of the four disciplines in the medieval quadrivium: arithmetic, geometry, music, and astronomy. And, just as astronomy was viewed as geometry in motion, music was considered arithmetic in motion. Music, along with its six sister studies, was prerequisite to the study of philosophy; as an applied mathematical art, it was considered an essential component of an education.

With the advent of the modern era, this work continued in the writings of such scholars as Kepler and Galileo (as well as Galileo's father, Vincenzo), largely in their descriptions of Harmonia

Mundi (or the Music of the Spheres). Through their efforts, mathematical applications in music became increasingly sophisticated. The work of René Descartes, particularly his *Compendium musicae*, served as a bridge between the ancient and modern worlds in terms of mathematical musical thought. It was during this time that the emphasis turned from the mere numerical properties of intervals to the acoustical properties of musical sounds and to rational description of the perception of these sounds. To that end, questions concerning which sounds we perceive as consonances or dissonances came to occupy a more central position, marking a shift from the belief that such things as musical intervals are themselves endowed with these properties, to an attitude that it is our experience of them that defines their qualities.

The formalism of these new ideas, representative of the Enlightenment in Europe, required new mathematical tools. Composers such as Jean-Phillipe Rameau applied these contemporary scientific ideas to their musical theories. Scientists such as Gottfried Leibniz (the founder of calculus) and Leonhard Euler (who published over 10,000 pages of research in all areas of mathematical thought) also applied their own original ideas to music. For example, answering the question of consonance versus dissonance now required measuring tools. Where does a particular interval fall on a continuous spectrum between pure consonance and pure dissonance? How do the acoustical properties of the instrument making the sound affect our perception of consonance or dissonance? What of the environment through which the sound travels? How does the human ear function in this regard? An interval incorporates two tones—what if we want to examine chords with more than two notes? This line of inquiry continued well into the nineteenth century in the work of Hermann von Helmholtz on "psychophysics," and its testing of the limits of human perception.

With the twentieth century came even greater strides in the application of mathematics to music. Arnold Schoenberg's development of the "Method of Composing with Twelve Tones" in the early part of the century inspired composers and music theorists alike for the next several decades with an essentially algebraic, group-theoretical approach to composition and analysis, based on permutations of pitch classes and row-order positions. (Granted, Schoenberg himself was not concerned with this mathematical basis. It was later that composer-theorists such as Milton Babbitt discussed Schoenberg's technique in these terms.) What is significant here is that the mathematics served not only to describe properties of musical events, such as whether they are consonant or dissonant, but to generate music itself as a compositional method. This had the effect of moving mathematical applications in music out of a merely descriptive role to a structural one. However, Schoenberg's method was not the only mathematically inspired technique from around this time: Josef Hauer arrived at a similar compositional method, using tropes (or unordered, complementary hexachords). Joseph Schillinger developed his own mathematical method of musical composition, which taught composers such as George Gershwin ways of organizing musical forms.

Returning to Babbitt, however, we observe a line of inquiry that leads directly into the mathematical music theories of today. Babbitt—as we observed previously—was the first to analyze the group-theoretical basis for Schoenberg's 12-tone method in the 1950s. The subsequent generation of composers and music theorists continued Babbitt's work, applying algebraic techniques not only to ordered rows (of pitch classes or attack points, etc.), but also to unordered sets. In the 1960s, Howard Hanson began the categorization of the power set of the aggregate, the set of all subsets of the chromatic scale. This work was furthered by Allen Forte, in his seminal work from 1973, *The Structure of Atonal Music* [17], which introduced the notion of pitch-class sets, interval vectors, set classes, the Z relation, etc., to a wide audience of music theorists. Meanwhile, John Clough was applying similar mathematical techniques to the study of the diatonic system. In addition to music theorists, composers were also publishing research along these lines: John

Rahn in his *Basic Atonal Theory*, and Robert Morris in his several significant articles and his book *Composition with Pitch Classes*.

In the 1980s and 1990s, the application of group theory to music also found a home in the study of historical music theories. David Lewin's work in transformational theory [32] included examples of generalized interval systems that derived from nineteenth-century and early twentieth-century models of tonal relations by Hugo Riemann, Arthur von Oettingen, Moritz Hauptmann, and others. Through this "neo-Riemannian theory," the mathematical analysis of earlier, tonal music came to occupy a prominent position, tantamount to the analysis of contemporary compositions. This work was furthered greatly though the efforts of Richard Cohn. The defining characteristic of much of the neo-Riemannian analysis, Cohn's notion of "parsimonious voice leading," engendered further investigation into efficient voice leading among not just consonant triads, but among and between other sonority types, leading ultimately to the geometric, orbifold models of voice leading developed by Clifton Callender, Ian Quinn, and Dmitri Tymoczko in the first decade of the twenty-first century.

Simultaneous to these developments in North America, the formation of another mathematical music theory took place in Europe. However, whereas much of the work in the Americas was conducted by musicians who had a keen interest in mathematics, the European practitioners were mostly mathematicians who were also highly skilled in music. Heading this movement was the Swiss mathematician and jazz pianist Guerino Mazzola. His teaching and research reached several other important figures in other European countries as well, including Thomas Noll in Germany, Moreno Andreatta in France, Domenico Vicinanza in Italy, Francisco Gomez in Spain, Anja Volk in the Netherlands, Christina Anagnostopoulou in Greece, and many others.

The recent mathematical music theories on both sides of the Atlantic Ocean have incorporated increasingly sophisticated mathematical models. In addition to the group-theoretical work that began in the middle twentieth century, composers and music theorists have applied many additional branches of mathematics to the study and creation of music: including, but not limited to, category theory, knot and braid theory, algebraic topology, combinatorics, topos theory, combinatorial word theory, module theory, homotopy theory, homology theory, graph theory, algebraic geometry, mathematical physics, machine learning, and computation. In fact, even ideas from mathematics education have had a bearing on musical studies.

Because of the plethora of concepts and techniques that have been introduced into our field, it became necessary to establish a central forum for the sharing of these ideas. That was largely the impetus behind the founding of the Journal of Mathematics and Music, and its associated Society for Mathematics and Computation in Music. In the next part of this article, I will describe briefly the process that led to their formation.

In January of 2001, the American Mathematical Society (AMS) and the Mathematical Association of America held their national meetings (as part of the annual Joint Mathematics Meetings) in New Orleans, Louisiana—a city that is 120 kilometers from my home. My wife's mother, Judith Baxter, and stepfather, Stephen Smith, are professional mathematicians, and they planned to attend that meeting. Because I had been working in the field of mathematical music theory, I thought it would be opportune for me to present some of my recent research at that meeting, so I submitted presentation proposals to both societies, which were accepted by the respective program committees. That experience was very positive, and I received interesting feedback from a number of mathematicians who were interested in music, and who were in attendance at the meeting. Consequently, when the American Mathematical Society announced that they would be holding a sectional meeting in Baton Rouge, Louisiana—my own city—in 2003, I planned to submit another proposal. Judith Baxter suggested that, because of the interest generated at the previous meeting, rather than a single presentation, she and I propose a whole special session devoted to mathematics and music. The proposal was accepted, and the first AMS special session on Mathematical Techniques in Musical Analysis was held in March of that year. It featured talks by twenty mathematicians and music theorists and composers, including Thomas Noll from Germany, and was attended by several mathematicians who were at the conference.

Based on the success of that special session at a sectional meeting, I proposed another special session at the next national meeting, in Phoenix, Arizona, in January of 2004. This session was somewhat larger, and generated a great deal of interest at the meeting. The session incorporated talks by several European mathematicians. One of the presenters at this session, Richard Cohn, suggested an organizational meeting during the conference to discuss the idea of an international journal devoted to the mathematical study of music. The idea was received with tremendous enthusiasm, and a working group was formed to begin the process of bringing this idea to life. That working group consisted of Thomas Noll and Moreno Andreatta from Europe, and Norman Carey, Ian Quinn, and myself from the Americas.

Another special session on Mathematical Techniques in Musical Analysis was accepted for the AMS sectional meeting in fall of 2004 in Evanston, Illinois. The working committee reconvened at that session, along with several members of the growing community of interested mathematicians and musicians. Another meeting was held to continue the work of creating the journal. Topics discussed at that meeting were the name of journal, possible publishers, and the election of the first editors-in-chief. It was decided that there would be two editors: one mathematician and one musician, and one from each side of the Atlantic. The first two editors were decided to be Thomas Noll and myself.

During the years 2005 and 2006, the editors, along with the working committee, continued the necessary planning for the journal. After much debate, it was decided that the publisher would be Taylor & Francis of London, England. Further, we had to choose how many issues we would publish per year, the approximate page count for issues and how many articles they should include, and related questions. We decided to publish the journal in both print and online versions. We also issued the initial call for submissions. Among the most significant of our plans for the journal was to devote one special issue per year to a topic of current research interest. We would invite experts in these areas to guest edit the special issues, allowing them to solicit papers or to invite authors to contribute articles.

Taylor & Francis, asked us if there was a related society whose members might receive a reduced subscription rate to the journal. At that time, there were various organizations for mathematicians (such as the AMS) and other organizations for musicians (such as the Society for Music Theory), but none dedicated solely to the interdisciplinary study of music and mathematics. So, we founded at that time the Society for Mathematics and Computation in Music. From the beginning, it was intended to be an international undertaking, and to hold its biennial meetings alternately on opposite sides of the Atlantic.

The Journal of Mathematics and Music launched in January 2007. The first issue, Volume 1, Number 1, included a ”Welcome” by the editors [44]; and research articles by John Rahn [49], Guerino Mazzola and Moreno Andreatta [36], and Jack Douthett and Richard Kranz [15]. The first meeting of the Society for Mathematics and Computation in Music followed in May of that year. It was held at the State Institute for Music Research in Berlin, Germany.

In July 2007, we published Volume 1, Number 2—the first special issue, devoted to “The Legacy of John Clough in Mathematical Music Theory.” The guest editor for this issue was David Clampitt (a former student of Clough’s). In addition to Clampitt’s guest editorial [12], it included articles by Norman Carey (“Coherence and sameness in well-formed and pairwise well-formed scales”) [8], Julian Hook (“Enharmonic systems: A theory of key signatures, enharmonic equivalence and diatonicism”) [21], and Thomas Noll (“Musical intervals and special linear transformations”)

[42]. John Clough's work has held much relevance for mathematical research in music theory since the late 1950s. His continued research into the diatonic system served as the basis for much of modern scale theory. It was a natural choice to devote this first special issue to his legacy. This issue investigated several mathematical-music topics that grew out of Clough's pioneering work, including generic and specific interval measures; Myhill's property (the condition in which intervals among the members of a scale come in two specific sizes), the property called "cardinality equals variety," which generalizes Myhill's property to subsets of a scale for any size, and which was one of the early examples of a significant mathematical result's being discovered first in a musical context; contextual operations; the theory of well-formed scales; and musical applications of continued fractions; the Farey series; the Stern-Brocot tree; the special linear group $SL(2, \mathbb{Z})$; and aspects of combinatorial word theory.

The second special issue, Volume 2, Number 2, was published in July of 2008. It was guest edited by Elaine Chew, Alfred Cramer, and Christopher Raphael, and was devoted to the topic of computation in music research - the first of a few special issues that dealt with aspects of computation. It featured articles by Chantal Buteau and John Vipperman ("Representations of motivic spaces of a score in *OpenMusic*") [7], Leigh M. Smith and Henkjan Honingh ("Time-frequency representation of musical rhythm by continuous wavelets") [50], and Anja Volk ("Persistence and change: Local and global components of metre induction using Inner Metric Analysis") [53]. Of particular concern in this issue are the processes and methods of computing data that represent musical information. Such a line of inquiry has implications not only for music theory and analysis, but also composition and improvisation, performance, and cognition and music perception. For instance, aspects of motivic manipulation can be generated or analyzed by machine - often more efficiently or thoroughly than by the human hand. Another application of computation could be found in determining algorithms. One may use computation to deduce the key of a piece or section of music, or its meter (based on finding and relating weights of adjacent and non-adjacent periodicities). Such determinations can be compared to those of more traditional music-theoretic analysis, or to the empirical results of cognitive research, thereby opening a new space for investigation that occupies the intersection of humanistic and scientific approaches to music study. One interesting question raised by the guest editors of this issue concerns the bottom-up versus top-down methods that are possible in a computational study [11]. Does one start with the data and build a theory up from there, or does one begin with a central idea that drives the pursuit of finer levels of detailed inquiry? The articles in this issue incorporate different approaches in this regard, which the editors acknowledge and offer to the reader for comparison.

The third special issue appeared in July 2009. Volume 3, Number 2, was devoted to "Tiling Problems in Music," and was guest edited by Moreno Andreatta and Carlos Agon. This issue included the following three articles: "New perspectives on rhythmic canons and the spectral conjecture," by Emmanuel Amiot [1]; "Algorithms for translational tiling," by Mihail N. Kolountzakis and Máté Matolcsi [30]; and "Tiling the integers with aperiodic tiles," by Franck Jedrzejewski [27].

In their introductory editorial to the issue [5], Andreatta and Agon quote Leibniz's famous letter to Christian Goldbach from April 17, 1712: "Musica est exercitium arithmeticae occultum nescientis se numerare animi" (which translates to: "music is a hidden arithmetic exercise of the soul, which does not know that it is counting"). In the case of tiling structures in music, the editors pararphrase Leibniz to say: "In some cases, mathematics is an 'exercitium musicae'." They continue: "What characterizes a 'mathemusical' problem is the fact that settling the originally musical problem in an appropriate mathematical framework not only gives rise eventually to new mathematical results, but also paves the way to new musical constructions that would have been impossible to conceive without the process of 'mathematization'. It is this double movement, from music to mathematics and backwards, which makes a 'mathemusical' problem so intriguing to both mathematicians and musicians" [5, pp. 63-64]. In this case, the problem is construction of rhythmic tiling canons: These are special musical forms which consist of a rhythmic pattern that completely tiles the musical time axis by temporal translations. Think of it as a strict rhythmic canon in which every beat receives one and only one impulse. Such tiling canons are not a new concept. Indeed, they have been in use since the Ars Nova though more recent American Minimalism and beyond.

The creation and study of tilings has a long and distinguished history. It incorporates results from several branches of mathematics: geometry, certainly, but also group factorizations and direct sums, characters and the discrete Fourier transform, Minkowski's (number-theoretical) tessellation problem, Fermat's last theorem, the classification of groups (they cannot be factorized into a direct sum of subsets without at least one of the factors being periodic), cyclotomic polynomials, and the use of machine computation. The study of tilings also has connections to at least one open mathematical conjecture, the Fuglede or spectral conjecture: This conjecture deals with the relation between the spectral property of a domain in n-dimensional Euclidean space and its tiling character. It states that such a domain admits a spectrum if it tiles n copies of $\mathbb{R}$ by translation. Another topic discussed in the issue is the means of testing of mathematical theories for determining the number of isomorphism classes of rhythmic tiling canons.

Of particular interest throughout this issue is the construction of Vuza canons, so-named after Dan Vuza, who discovered them, and who first published on them in the journal *Persepectives of New Music* in 1991 [55]. In fact, the mathematical problem in Vuza's *Perspective of New Music* article was rediscovered several years later by mathematicians. Vuza canons are one example of tiling canons that can be generated by means of cyclotomic polynomials. Other musical applications of tiling canons include homometric structures, such as the well-known Z-related pairs of sets classes in traditional pitch-class set theory; Boulez's chord multiplication, such as he used in his composition *Le marteau sans maître*; and transpositional combination.

In July 2010, Volume 4, Number 2 appeared—another special issue devoted to computation, titled: "Computational Music Analysis: Can computational music analysis be both musical and computational?" The guest editors for this issue were Christina Anagnostopoulou and Chantal Buteau. All of the articles in this issue incorporate analyses of the same piece of music: Brahms's String Quartet No. 1 in C Minor, Op. 51, No. 1. The articles are by Darrell Conklin ("Distinctive patterns in the first movement of Brahms' String Quartet in C minor") [13], Atte Tenkanen ("Tonal trends and α-motif in the first movement of Brahms' String Quartet op. 51 nr. 1") [51], and Philippe Cathé ("Harmonic vectors and stylistic analysis: a computer-aided analysis of the first movement of Brahms' String Quartet Op. 51-1") [10].

The guest editors take as a point of departure a quotation from Ian Bent: "music analysis is the means of answering the question: 'How does it work?"' [6, p. 5]. It relies on the comparison of data, and, via comparison, it discovers the structural elements of a piece of music and the specific functions of these elements. To conduct such an analysis requires a neutral level, which does not necessarily take into account the composer's intentions (the poetic level) or the listener's cognitive mechanisms, intuitions, aesthetic judgements, emotions, or reactions (the aesthesic level). In this regard, certain analytical methods, such as those of Jean-Jacques Nattiez, are cricitized: a human analysis may reflect the analyst's own perceptions, and would thus not be neutral. However, computational analysis can get closer to this neutrality. However, the human analyst cannot be eliminated altogether, when one seeks to combine objectivity and scientific rigor with the interpretative nature of music analysis. With this in mind, the computational side of music analysis has certain well-defined aims: to produce musicologically interesting results, to formulate a (neutral) analytical process, and to perform calculations that would have been difficult, tedious, or impossible by hand. Finally, an additional aim is to test computational methodologies.

Future work in computational music analysis should address various limitations and problems in the field:

1. The division that exists between analyses that start from various symbolic representations (like notation) and others from the audio signal. Often these two worlds do not meet.

2. The lack of emphasis on the representational issues, which nevertheless are crucial both for the formalization and the result aspects.

3. The plethora of approaches, with a distinct lack of comparisons and discussions between them—although the guest editors suggest that the formation of MIREX (the Music Information Retrieval Evaluation eXchange) is an important step towards this direction.

4. The lack of connections to the field of more traditional music analysis, a situation that has resulted in varying opinions from both sides represented in the special issue.

5. The means of musical evaluation of a system's results. For example, can the data alone reveal what a traditional motivic analysis proposes or the evolution of tonality in a piece that is evident in, say, a Schenkerian analysis?

In general, computational music analysis needs to address the delicate issue of the balance between computation, musicologically sound methodology, and the proper evaluation of results. Ultimately, the human factor is still crucial and necessary in any analytical approach, as is the inherent diversity in music analysis.

In 2011 (which coincided with Volume 5 of the journal), no special issue appeared: all three issues were devoted to contributed papers. However, Volume 6, Number 2, which appeared in July 2012, was another special issue, Mathematical and Computational Approaches to Music: Three Methodological Reflections, guest edited by Anja Volk and Aline Honingh. Unlike the previous special issues, this one followed closely from a workshop at the previous year's meeting of the Society for Mathematics and Computation in Music in Paris. In this workshop, three senior scholars in the field, Guerino Mazzola, Geraint A. Wiggins, and Alan Marsden presented their views on the current state of mathematical and computational analysis, and each was also invited to comment on the others' remarks. Essentially, the special issue reproduced this exchange, with position papers by all three authors: Guerino Mazzola ("Thinking music with precision, depth, and passion") [35], Geraint A. Wiggins ("Music, mind and mathematics: theory, reality and formality") [56], and Alan Marsden ("Counselling a better relationship between mathematics and musicology") [34], and responses from the others. The result is an interesting and challenging exchange of ideas.

In their introduction to the issue, the guest editors provide a background to the debate [54]. They cite Nicholas Cook, who states that we have been standing quite long at a moment of opportunity with respect to the relation between computational approaches and musicology, without reaching the full potential of the interdisciplinary enterprise. Mathematical music analysis seems too detailed for more traditional musicologists, and at the same time it is too vague for mathematical scientists. Nevertheless, it is a growing field, both in terms of computational and mathematical applications.

The first computational projects in music began in the 1960s: the idea was that machines could process more information than humans can process practically. More recently, computation has been a major tool in music theory and analysis, musical performance research, historical musicology, ethnomusicology, and cognitive musicology. The earliest known applications of mathematics to music were by the Pythagorean school, who associated intervals and numbers. Since that time, mathematics has been used to study aspects of tuning and temperament, consonance and dissonance, musical set theory, scale theory, transformational theory, musical topos theory, and various compositional models.

The articles focus on four main topics: what are the benefits of mathematical and computational music analysis? What are its failures? What are the challenges going forward? And what opportunities are available for interdisciplinary discourse among mathematical, computational, and musicological approaches to music research? With regard to these topics: in terms of benefits, What key contributions did mathematical and computational approaches bring to the field of music research according to your point of view? Each of the authors (with certain overlap) lists some specific benefits:

- key contributions to music technology, e.g., to mp3-format (Mazzola, Marsden)
- conceptual clarification of music theoretic concepts (Mazzola)
- mathematics allows to understand existing music from its positioning within a broader framework of possible musics (Marsden, Mazzola)
- mathematics affords explanations of the consequences of tuning (Marsden)
- generalizations in mathematics allow addressing the evolution of music (Mazzola)
- computation allows rigorous testing of evolutionary hypotheses (Wiggins)
- computation is a hard test of the efficiency and precision of conceptualization and operationalization of musical thought (Mazzola)
- computation allows creation of educational music tools (Marsden, Wiggins)

With regard to failures: What are examples of pitfalls that occurred within computational and/or mathematical approaches to music research in the past? What can we learn from them? Again, the authors list particular shortcomings of existing research in the field:

- it is a failure to study the musical products without the musical processes (Wiggins)
- it is a failure to treat music theory not as fundamentally perceptual (Wiggins)
- it is a failure to use information-theory approaches to music (Mazzola)
- it is a failure to neglect the sign-theoretical shape of musical phenomena in favor of purely formal aspects (Mazzola)
- it is a failure that mathematicians concentrate too much on scale theories; musicologists are more interested in musical pieces (Marsden, Mazzola)
- it is a failure to reject higher math as a means to understand complex musical structures on the grounds that everybody understands music in some sense (Mazzola)

Next, what challenges are we facing now within computational and/or mathematical approaches to music? What unexplored fields and questions have the potential to move our understanding of music forward with the help of computational and/or mathematical approaches? What steps need to be taken now and in the near future in order to fully unfold the potential of computational and/or mathematical approaches to music? The authors identify the following challenges that we are facing in computational and mathematical approaches to music:

- musicologists have not yet taken up the tools offered by mathematics and computation (Marsden)
- accounting for the 'messiness' of humans is a challenge for the preciseness in mathematics and computation (Marsden, Wiggins)
- classification of musical objects in mathematics alone is not enough; we need to apply them to musical pieces within computational experiments (Mazzola)
- to achieve a meta-theory of music, we need to rise above the study of particular examples of music tying all musical cultures together through time (Wiggins)

- we need to model musical behavior rather than merely music products (Wiggins)
- mathematicians need to consider more empirical, statistical work for a better connection between rationalism and empiricism (Marsden)

And, finally, how can we strengthen the connections among the three fields of mathematical, computational, and musicological approaches to music? Are there different ways of 'understanding' music in these three fields? In what context are the differences between the disciplines (mathematics, music research, computer science) a useful source for innovative research on music-related questions? When do the differences between the disciplines become a stumbling block for interdisciplinary research, and what needs to be done to overcome that? In the interdisciplinary discourse, we need to consider:

- humility is essential for interdisciplinary work (Marsden)
- honesty about what is within scope and what not is important (Marsden)
- using the fuzzy concept of 'musicality' as an argument to dismiss mathematical or computational models of music is not appropriate (Mazzola)
- achieving mathematical theories that are computational and can be tested by comparison with humans requires substantial interdisciplinarity (Wiggins)

In answering all these questions, the three authors discuss the use of models and the issue of abstraction; the precision of mathematics and computation versus the imprecision of humans; and, ask what kind of theory is music theory—what are the ultimate consequences for mathematical and computational approaches to music?

The special issue in 2013, Volume 7, Number 2, was devoted to the subject of "Mathematical Theories of Voice Leading." It followed from the John Clough Memorial Conference in 2013, which dealt with the same topic. The conference and the issue endeavored to seek a reconciliation of mathematical theories of voice leading by the authors who presented work at that conference, and whose articles appeared in the issue: Julian Hook ("Contemporary methods in mathematical music theory: a comparative case study") [23], Richard Plotkin and Jack Douthett ("Scalar context in musical models") [46], Dmitri Tymoczko ("Geometry and the quest for theoretical generality") [52], and Jason Yust ("Tonal prisms: iterated quantization in chromatic tonality and Ravel's 'Ondine'") [57].

In his introductory editorial, guest editor Marek Žabka notes that this issue marked a shift away from the algebraic basis of the mathematical music theory of the previous decades, particularly the transformational theories of Lewin. However, Lewin's notion of "abstract (musical) spaces as extratemporal universe[s] of quasi-spatial potentialities," where actual pieces of music are conceived as "human gestures that move through chronological time" continues to serve as a central theme in these theories of voice leading [33, p. 41]. Specifically, Hook's 'cross-type transformations,' Tymoczko's 'paths in pitch class space,' and Douthett's 'stroboscopic portraits' all consider musical motion as a form of gesture in an abstract space—although their specific means of describing these gestures differ significantly from Lewin's. Another point of commonality among the articles is their reliance on Clough and Douthett's theory of maximal evenness. This property remains a central issue in the types of voice-leading structures employed in much music, and suggests its universal significance.

The next special issue, Volume 8, Number 2, which was published in July 2014, represented a different direction: guest editors Jason Yust and Thomas M. Fiore put together an issue that does not engage directly with a research technique, but rather with "Pedagogies of Mathematical Music Theory." They invited six articles by scholars who have taught courses in the field: Jon Kochavi ("*Musica speculativa* for the twenty-first century: integrating mathematics and music in the liberal arts classroom") [29], Rachel Wells Hall ("Acoustics labs for a general education math and music course") [20], James R. Hughes ("Creative experiences in an interdisciplinary Honors course on mathematics in music") [25], Robert Peck ("Mathematical music theory pedagogy and the 'New Math"') [45], Mariana Montiel and Francisco Gómez ("Music in the pedagogy of mathematics") [39], and Thomas Noll ("Getting involved with mathematical music theory") [43]. The first three of these articles describe specific courses, their goals, and pedagogical philosophies, provide samplings of their contents, and reflect on successes and challenges. The next three papers are essays of a more general nature about the teaching and learning of mathematical music theory, and its place in academic institutions and the public sphere at large.

The editors proceed from the adage that "a discipline defines itself by its pedagogy" [59]. In this connection, curricula provide a basis of shared knowledge, and a means to organize, classify, and categorize that knowledge. Further, it records and passes along the history of the discipline, defining along the way its classic results and its most significant contributors. In the case of mathematics and music theory, both fields require individually a great deal of specialized knowledge and skills to access basic research. When these subjects are combined in the interdisciplinary context of mathematical music theory, these challenges of accessibility are multiplied substantially. Several themes emerged across the articles in this issue. Among the most ubiquitous is that creative and discovery-based learning seems to be an effective method for the teaching of mathematical music theory. Certain challenges also reappear in several of the articles: these problems center primarily around the issue of accessibility and making mathematical music theory available to a general audience. Solutions to these problems include popularization, such as creating museum exhibits and other public institutional venues that promote the topic of music and mathematics.

The special issue from July 2015, Volume 9, Number 2, is another unique issue. It is the only special issue (indeed the only issue of the *Journal of Mathematics and Music*) that is devoted primarily to a single paper: Harald Fripertinger's and Peter Lackner's "Tone rows and tropes" [18]. Guest Editors Julian Hook and Robert Peck also invited responses by Robert Morris [41] and Andrew Mead [37], both composers of 12-tone music and leading experts in its theory.

The main article deals with a classification of the set of 479,001,600 12-tone rows. The authors complete the categorization of tone rows and tropes using Pólya's Theorem, which derives from and essentially generalizes Burnside's lemma on the number of orbits of a group action on a set. In this case, the set is the collection of all 12-tone rows, and the groups the authors employ involve the canonical operations of transposition, inversion, retrograde, rotation, and multiplication of pitch classes and order position numbers. Whereas classifying and counting musical objects have long been among music theorists' typical endeavors, using sophisticated techniques from the field of combinatorics is rather novel. Julian Hook published an article that contains an application of Pólya's Theorem to the universe of pitch-class sets in 2007 [22], but that is a significantly smaller set.

As the guest editors discuss [24], bringing such mathematical complexity to music also creates certain problems. The theorems and proofs of the mathematicians leave the musicians nonplussed; even those few who can follow the mathematical detail often find that it falls short in musical relevance or sensitivity, in historical or cultural reference, and in connections to the ways they have learned to think about music. The writings of the musicians, meanwhile, may be filled with revealing musical examples and analyses but frustrate the mathematicians via imprecise prose, mathematical terms and notations deployed in ways that are non-standard or outright wrong, and rambling commentary in the place of logical demonstration. The divide is both made evident and also mended to a certain degree in Morris's and Mead's responses to the main article. Both responders speak to earlier results in the world of 12-tone composition that anticipate many of the paper's findings.

A special issue on "Machine Learning and Music Generation" followed in 2016: Volume 10, Number 2. It was guest edited by José M. Iñesta, Darrell Conklin, and Rafael Ramírez, and featured articles by Darrell Conklin ("Chord sequence generation with semiotic patterns") [14], Sergio Giraldo and Rafael Ramírez ("A machine learning approach to ornamentation modeling and synthesis in jazz guitar") [19], Phillip B. Kirlin and Jason Yust ("Analysis of analysis: Using machine learning to evaluate the importance of music parameters for Schenkerian analysis") [28], Katerina Kosta, Rafael Ramírez, Oscar F. Bandtlow, and Elaine Chew ("Mapping between dynamic markings and performed loudness: a machine learning approach") [31], and Pedro J. Ponce de León, José M. Iñesta, Jorge Calvo-Zaragoza, and David Rizo ("Data-based melody generation through multi-objective evolutionary computation") [47].

Music generation research has generally employed one of two strategies: first, knowledge-based methods that model style through explicitly formalized rules; and, second, data mining methods that apply machine learning to induce statistical models of musical style, training data, music representation, candidate generation, and evaluation, and modeling and generating music, including melody, chord sequences, ornamentation, and dynamics. In such machine learning, models are induced from either audio data or symbolic data. The guest editors identify a challenging issue in computational music generation, especially if fully automated: the determination of concrete musical events that cohere with certain more abstract or underlying structures [26]. Examples of these situations occur as metrical regularities or as both adjacent and distant repetitions.

In general, machine learning is incorporated in the generation of events and event sequences through statistical models. For instance, one might employ model-based prediction or sampling from a statistical model, or in evaluating candidate-event sequences or ranking generated sequences that are based on a statistical model. In studies of repetition in music, it develops a way to represent and generate repetition using statistical models. Further, it develops a pattern representation that explicitly describes relations between events using variables that are instantiated during generation. A semiotic pattern describes a formal language of sequences, and sequences satisfying the pattern are generated by sampling from a statistical model. The statistical model derives from a corpus and is an instance of the viewpoint-modeling approach used previously with success for prediction, classification, and music generation. Of particular interest are genetic algorithms. These provide an optimization technique that mimics the biological evolution of living beings and natural selection. A population of individuals, representing different possible solutions, are subjected to crossovers and mutations, and a selection stage decides which individuals might best solve the problem. Those individuals are allowed to procreate a new generation of, supposedly, better individuals until convergence.

The articles in this issue involve topics such as creating a machine-learning system that synthesizes a notated jazz melody, while adding ornamentations that are characteristic of a natural musical interpretation. Such a system might rely on a classifier that learns from a dataset of recorded performances in deciding which notes to ornament and which notes not to ornament. Another topic is experiments with learning probabilities for melodic reductions, such as in modeling Schenkerian analysis, directed at discovering the importance of different features that most closely match an accepted analysis. Another article examines the relationship between dynamic markings in music scores and performed loudness by applying machine-learning techniques to induce predictive models of loudness levels corresponding to dynamic markings, and to classify dynamic markings given loudness values.

2017 produced a double special issue: both Numbers 2-3 in Volume 11 dealt with the topic of "Perfect Balance and the Discrete Fourier Transform." This issue was edited by Thomas M. Fiore, and included two articles by Emmanuel Amiot ("Decompositions of nil sums of roots of unity. An adaptation of 'Sommes nulles de racines de l'unité"' and "The discrete Fourier transform of distributions") [2] and [3], Andrew J. Milne, David Bulger, and Steffen A. Herff ("Exploring the space of perfectly balanced rhythms and scales") [38], Norman Carey ("Perfect balance and circularly rich words") [9], and Jason Yust ("Harmonic qualities in Debussy's 'Les sons et les parfums tournent dans l'air du soir"') [58].

Fiore's editorial observes that applications of the discrete Fourier transform in music scholarship span decades, countries, and continents [16]. This issue develops the mathematical framework of the discrete Fourier transform for music theory, and it connects it to salient music-theoretical problems and questions. In particular, the authors in the issue examine the mathematical ramifications of perfectly balanced scales and rhythms via the discrete Fourier transform, and musically explore the smooth manifold they inhabit. They study perfectly balanced scales from a word-theoretic perspective, focusing on those perfectly balanced scales with circular palindromic "richness" and relatively few step "differences." The final article analyzes Debussy's prelude "Les sons et les parfums tournent dans l'air du soir" from a Fourier perspective, highlighting perfectly balanced pitch-class sets.

While it has not appeared in print yet, 2018's special issue examines "Combinatorics on Words." It is guest edited by Marc Chemillier, Christophe Reutenauer, and Srecko Brlek, and features articles by Jean-Paul Allouche, Tom Johnson, David Clampitt, Norman Carey, Thomas Noll, Marc Chemillier, Lorraine Ayad, and Solon Pissis. Exciting new work is being done in the application of combinatorial word theory to music, as several results in the mathematical field have been anticipated by the work of these music researchers.

* * *

The respective fields of music and mathematics are each tremendously rich, and they represent some of the greatest achievements of individuals, as well as society. Their interdisciplinary combination highlights aspects of each field that are not always in evidence. Such amalgamation also presents certain problems: particularly, we do not always have the language needed to describe these newfound facets. The work of the special issues of the *Journal of Mathematics and Music*, as well as of this conference, is to find new and successful ways of expressing and disseminating these ideas and thereby laying the groundwork for future studies in our field.

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Axis of Contextual Inversion

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Abstract: *contextual inversion operations are commonly associated with neo-Riemannian transformations, but the labels **P**, **L** and **R** and their obverse versions **P'**, **L'** and **R'** only map sets with at least one common pitch-class. This article shows how contextual inversion operations can be mapped by axes in a similar manner to I_n-operations, and how the positions of these axes can also be labeled. The advantage of this approach is that it will make the representation of any contextual inversion operation between members of any set class possible, which will be useful for both musical analysis and pre-compositional processes. The theoretical concepts developed in this article will be demonstrated in analyses of passages of works by Webern, Stravinsky and Villa-Lobos.*

Keywords*: Contextual inversion. Neo-Riemannian Theory. Graph Theory. Voice-leading.*

I. Introduction

My interest in contextual inversions started when I was trying to wholly adapt Cube Dance[1] to trichords besides sc. (037) and sc. (048) and I ended up finding some difficulties in labelling the connections between the sets. The sets in Cube Dance are organized in voice-leading zones, meaning that the entire graph is within a clock face and a single or a group of trichords are positioned on the radius next to the number relative to their sum class[2]. Richard Cohn calls these numbers as "voice-leading zones" ([3, p.102]). On the original Cube Dance, augmented triads, members of sc. (048), are on voice-leading zone 0, 3, 6 and 9 and consonant triads, members of sc. (037), are on voice-leading zone 1, 2, 4, 5, 7, 8, 10 and 11. Two members of sc. (037) are always connected by neo-Riemannian P or L transformations in two adjacent voice-leading zones of Cube Dance, which means that these sets are both related by contextual inversion and by parsimonious voice-leading. In order to substitute the original sets on cube dance for any type of trichord I had to give up the parsimonious connection between all members of the same set class, since the consonant triads are the only set class in which a member can be connected to two others by contextual inversion and parsimonious voice-leading. In many of the Cube Dances I have built, members of the same set class were connected in two adjacent voice-leading zones, even though they were related by contextual inversion, not sharing a common pitch-class, which means that there is no transformation label for these connections. To overcome this problem I began working on a way to label the contextual inversions in which it was not necessary for the sets to share any pitch-class, which resulted in the theoretical concepts described in this article.

[1] The original Cube Dance is a graph created by Jack Douthett and Peter Steinbach ([4, p.254].

[2] "Sum Class" is a concept created by Joseph Straus: "Two pitch sets are equivalent as members of the same sum class if their pitch integers have the same sum; two pitch- class sets are equivalent as members of the same sum class if their pitch-class integers have the same sum. ([10, p.279]).

The difference between I_n and contextual inversion operations will be discussed in section II, this question will be clarified in the comparison between I_n-operations and neo-Riemannian transformations **P**, **L**, **R**, **P'**, **L'**, and **R'**. In section III it will be argued that transformations and contextual inversion are not necessarily the same. Some authors, like Jack Douthett and Peter Steinbach, consider that the transformations can occur between any two parsimonious sets, even if they are not members of the same set class. Others, like Joseph Straus, take an opposite approach and consider that all transformations are contextual inversions and therefore they must occur between two members of same set class, even if they are not parsimonious. However, the association of contextual inversions with neo-Riemannian transformations generates labels only for those contextual inversions between sets that share pitch-classes.

In Section IV, contextual inversions will be represented by axes in a similar manner to In-operations, but with the difference that they will move according to the pitch-classes of the pair of sets they connect. Also, in the same section, it will be shown that there are twelve contextual inversion axes and that they will also be labeled by letters. Section V will show examples of the usefulness of labels for musical analysis. There will be shown passages in which the sets are related by contextual inversion in Webern's *Concerto for Nine Instruments, op. 24*, in Stravinsky's "Musick to Heare," from *Three Shakespeare Songs* and in Villa-Lobos *Etude 10 for Guitar*. All appendixes for this article are available as supplementary material at https://axesofcontextualinversion.com

II. I_N-operation *versus* Contextual Inversion

Inversion, as well as transposition, is an operation that relates two sets with the same interval content and therefore belonging to the same set class. Following Joseph Straus's approach to this matter ([9, p.53]), we will represent the operation of inversion as I_n, where I stands for inversion and n is the index number (or index of inversion) that is the result of the sum (in mod 12) of the two pitch-classes related by inversion. Thus, for example, I_9 maps C onto A (0 + 9), E onto F (4 + 5) and B onto B♭(11 + 10); I_4 maps D onto D (2 + 2), F♯onto A♯(6 + 10) and A onto G (9 + 7). Inversions with any index number may be represented by axes in the clock face. If the index number is even, the axis crosses two opposing pitch-classes, if the index number is odd, it crosses the point halfway between pitch-classes. The axes for all index numbers are shown in Figure1.

However, contextual inversions are a different type of operation and they are not represented by the axes shown in Figure 1. According David Lewin "'contextual' inversion operation is not defined with reference to any pitch-classes whatsoever" but rather "with respect to a 'contextual' feature of the configuration (s) upon which it operates" ([5, p.7]). The difference between the inversion and the contextual inversion can be clarified in the following examples with the consonant triads: I_7 maps C [0,4,7] onto Cm [0,3,7] and I_1 maps A [9,1,4] onto Am [9,0,4], although in both cases the triads are built over the same root;

I_{11} maps C [0,4,7] onto Em [4,7,e] and I_5 maps A [9,1,4] onto C♯[1,4,8], although in both cases the triads roots are separated by 4 semitones;

I_4 maps C [0,4,7] onto Am [9,0,4] and I_{10} maps A [9,1,4] onto F♯m [6,9,1], although the triads, in both cases, are relative;

I_0 maps C [0,4,7] onto Fm [5,8,0] and I_6 maps A [9,1,4] onto Dm [2,5,9], although in both cases the triads roots are 5 semitones apart;

I_8 maps C [0,4,7] onto C♯m [1,4,8] and I_2 maps A [9,1,4] onto A♯m [t,1,5], although in both cases the triads roots are one semitone apart;

I_2 maps C [0,4,7] onto Gm [7,t,2] and I_8 maps A [9,1,4] onto Em [4,7,e], although in both cases the triads roots are 7 semitones apart.

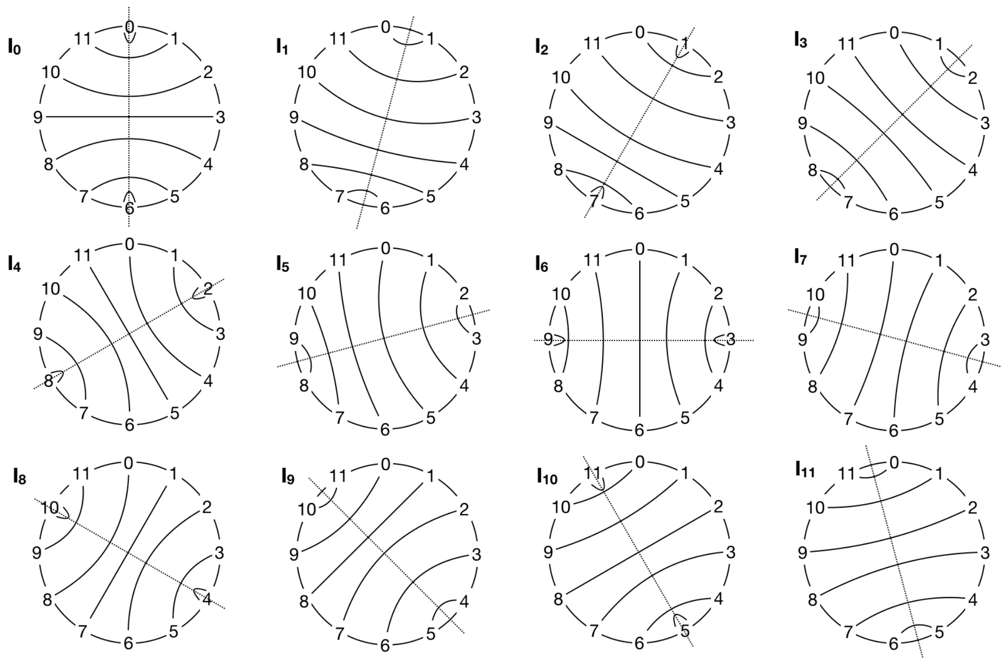


Figure 1: *axes of inversion operations.*

One can note in these six examples that the context between the both pairs of consonant triads does not change, but the index number does so every time. Since connections between consonant triads are a great deal in neo-Riemannian theory, it provided transformational labels for all these six contextual inversions: **P**, **L**, **R**, **P′**, **L′** and **R′**, respectively[3]. **P** maps two parallel triads because it is the contextual inversion that preserves ic 5 with the remaining note flipping around it (Figure 2a); **L** maps a major onto a minor triad preserving ic 3 with the remaining note flipping around it (Figure 2b); **R** maps two relative triads because it is the contextual inversion that preserves ic 4 with the remaining note flipping around it (Figure 2c); **P′** maps a major onto a minor triad flipping ic 5 around the remaining note (Figure 2d); **L′** maps a major onto a minor triad flipping ic 3 around the remaining note (Figure 2e); **R′** maps a major onto a minor triad flipping ic 4 around the remaining note (Figure 2f). Figure 2 shows all these contextual inversions in the consonant triads of the six previous examples.

The transformations used by the neo-Riemannian theory shown in Figure 2 are a nice example of contextual inversion and how it differs from the I_n operations whose axes were shown in Figure 1. However, in the following section, we will show how not all the possibilities of contextual inversions have a transformation label in neo-Riemannian theory and that these two concepts may not be associated when they relate sets of cardinality greater than 3.

III. Transformation *versus* Contextual Inversion

Section II shows how the neo-Riemannian transformations can relate to contextual inversion, but not all authors employ this approach. Douthett and Steinbach, for example, use **P***, **L*** and **R***

[3] The labels **P′**, **L′**, and **R′** is from Robert Morris and he refers to them as "obverse operations”. ([6, p.185]). Cohn uses labels **S** and **N** instead **P′** and **L′**, respectively ([3, p.61-64].

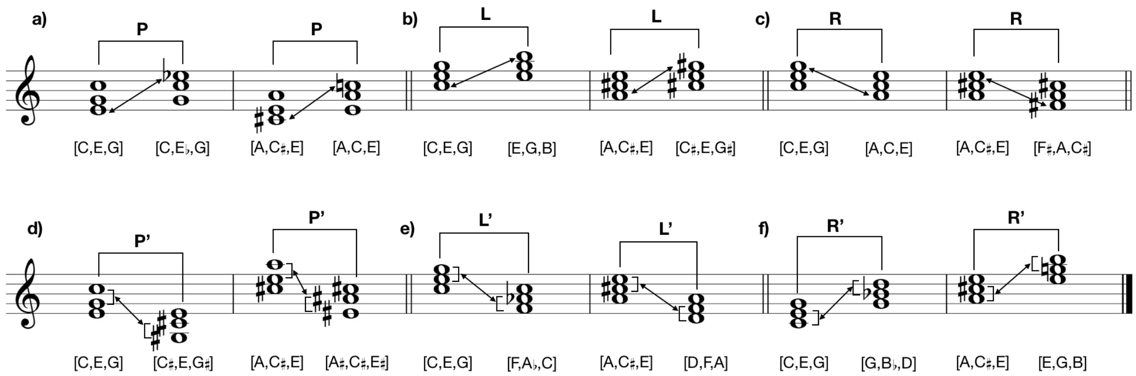


Figure 2: *the contextual inversions of the six main transformations of neo-Riemannian theory.*

(an adaptation of **P**, **L** and **R**) for seventh chords connections that are not related by contextual inversion:

> In the case of seventh chords there are two Parallel* transformations. The transformation **P*₁** exchanges the half-diminished and minor seventh chords that have the same root, and **P*₂** exchanges the minor and dominant-seventh chords with the same root (...). For seventh chords there are two Leittonwechsel* transformations; **L*₁** exchanges root-distinc $P_{1,0}$-related[4] half-diminished and minor seventh chords, and **L*₂** exchanges root-distinc $P_{1,0}$-related dominant and minor seventh chords (...). the Relative* transformation **R*** exchanges two seventh chords[5] that are $P_{0,1}$-related. ([4, p.250]).

It can be noted in this adaptation of the **P**, **L** and R* transformations for seventh chords that the authors are more interested in the parsimonious voice-leading between the chords than in the contextual inversion, since **P*** and **L*** relate members of two different set classes, namely sc. (0258) and sc. (0358). In fact, **P**, **L** and **R** connect two consonant triads that, besides being contextual inversions, are parsimonious, that is, they have two pitch-classes in common and the remaining pitch-class move by a half or a whole step. However, parsimony among members of the same set class is a rather rare feature found only in few classes of trichords, pentachords, heptachords and nonachords, so in order to adapt the labels of transformations for all remaining set classes it is necessary to choose between parsimony and contextual inversion. Straus approaches the problem differently from Douthett and Steinbach. He adapted the 6 transformations labels used in the neo-Riemannian theory for all trichords classes ([8, p.53-56]). He shows how **P** is a contextual inversion that retains the dyad with the largest trichord interval and flips the remaining pitch around it; **L** is a contextual inversion that retains the dyad with the smallest trichord interval and flips the remaining pitch around it; **R** is a contextual inversion that retains the dyad with the second largest trichord interval and flips the remaining pitch around it. He also shows the obverse operation **P'**, **L'** and **R'** and provides a table with the 6 transformations for all the 12 trichords classes ([8, p.56], table 1). This approach can be easily transported to set classes with greater cardinalities. Figure 3 shows how the 6 neo-Riemannian transformations can relate members of sc.

[4] The authors use $P_{m,n}$ notation for parsimonious chords with same cardinality. P stands for parsimony, m is the number of pitches that move by semitone and n is the number of pitches that move by a whole step. Thus, $P_{1,0}$-related chords have all pitches in common, except one that moves by semitone; $P_{0,1}$-related chords have all pitches in common, except one that moves by a whole step.

[5] In this case, both chords are members of sc. (0258).

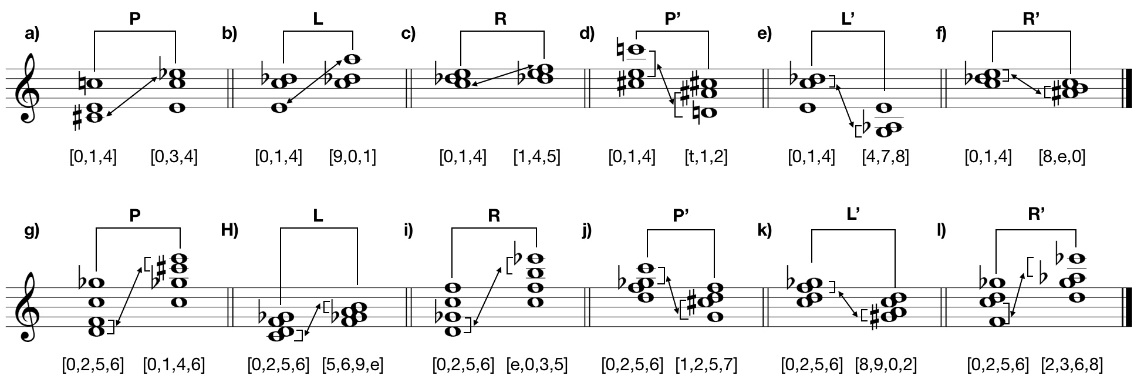


Figure 3: ***P**, **L**, **R**, **P'**, **L'** and **R'** between members of sc. (014) and sc. (0146).*

(014) and members of sc. (0146). For members of sc. (014): **P** retains ic 4 and flips the remaining pitch around it, **L** retains ic 1 and flips the remaining pitch around it, **R** retains ic 3 and flips the remaining pitch around it, **P'** flips ic 4 around the remaining pitch, **L'** flips ic 1 around the remaining pitch and **R'** flips ic 3 around the remaining pitch. For members of sc. (0146): **P** retains ic 6 and flips the remaining dyad around it, **L** retains ic 1 and flips the remaining dyad around it, **R** retains ic 5 and flips the remaining dyad around it, **P'** flips ic 6 around the remaining dyad, **L'** flips ic 1 around the remaining dyad and **R'** flips ic 5 around the remaining dyad.

One can see in Straus's approach that he is interested in contextual inversion rather than in the parsimonious voice-leading, since none of pairs of sets shown in Figure 3 are parsimonious. The advantage of this approach is that it settles the neo-Riemannian labels to any set of any cardinality, however those labels are limited to pairs of sets that have common pitch-classes to each other. In other words, following this approach, all transformations are contextual inversions, but the inverse is not true because the context of the inversions in **P**, **L**, **R**, **P'**, **L'** and **R'**, are given by an axis that must be a dyad or a pitch-class common to both sets. Therefore, there are contextual inversions that cannot be labeled as transformations because they connect sets that do not have pitches in common.

One example of contextual inversions mapping two sets with no common pitches can be show in the Anton Webern's *Concerto for nine Instruments, Op. 24*. Figure 4 shows nine tone rows used in the first 16 measures of the first movement, they are related by four basic operation of twelve-tone system: prime (P), inversion (I), retrograde-prime (RP) and retrograde-inversion (RI). The initial row (Figure 4a) is P_{11}, and the following rows are RI_2 (Figure 4b), RI_1 (Figure 4c), P_0 (Figure 4d), I_0 (Figure 4e), I_5 (Figure 4f), RP_3 (Figure 4g), RP_4 (Figure 4h) and I_6 (Figure 4i). Figure 4 also shows how each row can be segmented into four trichords members of sc. (014)[6] and how the pitches sequence in each trichord also follows operations P, RP, I and RI, as already has observed by Milton Babbitt ([1, p.90-91]). Furthermore, the members of sc. (014) in all tone rows are related by inversion (the index numbers of all inversions are also shown in Figure 4) where the pitches of the first and last pair of trichords in each row always make one of the four hexatonic collections[7], and pitches of the middle pair and the pair with last and first set always make a chromatic hexachord.

It may be noted that inversion index numbers (I_1, I_3, I_5, I_7, I_9 and I_{11}) that map all trichords do not show the uniformity of the criterion according to which their ordering was chosen in each

[6] This segmentation follows the instrumentation choice used by Webern.

[7] In Figure 4, the hexatonic collections are labelled following Straus's notation: $HEX_{0,1}$ [0,1,4,5,8,9], $HEX_{1,2}$ [1,2,5,6,9,t], $HEX_{2,3}$ [2,3,6,7,t,e] and $HEX_{3,4}$ [3,4,7,8,e,0] ([9, p.257]).

row and this is a situation similar to that of the six contextual inversion examples that map the triad consonants shown in Figure 2. But despite the consistency in the connection between the trichords in this example, there is no transformation labels for the contextual inversions mapping the members of sc. (014) because they do not share any common tone. The next section will show how contextual inversions can also be determined by axes, as well as the iI_n-operations, and also can be labelled, as well transformations. In this way, any pair of sets that map onto themselves by contextual inversion, even if they do not share common pitches, can be represented by a label. The Webern's tone rows shown in Figure 4 will be on focus again as examples of axes of contextual inversion.

IV. Axes of Contextual Inversion

Straus states "when a pitch-class set is transposed or inverted, its content will change entirely, partially or not at all" ([9, p.96]). Sets that map entirely onto themselves by transposition are called "transpositionally symmetrical" ([9, p.100]), if they map entirely onto themselves by inversion they are called "inversionally symmetrical" ([9, 107]). According to Straus, there are 14 Tn_n-symmetrical and 79 I_n-symmetrical among the 232 set classes on the Forte List ([9, p.101 and 107]). Since 13 T_n-symmetrical set classes are also I_n-symmetrical[8] , there are 153 set classes which members do not map onto themselves by inversion. These 153 set classes have their members divided into two groups with different OPTC normal forms[9] , following Larry Solomon [7] I will label them as group A and B[10] . For example, sc. (037) is the set class of all consonant triads and they are divided into group A, with 12 minor triads that may be represented by OPTC normal form [0,3,7], and group B, with 12 major triads that may be represented by OPTC normal form [0,4,7]. Members of sc.(014) are also divided into to two groups: A = [0,1,4] and B = [0,3,4], both with 12 members. Any inversion or contextual inversion operation for these set classes will map any member of group A onto a member of group B and vice versa, in this way: I_9 maps C major triad [0,4,7] (group B) onto D minor triad [2,5,9] (group A) and **P** maps B♭minor triad [t,1,5] (group A) onto B♭major triad [t,2,5] (group B); I_7 maps [0,1,4] (group A) onto [3,6,7] (group B) and **L** maps [9,0,1] (group B) onto [9,t,1] (group A). In contrast, members of I_n-symmetrical set classes are not divided into two groups with different OPTC normal forms, since any inversion or contextual inversion operation will map two sets that can also be mapped by inversion. For example: I_0 maps [0,3,6] onto [8,e,2], but these sets are also mapped by T_8; **P** maps [0,3,6] onto [6,9,0], but these sets are also mapped by T_0.

In order to provide positions of axes of contextual inversion it is necessary to determine which one is the fixed point around which a set pair maps themselves by inversion, in other words, it is necessary to determine the context of the inversion. Like the axes of In operations (shown in Figure 1), the axes of contextual inversion operations may cross a pitch-class or halfway between pitch-classes. The difference between these two operations is that the positions of axes in In operations are fixed for each index number (that is, I_0 axis always crosses pitch-classes 0 and 6, I_1 axis always crosses the point between the pitch-classes 0 -1 and 6-7, and so on) while the contextual inversion axes move according to the set pair that is mapped by the operation.

[8] Sc. (013679) is the only one T_n-symmetrical set class that is not I_n-symmetrical.

[9] Callender, Quinn and Tymoczko use the mnemonic OPTIC to refer to octave (O), permutation (P), transposition (T), inversion (I), and cardinality (C) equivalences. The authors explain that there are 32 ways to combine these five equivalences to create different kind of normal forms to represent a set ([2, p.346-348]). OPTC normal forms differ to the prime forms because they do not have the inversion equivalence.

[10] Salomon, in his table of pitch-class sets, labels all the inverses Forte's prime form (which are the OPTC normal form) as "B", so is logical to label the prime form as "A".

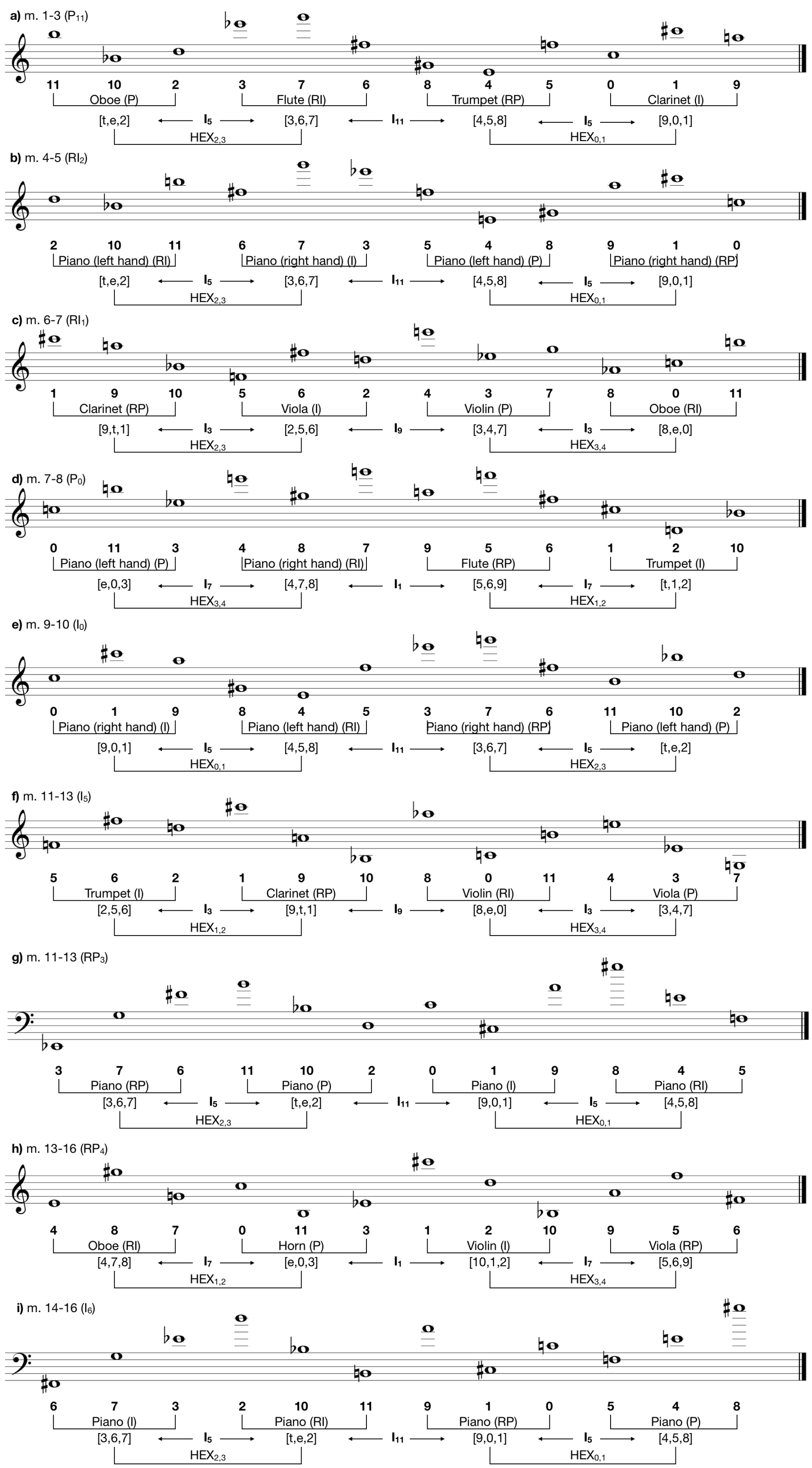


Figure 4: *nine tone rows used in the beginning of the first movement of Anton Webern's Concerto for nine Instruments, Op. 24.*

To determine the position of a contextual axis that maps two sets of same cardinality it is necessary to put both sets in normal form. For sets that are not I_n-symmetrical the position of the contextual inversion axis relative to the first pitch-class of the normal form A is always mirrored to the last pitch-class of the normal form B, this position changes according to the sets. There are 12 different contextual inversion axes that I will label with bold capital letters:

- **A** maps two sets with the axis crossing the first pitch-class of the set in the normal form A and through the last pitch-class of the set in the normal form B;
- **B** maps two sets with the axis crossing the point that is 1/2 semitone over the first pitch-class of the set in the normal form A and the point that is 1/2 semitone below the last pitch-class of the set in the normal form B;
- **C** maps two sets with the axis crossing the point that is 1 semitone over the first pitch-class of the set in the normal form A and the point that is 1 semitone below the last pitch-class of the set in the normal form B;
- **D** maps two sets with the axis crossing the point that is 11/2 semitone over the first pitch-class of the set in the normal form A and the point that is 11/2 semitone below the last pitch-class of the set in the normal form B;
- **E** maps two sets with the axis crossing the point that is 2 semitones over the first pitch-class of the set in the normal form A and the point that is 2 semitones below the last pitch-class of the set in the normal form B;
- **F** maps two sets with the axis crossing the point that is 21/2 semitone over the first pitch-class of the set in the normal form A and the point that is 21/2 semitones below the last pitch-class of the set in the normal form B;
- **G** maps two sets with the axis crossing the point that is 3 semitones over the first pitch-class of the set in the normal form A and the point that is 3 semitones below the last pitch-class of the set in the normal form B;
- **H** maps two sets with the axis crossing the point that is 31/2 semitone over the first pitch-class of the set in the normal form A and the point that is 31/2 semitones below the last pitch-class of the set in the normal form B;
- **I** maps two sets with the axis crossing the point that is 4 semitones over the first pitch-class of the set in the normal form A and the point that is 4 semitones below the last pitch-class of the set in the normal form B;
- **J** maps two sets with the axis crossing the point that is 41/2 semitone over the first pitch-class of the set in the normal form A and the point that is 41/2 semitones below the last pitch-class of the set in the normal form B;
- **K** maps two sets with the axis crossing the point that is 5 semitones over the first pitch-class of the set in the normal form A and the point that is 5 semitones below the last pitch-class of the set in the normal form B;
- **L** maps two sets with the axis crossing the point that is 51/2 semitone over the first pitch-class of the set in the normal form A and the point that is 51/2 semitones below the last pitch-class of the set in the normal form B.

In this way, the context of the inversion is always determined by the position of the axis in relation to one given pitch-class (the first or the last) of a set written in normal form. One can see in Figure 5 some examples of axes of contextual inversion mapping two members of same set classes in clock faces. The pitch-classes of members in the normal form A are connected by lines drawn outside the circle, while the pitch-classes of members in the normal form B are connected by lines drawn inside the circle.

For I_n-symmetrical set classes, the 12 axes of contextual inversion will map members that are

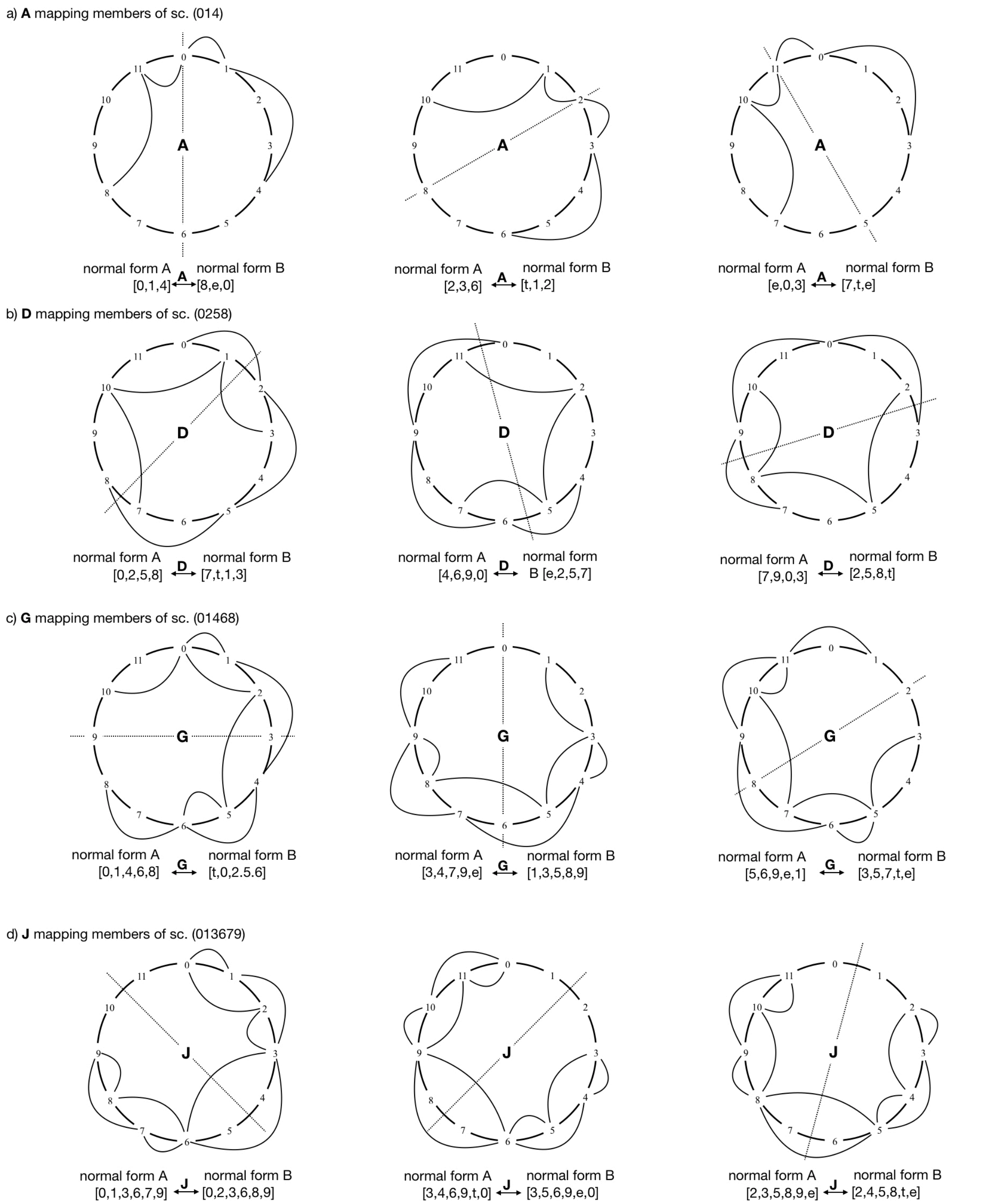


Figure 5: *examples of **A**, **D**, **G** and **J** mapping members of sc. (014), (0258), (01468) and (013679), respectively.*

Table 1: *all members of sc. (014) connect by the axes of contextual inversion.*

[0,1,4]		[8,e,0]	[0,1,4]		[t,1,2]	[0,1,4]		[0,3,4]	[0,1,4]		[2,5,6]	[0,1,4]		[4,7,8]	[0,1,4]		[6,9,t]
[1,2,5]		[9,0,1]	[1,2,5]		[e,2,3]	[1,2,5]		[1,4,5]	[1,2,5]		[3,6,7]	[1,2,5]		[5,8,9]	[1,2,5]		[7,t,e]
[2,3,6]		[t,1,2]	[2,3,6]		[0,3,4]	[2,3,6]		[2,5,6]	[2,3,6]		[4,7,8]	[2,3,6]		[6,9,t]	[2,3,6]		[8,e,0]
[3,4,7]		[e,2,3]	[3,4,7]		[1,4,5]	[3,4,7]		[3,6,7]	[3,4,7]		[5,8,9]	[3,4,7]		[7,t,e]	[3,4,7]		[9,0,1]
[4,5,8]		[0,3,4]	[4,5,8]		[2,5,6]	[4,5,8]		[4,7,8]	[4,5,8]		[6,9,t]	[4,5,8]		[8,e,0]	[4,5,8]		[t,1,2]
[5,6,9]	**A**	[1,4,5]	[5,6,9]	**C**	[3,6,7]	[5,6,9]	**E**	[5,8,9]	[5,6,9]	**G**	[7,t,e]	[5,6,9]	**I**	[9,0,1]	[5,6,9]	**K**	[e,2,3]
[6,7,t]		[2,5,6]	[6,7,t]		[4,7,8]	[6,7,t]		[6,9,t]	[6,7,t]		[8,e,0]	[6,7,t]		[t,1,2]	[6,7,t]		[0,3,4]
[7,8,e]		[3,6,7]	[7,8,e]		[5,8,9]	[7,8,e]		[7,t,e]	[7,8,e]		[9,0,1]	[7,8,e]		[e,2,3]	[7,8,e]		[1,4,5]
[8,9,0]		[4,7,8]	[8,9,0]		[6,9,t]	[8,9,0]		[8,e,0]	[8,9,0]		[t,1,2]	[8,9,0]		[0,3,4]	[8,9,0]		[2,5,6]
[9,t,1]		[5,8,9]	[9,t,1]		[7,t,e]	[9,t,1]		[9,0,1]	[9,t,1]		[e,2,3]	[9,t,1]		[1,4,5]	[9,t,1]		[3,6,7]
[t,e,2]		[6,9,t]	[t,e,2]		[8,e,0]	[t,e,2]		[t,1,2]	[t,e,2]		[0,3,4]	[t,e,2]		[2,5,6]	[t,e,2]		[4,7,8]
[e,0,3]		[7,t,e]	[e,0,3]		[9,0,1]	[e,0,3]		[e,2,3]	[e,0,3]		[1,4,5]	[e,0,3]		[3,6,7]	[e,0,3]		[5,8,9]
[0,1,4]		[9,0,1]	[0,1,4]		[e,2,3]	[0,1,4]		[1,4,5]	[0,1,4]		[3,6,7]	[0,1,4]		[5,8,9]	[0,1,4]		[7,t,e]
[1,2,5]		[t,1,2]	[1,2,5]		[0,3,4]	[1,2,5]		[2,5,6]	[1,2,5]		[4,7,8]	[1,2,5]		[6,9,t]	[1,2,5]		[8,e,0]
[2,3,6]		[e,2,3]	[2,3,6]		[1,4,5]	[2,3,6]		[3,6,7]	[2,3,6]		[5,8,9]	[2,3,6]		[7,t,e]	[2,3,6]		[9,0,1]
[3,4,7]		[0,3,4]	[3,4,7]		[2,5,6]	[3,4,7]		[4,7,8]	[3,4,7]		[6,9,t]	[3,4,7]		[8,e,0]	[3,4,7]		[t,1,2]
[4,5,8]		[1,4,5]	[4,5,8]		[3,6,7]	[4,5,8]		[5,8,9]	[4,5,8]		[7,t,e]	[4,5,8]		[9,0,1]	[4,5,8]		[e,2,3]
[5,6,9]	**B**	[2,5,6]	[5,6,9]	**D**	[4,7,8]	[5,6,9]	**F**	[6,9,t]	[5,6,9]	**H**	[8,e,0]	[5,6,9]	**J**	[t,1,2]	[5,6,9]	**L**	[0,3,4]
[6,7,t]		[3,6,7]	[6,7,t]		[5,8,9]	[6,7,t]		[7,t,e]	[6,7,t]		[9,0,1]	[6,7,t]		[e,2,3]	[6,7,t]		[1,4,5]
[7,8,e]		[4,7,8]	[7,8,e]		[6,9,t]	[7,8,e]		[8,e,0]	[7,8,e]		[t,1,2]	[7,8,e]		[0,3,4]	[7,8,e]		[2,5,6]
[8,9,0]		[5,8,9]	[8,9,0]		[7,t,e]	[8,9,0]		[9,0,1]	[8,9,0]		[e,2,3]	[8,9,0]		[1,4,5]	[8,9,0]		[3,6,7]
[9,t,1]		[6,9,t]	[9,t,1]		[8,e,0]	[9,t,1]		[t,1,2]	[9,t,1]		[0,3,4]	[9,t,1]		[2,5,6]	[9,t,1]		[4,7,8]
[t,e,2]		[7,t,e]	[t,e,2]		[9,0,1]	[t,e,2]		[e,2,3]	[t,e,2]		[1,4,5]	[t,e,2]		[3,6,7]	[t,e,2]		[5,8,9]
[e,0,3]		[8,e,0]	[e,0,3]		[t,1,2]	[e,0,3]		[0,3,4]	[e,0,3]		[2,5,6]	[e,0,3]		[4,7,8]	[e,0,3]		[6,9,t]

also mapped by transposition, meaning that they cannot be expressed by two different OPTC normal forms. Because of that, the position of any axis relative to the first pitch-class will be mirrored to the last pitch-class of the same normal form and this will result, in some cases, in a single axis mapping two different pairs of sets. For example, A maps set [0,1,2] onto [t,e,0], since the axis crosses the first pitch-class of its normal form, but A also maps [0,1,2] onto [2,3,4] because the axis also crosses the last pitch-class of its normal form. In order to differentiate these two connections, I will use A_0 to label the axis between [0,1,2] and [t,e,0], since it crosses pitch-class 0, and A_2 to label the axis between [0,1,2] onto [2,3,4], since it crosses pitch-class 2. For axes that cross points halfway between two pitch-classes, I will use the integers that represent these two pitch-classes separated by a comma after the label letter, so $F_{2,3}$ maps [0,1,2] onto [3,4,5], since the axis crosses halfway between pitch-classes 2 and 3, and $F_{e,0}$ maps [0,1,2] onto [9,t,e], since the axis crosses halfway between pitch-classes 11 and 0. Figure 6 shows some examples of axes of contextual inversion mapping two members of same In-symmetrical set classes in clock faces.

One can see in Table 1 an 2, how all members of same set class connect to each other by the 12 contextual inversion axes. Table 1 shows the axes (labels **A** to **L**) connecting all members of sc. (014), with the sets represented by normal form A to the left and those represented by normal form B to the right of each label. Table 2 shows the axes connecting all members of sc. (0158)[11] . Since this is a In-symmetrical set class, a pair of members can be connected by two different axes, resulting in 7, instead of 12, different labels[12].

V. Analytical examples

In this section, examples of how labels for contextual inversion axis discussed on previous section are helpful for analyses will be discussed. I will show some analytical examples of passages

[11] Access www.axesofcontextualinversion.com to download the tables for all each set-classes.

[12] One can see in Table 2 that C maps the same pairs of sets that A, and the same happens with I and G, J and F, K and E and L and D. All In-symmetrical set classes will have pairs of sets that can be map by two axes.

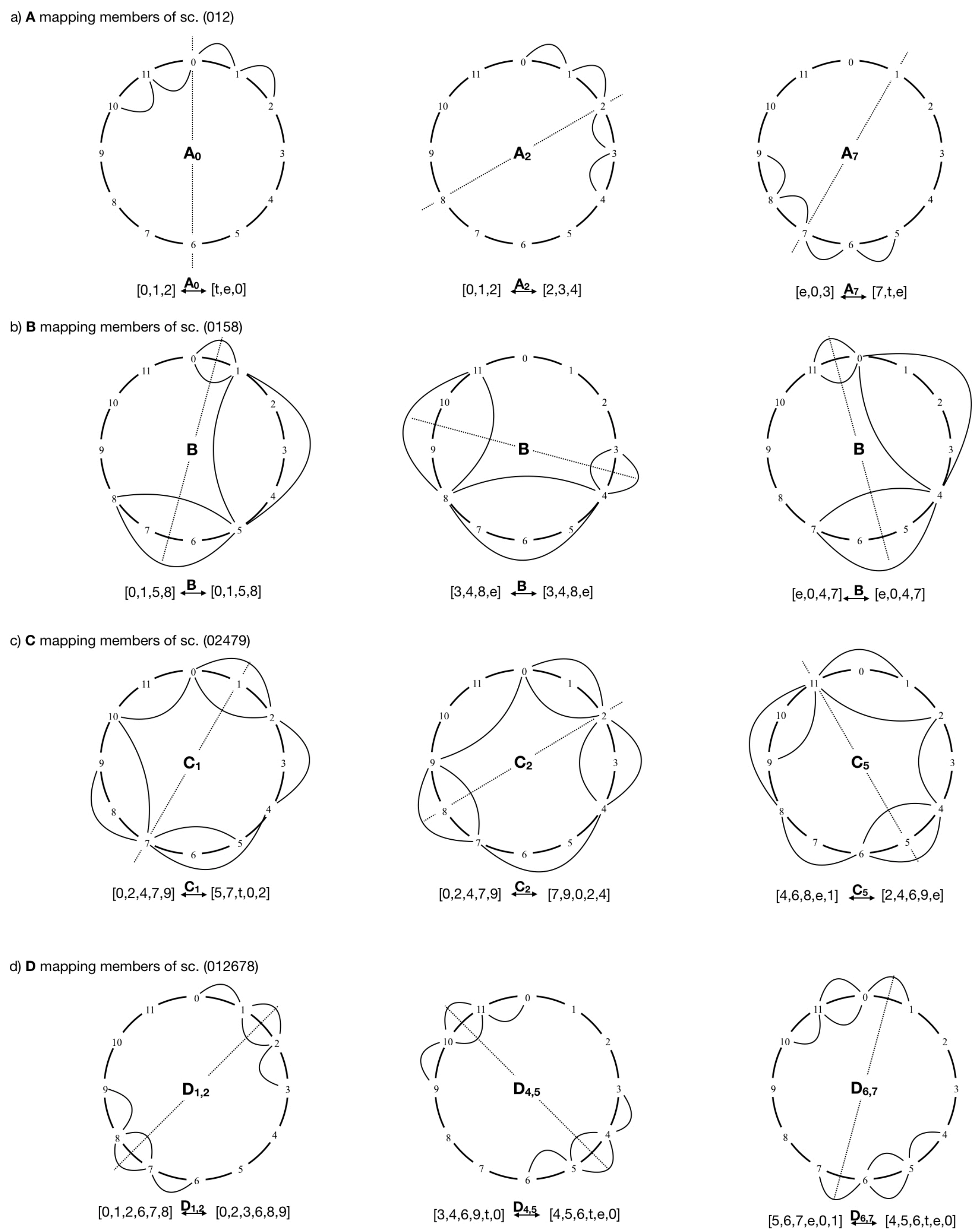


Figure 6: *examples of A, B, C and D mapping members of sc. (012), (0158), (02479) and (012678), respectively.*

Table 2: *all members of sc. (0158) connect by the axes of contextual inversion.*

	A			C (=A)			E			G		I (=G)	K (=E)
[0,1,5,8]	A_0	[e,0,4,7]				[0,1,5,8]	E_2	[3,4,8,e]	[0,1,5,8]	G_3	[5,6,t,1]		
[1,2,6,9]	A_1	[0,1,5,8]				[1,2,6,9]	E_3	[4,5,9,0]	[1,2,6,9]	G_4	[6,7,e,2]		
[2,3,7,t]	A_2	[1,2,6,9]				[2,3,7,t]	E_4	[5,6,t,1]	[2,3,7,t]	G_5	[7,8,0,3]		
[3,4,8,e]	A_3	[2,3,7,t]				[3,4,8,e]	E_5	[6,7,e,2]	[3,4,8,e]	G_6	[8,9,1,4]		
[4,5,9,0]	A_4	[3,4,8,e]				[4,5,9,0]	E_6	[7,8,0,3]	[4,5,9,0]	G_7	[9,t,2,5]		
[5,6,t,1]	A_5	[4,5,9,0]				[5,6,t,1]	E_7	[8,9,1,4]	[5,6,t,1]	G_8	[t,e,3,6]		
[6,7,e,2]	A_6	[5,6,t,1]				[6,7,e,2]	E_8	[9,t,2,5]	[6,7,e,2]	G_9	[e,0,4,7]		
[7,8,0,3]	A_7	[6,7,e,2]				[7,8,0,3]	E_9	[t,e,3,6]	[7,8,0,3]	G_t	[0,1,5,8]		
[8,9,1,4]	A_8	[7,8,0,3]				[8,9,1,4]	E_t	[e,0,4,7]	[8,9,1,4]	G_e	[1,2,6,9]		
[9,t,2,5]	A_9	[8,9,1,4]				[9,t,2,5]	E_e	[0,1,5,8]	[9,t,2,5]	G_0	[2,3,7,t]		
[t,e,3,6]	A_t	[9,t,2,5]				[t,e,3,6]	E_0	[1,2,6,9]	[t,e,3,6]	G_1	[3,4,8,e]		
[e,0,4,7]	A_e	[t,e,3,6]				[e,0,4,7]	E_1	[2,3,7,t]	[e,0,4,7]	G_2	[4,5,9,0]		
[0,1,5,8]		[0,1,5,8]	[0,1,5,8]	$D_{1,2}$	[2,3,7,t]	[0,1,5,8]	$F_{2,3}$	[4,5,9,0]	[0,1,5,8]	$H_{3,4}$	[6,7,e,2]		
[1,2,6,9]		[1,2,6,9]	[1,2,6,9]	$D_{2,3}$	[3,4,8,e]	[1,2,6,9]	$F_{3,4}$	[5,6,t,1]	[1,2,6,9]	$H_{4,5}$	[7,8,0,3]		
[2,3,7,t]		[2,3,7,t]	[2,3,7,t]	$D_{3,4}$	[4,5,9,0]	[2,3,7,t]	$F_{4,5}$	[6,7,e,2]	[2,3,7,t]	$H_{5,6}$	[8,9,1,4]		
[3,4,8,e]		[3,4,8,e]	[3,4,8,e]	$D_{4,5}$	[5,6,t,1]	[3,4,8,e]	$F_{5,6}$	[7,8,0,3]	[3,4,8,e]	$H_{6,7}$	[9,t,2,5]		
[4,5,9,0]		[4,5,9,0]	[4,5,9,0]	$D_{5,6}$	[6,7,e,2]	[4,5,9,0]	$F_{6,7}$	[8,9,1,4]	[4,5,9,0]	$H_{7,8}$	[t,e,3,6]		
[5,6,t,1]	B	[5,6,t,1]	[5,6,t,1]	$D_{6,7}$	[7,8,0,3]	[5,6,t,1]	$F_{7,8}$	[9,t,2,5]	[5,6,t,1]	$H_{8,9}$	[e,0,4,7]		
[6,7,e,2]		[6,7,e,2]	[6,7,e,2]	$D_{7,8}$	[8,9,1,4]	[6,7,e,2]	$F_{8,9}$	[t,e,3,6]					
[7,8,0,3]		[7,8,0,3]	[7,8,0,3]	$D_{8,9}$	[9,t,2,5]	[7,8,0,3]	$F_{9,t}$	[e,0,4,7]					
[8,9,1,4]		[8,9,1,4]	[8,9,1,4]	$D_{9,t}$	[t,e,3,6]	[8,9,1,4]	$F_{t,e}$	[0,1,5,8]					
[9,t,2,5]		[9,t,2,5]	[9,t,2,5]	$D_{t,e}$	[e,0,4,7]	[9,t,2,5]	$F_{e,0}$	[1,2,6,9]					
[t,e,3,6]		[t,e,3,6]	[t,e,3,6]	$D_{e,0}$	[0,1,5,8]	[t,e,3,6]	$F_{0,1}$	[2,3,7,t]					
[e,0,4,7]		[e,0,4,7]	[e,0,4,7]	$D_{0,1}$	[1,2,6,9]	[e,0,4,7]	$F_{1,2}$	[3,4,8,e]					
	B			D			F			H		J (=F)	L (=D)

from Webern, *Concerto for Nine Instruments, op. 24*, first movement, Stravinsky, "Musick to Heare," from *Three Shakespeare Songs* and Villa-Lobos, *Etude nº 10*. In these examples, the labels for the contextual inversion axis will be essential both to map two sets and to construct graphs that visually represent a sequence of connections.

a) Webern, *Concerto for Nine Instruments, op. 24*, first movement

With these new labels it is possible to analyze any passages which the sets connect by contextual inversion, especially those passages in which the sets do not share any common pitch-class. One example is the nine tone rows of the first sixteen bars of Webern's *Concerto for nine Instruments, Op. 24* shown in Figure 4. As seen in the previous section, there is no transformation to label the connections between the trichords that Webern choose for each tone row, in addition, the index numbers for the I_n-operations that map those trichords do not help to understand why their connection is so consistent. However, using the labels previously presented in this section it is possible to verify that the members of sc. (014) are mapped only by two axes of contextual inversion in all tone rows. Figure 7 shows the sets of the same nine Webern's tone rows discussed in previous section, one can see that all the members of sc. (014) are mapped by **J** and **D**. Since all trichords are written in the normal form, it is easier to notice that some of these tone rows use same sets.

It can seen in Figure 7 how the first pair of members of sc. (014) in all tone rows used in the beginning of *Concerto* is connected by **J**, the second pair is connected by **D** and the last pair is connected by **J** again. That means that **J** connects all pairs of members of sc. (014) that make a hexatonic collection and **D** connects all pairs of members of sc. (014) that make a chromatic hexachord. Since the connection between the last and the first set of each row is also by **D**, all tone rows sets became a cycle. Based on these tone rows it is possible to build six different cycles induced by a <**DJ**> chain with all 24 members of sc. (014). I will call these cycles as Webern's Graphs and each of them will be numbered according to the first pitch-class of the set in its North Pole, thus: WG_0, cycle with [0,1,4], [5,8,9], [6,7,t] and [e,2,3]; WG_1, cycle with [1,2,5], [6,9,t], [7,8,e]

a) m. 1-3 (P_{11})

Oboe)		Flute		Trumpet		Clarinet
[t,e,2]	← J →	[3,6,7]	← D →	[4,5,8]	← J →	[9,0,1]

b) m. 4-5 (RI_2)

Piano (left hand)		Piano (right hand)		Piano (left hand))		Piano (right hand)
[t,e,2]	← J →	[3,6,7]	← D →	[4,5,8]	← J →	[9,0,1]

c) m. 6-7 (RI_1)

Clarinet		Viola		Violin		Oboe
[9,t,1]	← J →	[2,5,6]	← D →	[3,4,7]	← J →	[8,e,0]

d) m. 7-8 (P_0)

Piano (left hand)		Piano (right hand)		Flute		Trumpet
[e,0,3]	← J →	[4,7,8]	← D →	[5,6,9]	← J →	[t,1,2]

e) m. 9-10 (I_0)

Piano (right hand)		Piano (left hand)		Piano (right hand)		Piano (left hand)
[9,0,1]	← J →	[4,5,8]	← D →	[3,6,7]	← J →	[t,e,2]

f) m. 11-13 (I_5)

Trumpet		Clarinet		Violin		Viola)
[2,5,6]	← J →	[9,t,1]	← D →	[8,e,0]	← J →	[3,4,7]

g) m. 11-13 (RP_3)

Piano		Piano)		Piano		Piano
[3,6,7]	← J →	[t,e,2]	← D →	[9,0,1]	← J →	[4,5,8]

h) m. 13-14 (RP_4)

Oboe		Horn		Violin		Viola
[4,7,8]	← J →	[e,0,3]	← D →	[10,1,2]	← J →	[5,6,9]

i) m. 14-16 (I_6)

Piano		Piano		Piano		Piano
[3,6,7]	← J →	[t,e,2]	← D →	[9,0,1]	← J →	[4,5,8]

Figure 7: *all trichords in Webern's tone rows are connect by **D** or **J**.*

and [0,3,4]; WG_2, cycle with [2,3,6], [7,t,e], [8.9,0] and [1,4,5]; WG_3, cycle with [3,4,7], [8,e,0], [9,t,1] and [2,5,6]; WG_4, cycle with [4,5,8], [9,0,1], [t,e,2] and [3,6,7]; and WG_5, cycle with [5,6,9], [t,1,2], [e,0,3] and [4,7,8]. The members of sc. (014) in each cycle are the set's content of 8 different tone rows that are listed below them, once the sequence of sets can start with the member to the North, to the East, to the South or to the West of a cycle and move clockwise or counterclockwise. The matrix with all 48 forms of the tone rows for the Webern's Concerto is shown in the bottom of Figure 8.

One can see in Figure 8 that all nine Webern's tone rows shown in Figure 4 are built with WG_3, WG_4 or WG_5 sets. Both P_{11} and RI_2 start with the set to the South Pole in WG_4 and move clockwise in the cycle; RI_1 and P_0 start with the set to the South Pole in WG_3 and in WG_5, respectively, and move clockwise in the cycles; I_0 starts with the set to the East Pole in WG_4 and moves counterclockwise in the cycle; RP_3 and I_6 start with the set to the West Pole in WG_4 and move counterclockwise in the cycle; I_0 and RP_4 start with the set to the West Pole in WG_3 and in WG_5, respectively, and move counterclockwise in the cycles.

Figure 9 shows a space with these graphs where we can see the sequence of the sets and tone rows of the beginning of Webern's *Concerto* in timeline[13]. The advantage of creating a space for these passage is that some relationships between the sets are easier to understand with this visual feature.

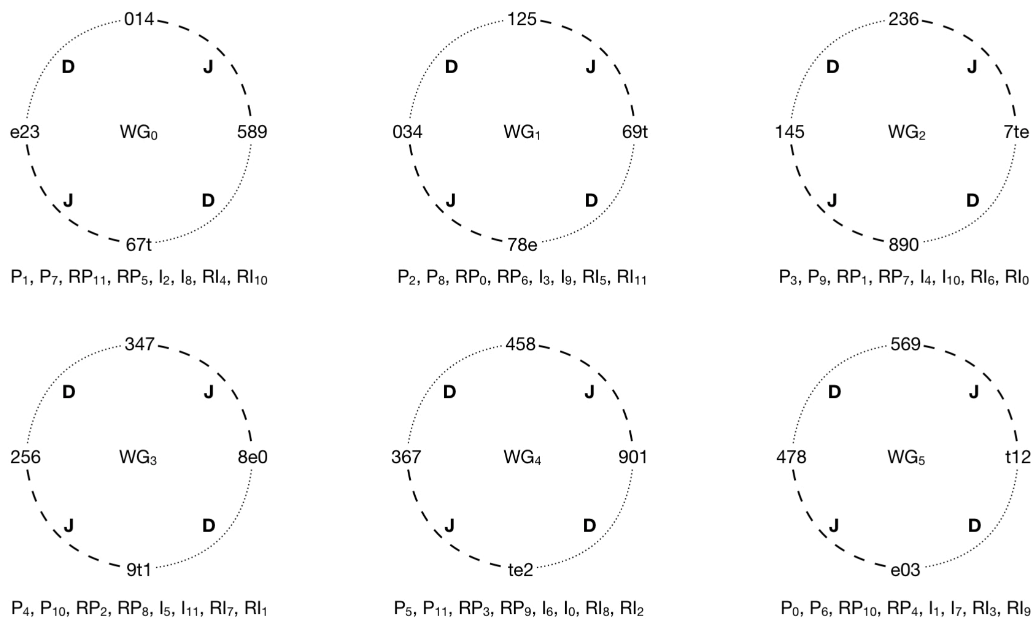
The space in figure 9 shows all the tone rows of the beginning of Webern's *Concerto* divided in three Webern's Graph: WG_3, WG_4 and WG_5. The sequence starts with the oboe playing the member of sc. (014) placed to the South Pole in the first cycle to the left and follows the arrows along all the remaining cycles. Note how the first and the second tone rows use sets in the same cycle (WG_4) in bars 1-5; then in bars 6-10, the three cycles are heard in sequence (WG_3, WG_5 and WG_4); and in the end, they are heard simultaneously in pairs (WG_3 with WG_4 and WG_4 with WG_5). Figure 9 also shows the labels of contextual inversion axes for sets that connect two different tone rows. D also connects last set of first row (P_{11}) and first set of the second row (RI_2), once these two tone rows share a same cycle (WG_4); **E** connects last set of second row (RI_2) and first set of the third row (RI_1); the last and the first sets of next two rows are connected by **B**; the last and the first sets of the fourth and the fifth tone rows are connected by transposition (T_{11}) this is the only exception for the contextual inversion connections in this passage of the *Concerto*; the last and the first sets of next two rows are connected by I; then the second set of the I_0 tone row connects to first set of RP_3 by **K** and its last set connects with the first set of RP_4 by **F**; finally, **D** connects the last and the first set of the final two piano tone rows (they share a same cycle). By observing the paths of tone rows in their cycle, it is possible to divide them into two different groups. The first group has the first four tone rows of these passage as they share a same path in their cycle, starting in the set placed on the South Pole and moving clockwise. The second group has the last four tone rows of these passage as they also share a same path in their cycle, starting in the set placed on the West Pole and moving counterclockwise. Between these two groups is the only tone row that does not share its path with any other, once it starts in the East Pole of the cycle and moves counterclockwise. One can see how this tone row between those two groups also functions as a pivot on the texture change that occurs between the first part of that passage (m. 1 to 8), where each tone row are heard alone without harmonic background, and the second (m, 11 to 16), where two tone rows are heard simultaneously.

b) Stravinsky, "Musick to Heare," from *Three Shakespeare Songs*

Straus also shows how transformations **P**, **L**, **R**, **P'**, **L'**, and **R'** map members of tetrachords

[13] In figure 9, abbreviations for instruments are: ob = oboe; fl = flute; tpt = trumpet; cl = clarinet; pn = piano; va = viola; vn = violin; hn = horn. In the same figure, (LH) and (RH) stands for left hand and right hand for piano, respectively.

a) six cycles with members of sc. (014) induced by <**DJ**> chain.



b) matrix for Webern's *Concerto*

	I_0	I_{11}	I_3	I_4	I_8	I_7	I_9	I_5	I_6	I_1	I_2	I_{10}	
P_0	0	11	3	4	8	7	9	5	6	1	2	10	RP_{10}
P_1	1	0	4	5	9	8	10	6	7	2	3	11	RP_{11}
P_9	9	8	0	1	5	4	6	2	3	10	11	7	RP_7
P_8	8	7	11	0	4	3	5	1	2	9	10	6	RP_6
P_4	4	3	7	8	0	11	1	9	10	5	6	2	RP_2
P_5	5	4	8	9	1	0	2	10	11	6	7	3	RP_3
P_3	3	2	6	7	11	10	0	8	9	4	5	1	RP_1
P_7	7	6	10	11	3	2	4	0	1	8	9	5	RP_5
P_6	6	5	9	10	2	1	3	11	0	7	8	4	RP_4
P_{11}	11	10	2	3	7	6	8	4	5	0	1	9	RP_9
P_{10}	10	9	1	2	6	5	7	3	4	11	0	8	RP_8
P_2	2	1	5	6	10	9	11	7	8	3	4	0	RP_0
	RI_2	RI_1	RI_5	RI_6	RI_{10}	RI_9	RI_{11}	RI_7	RI_8	RI_3	RI_4	RI_0	

Figure 8: *a) Webern Graphs, cycles with members of sc. (014) induced by <DJ>; b) matrix with all form of tone rows for Webern's Concerto.*

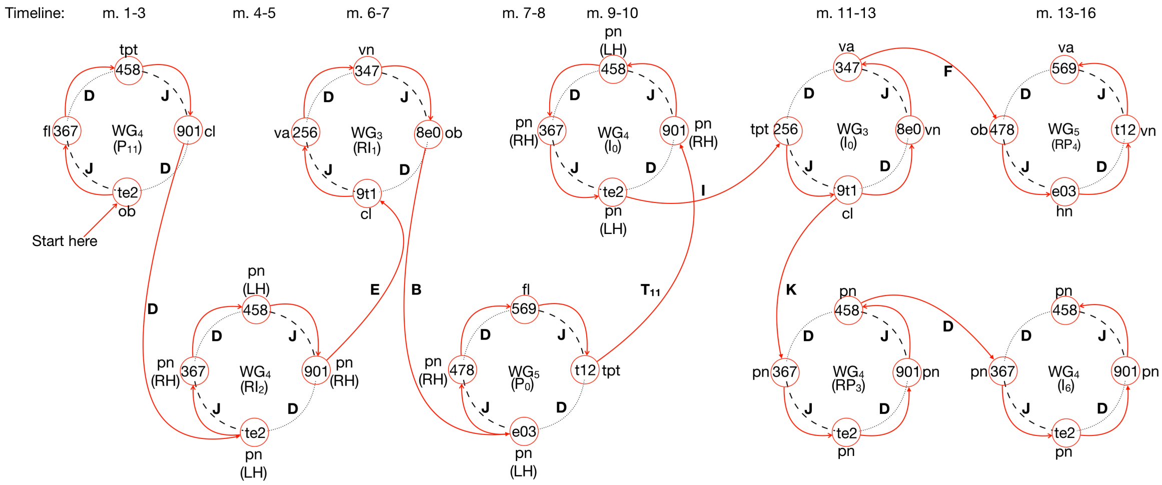


Figure 9: *space that shows the tone rows and the sets of the beginning of Webern's Concerto in the Webern Graphs (see the animation movie at https://axesofcontextualinversion.com).*

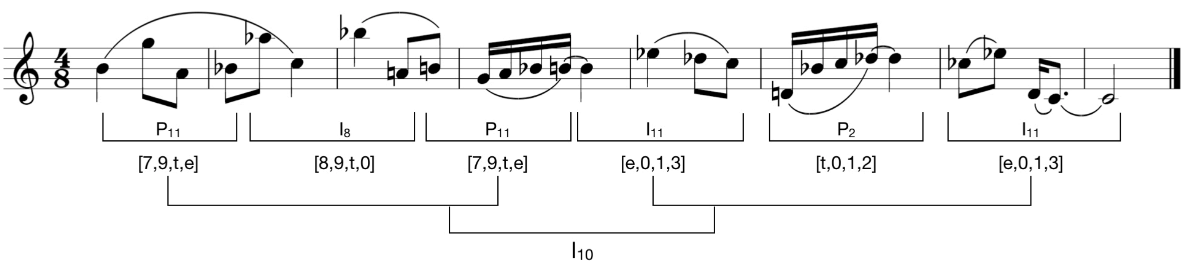


Figure 10: *melody played by flute in the introduction of Stravinsky's "Musick to Heare" (m1-8).*

set classes ([8, p.63-67]). All transformations preserve two pitch-classes in common between the sets. This means that in addition to inversions that do not have pitch-class in common, there are also no transformation labels for inversions that preserve only one pitch-class. In his analysis of Stravinsky's "Musick to Heare", from *Three Shakespeare Songs*, Straus provides a good example of a musical passage with tetrachords of the same set classes connected by contextual inversion sharing only one pitch-class. He explains: "with the exception of a brief diatonic reference in opening and closing measures, this song is based entirely on a four-note series: B–G–A–B♭"([8], p. 67), which is a member of sc. (0124)". Straus segmented the melodic line played by the flute in the introduction (mm. 1-8) in six statements of the series: three forms, P_{11}–I_8–P_{11}, followed by I_{11}–P_2–I_{11} ([8, p.67-69]), the first and the second segments are mapped by I_{10}. Figure 10 shows this melody and its segmentation made by Straus.

One can see in Figure 10 that there are four different members of sc. (0124) in this passage: [7,9,t,e], [8,9,t,0], [e,0,1,3] and [t,0,1,2], they are all related by context inversions that preserve only one pitch-class in common. Since there are no neo-Riemannian's transformations labels for these connections, we will use the labels for axes of contextual inversion presented on section IV in these passage. The clock faces at the top of figure 11 show how **D** maps [7,9,t,e] onto [8,9,t,0] and [e,0,1,3] onto [t,0,1,2], and **A** maps [7,9,t,e] onto [e,0,1,3]. In the bottom of the same figure we can see the sequence of sets of the Stravinsky's flute melody connected by these labels of contextual inversions.

Since **D** and **A** map all sets in these passage, we will build cycles with the members of sc.

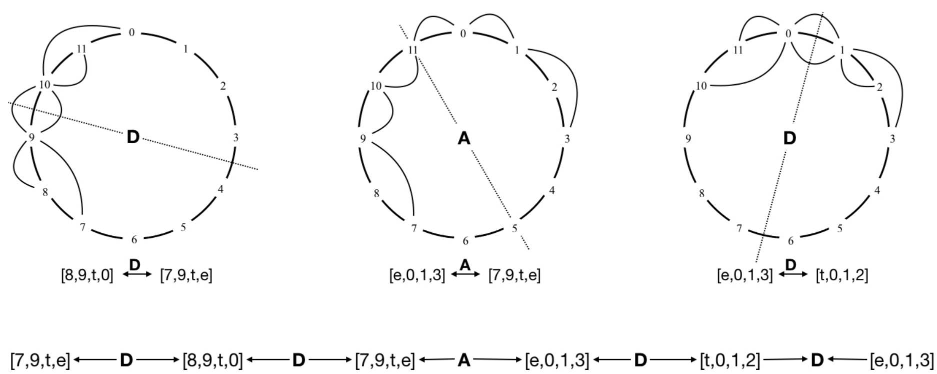


Figure 11: ***D** and **A** map all sets of the melody played by flute in the introduction of Stravinsky's "Musick to Heare".*

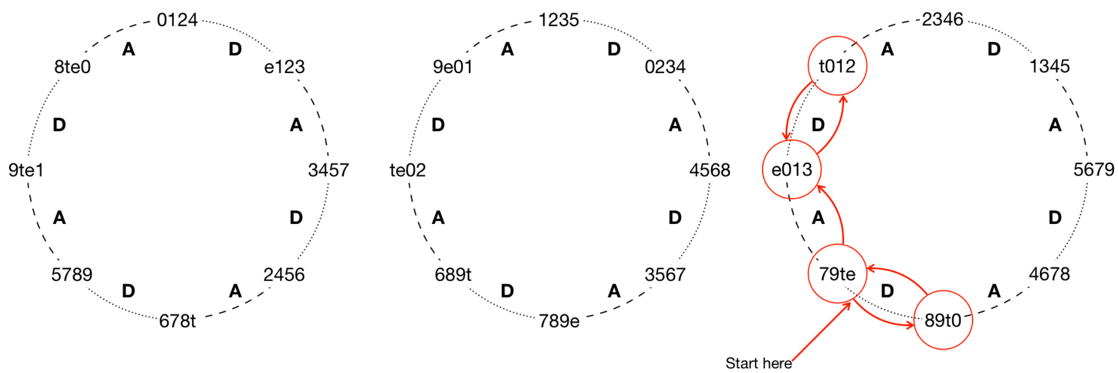


Figure 12: *all three cycles of members of sc. (0124) induced by <**DA**> chain, the connections between the sets from introduction of "Musick to Heare" are shown in the right cycle (see the animation movie at https://axesofcontextualinversion.com).*

(0124) induced by <**DA**> chain in the same way we did for members of sc. (014) in the Webern Graphs (Figure 8). There are three of these cycles, each one with eight members of sc. (0124), they are all shown in Figure 12. All fours sets from Stravinsky's "Musick to Heare" flute introduction are embedded in the third cycle on the right and the path traced by the connection between them starts in the southwestern set of the cycle and follows the neighbouring sets.

c) Villa-Lobos, Etude n^0 10

In the last example of this section I will focus on contextual inversion between sets that are I_n-symmetrical. As has been said previously, inversion or contextual inversion operations map two I_n-symmetrical sets that are also mapped by transposition. In a previous analysis of Villa-Lobos *Etude n_o 10* ([11, p.83-94]), I mentioned the great recurrence of members of sc. (0257) in its section B relating these sets by transposition. I also divided this section into three distinct parts: in the first part (m. 21-42) the texture has two simultaneous layers; the second part has mostly only one layer that only splits itself into two in the five final bars; the third part (m. 57-65) is a retransition to section A' which then returns. For this example I will consider just the first and the second part of section B. In the first part there is a layer with high notes in an ostinato in the rhythm of sixteenth notes and another layer with a lower melody with greater note values. These two layers are heard simultaneously and I will call them as first and second layer, respectively. With exception of the last set in the first layer, all sets are members of sc. (0257), in my previous analysis

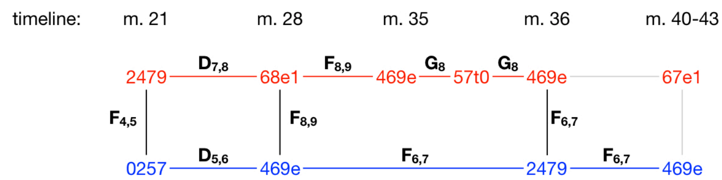


Figure 13: *the sets in first part of Section B of Villa-Lobos's Etude 10.*

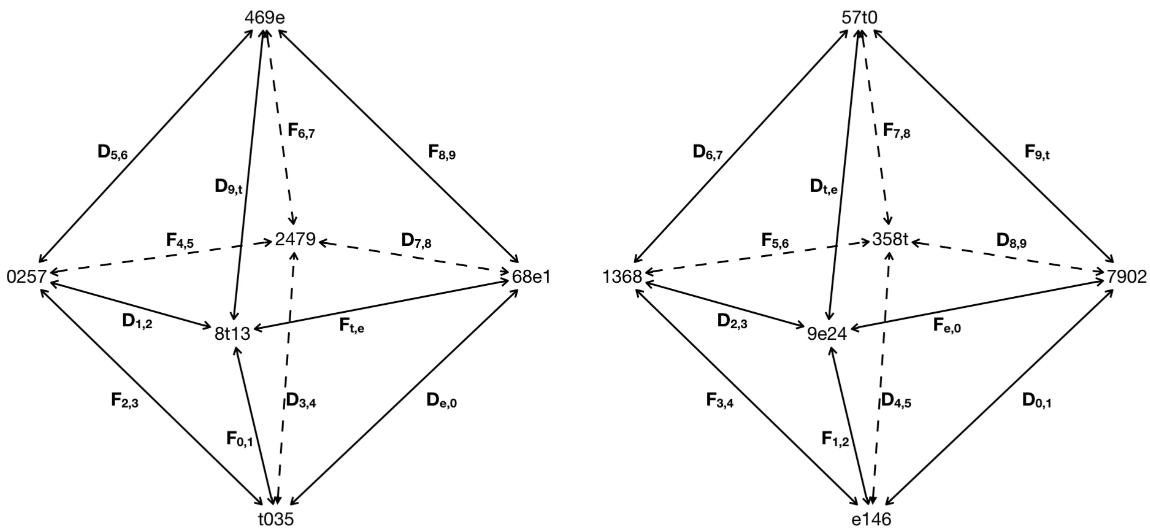


Figure 14: *two octahedron-shaped graphs with members of sc. (0257), each one is the union of two cycles induced by <**DF**> chain.*

I related them by transposition, but now I will focus on the contextual inversion operations that relate them. See in Figure 13 all the sets in both layers of this first part of section B.

All sets of first layer in figure 13 are written in red and all sets of second layer are written in blue. The lines represent the connection between the sets: red lines are the connections between members of sc. (0257) of first layer; blue lines are the connections between members of sc. (0257) of second layer; the black lines represent connections between members of sc. (0257) of different layers that are heard simultaneously; gray lines represent connections of members of different set classes. We can see how almost all sets are connect by **D** and **F** in this passage (with exception of connection between [4,6,9,e] and [5,7,t,0] mapped by G_8) and thus, I will, however, build a graph induced by <**DF**> chain in the same way I did in last two analyses. Once **D** and **F** can connect one member of sc. (0257) to two others, all members will be part of several different cycles. All of these cycles can be joined into a two octahedron-shaped graph that incorporate all members of sc. (0257). See the graphs of members of sc. (0257) induced by <**DF**> chain in Figure 14.

It is possible to trace a path of the connections between all members of sc. (0257) of this passage in this new two graphs in the same way we did with the cycles in previous analyses. One can see in Figure 13 that this type of set is the pitch content of both layers between mm. 21 and 36. In the top of figure 15 we can see how these sets are connected by contextual inversion, this figure has the same color pattern used in Figure 13 red circles indicate sets of the first layer and red arrows indicate connections between these sets; blue circles indicate sets of the first layer and arrows indicate connections between these sets; black line connecting two sets of different layers indicate that they are heard simultaneously. One can see in Figure 15 that all members are connect by **D** or

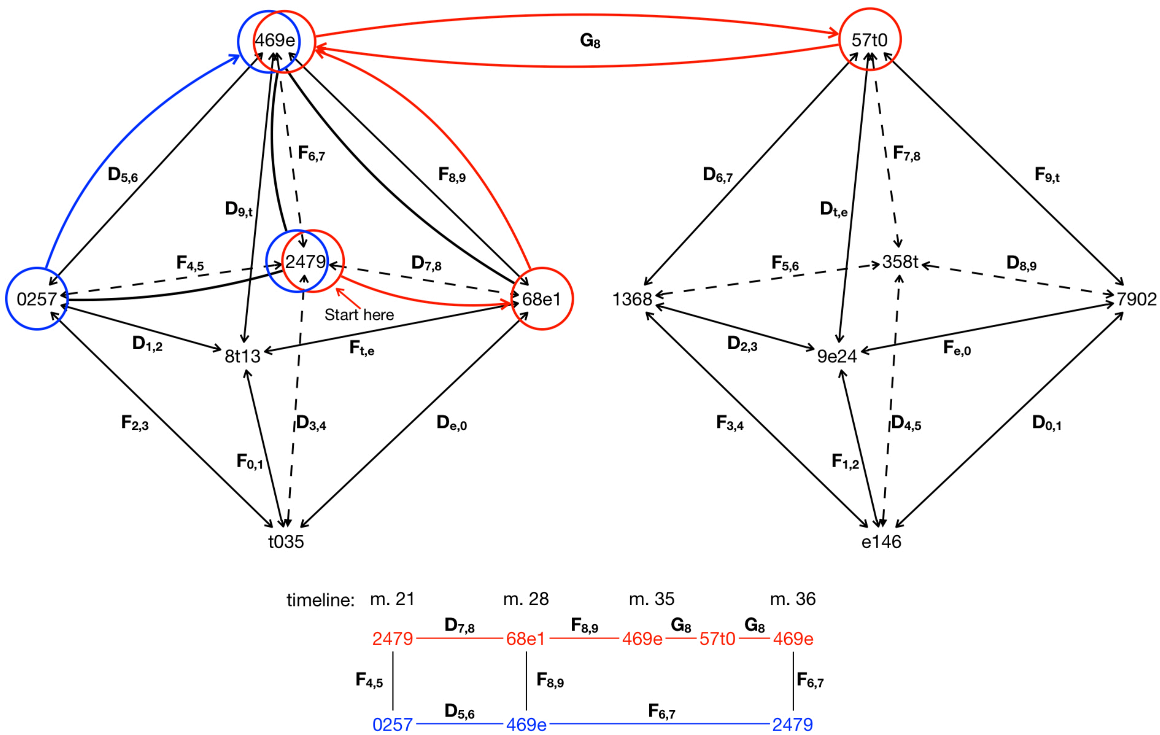


Figure 15: *connections between members of of sc. (0257) between bar 21 and 36 (see the animation movie at https://axesofcontextualinversion.com).*

F and belongs to the left octahedron-shaped graph except [5,7,t,0] that are connected to [4,6,9,e] by G_8, these two sets are in the same position in its graph (north pole). Note that because both graphs are related by T_1, **G** always maps two sets in the same position in each octahedron, since it is the contextual inversion that connects two members of sc. (0257) that are semitone apart.

In the beginning of the second part of section B of *Etude 10*, the texture has just one layer. From bar 48 a sequence of connections between five set class members (0257) begins. Next (m. 52), the texture is divided again into two simultaneous layers with the same two sets that started section B. In the last bar of section B, the texture has a single layer again; notice how the sequence of the latter sets make a palindrome. The path representing these connections over the graph is quite different from that shown in the previous figure, once most sets belong to the right octahedron, in addition, the connections from one graph to the other are mapped by **E** instead of **G**. Figure 16 shows on the bottom all these connections in timeline and on the top their representation on the octahedron-shaped graphs.

VI. Conclusion

Straus states that the separation of parsimonious voice-leading and contextual inversion concepts, which "sometimes has been assumed that they are inseparable (...), has led to research in two profitable directions" ([8, p.84]): (1) an interest in parsimonious voice-leading between members of different set classes; (2) an interest in common-tone contextual inversion between sets that are not parsimonious ([8, p.84]). This article proposed a third direction in this research, an interest in contextual inversion operations between sets that have no pitch-class in common or that are

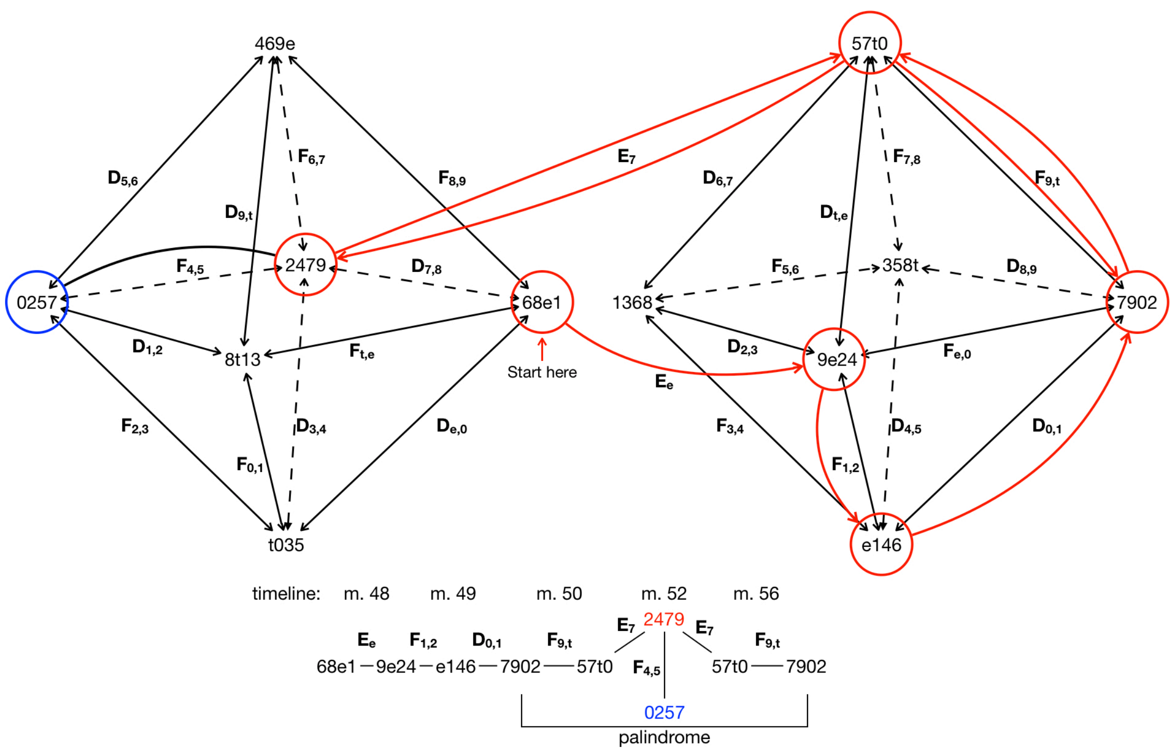


Figure 16: *connections between sets in second part of section B of Etude 10. (see the animation movie at https://axesofcontextualinversion.com).*

not mapped by any of the neo-Riemannian transformations. Actually, the labels for the axes of contextual inversion presented in section IV can also be used in passages where the connections are mapped by neo-Riemannian transformations, since **P**, **L**, **R** and their obverse versions can also be defined by an axis. But in addition, with these new concepts it is possible to go beyond and label any kind of connection by contextual inversion between sets of any cardinality. The analyses in section V have shown that the labels for the axes of contextual inversion are useful not only for designating a connection between two sets, as in Figures 7 and 13, but they can also be used to create new graphs and spaces, as in the Webern case Graphs (Figures 8 and 9), in cycles of members of sc. (0124) induced by <**DA**> chain (figure 12) and in two octahedron-shaped graph that incorporate all members of sc. (0257) (figures 14, 15 and 16). Of course, a wide variety of these graphs can be created with these axis and labels for using by both analysts, especially those focused on post-tonal works, and composers in the pre-compositional stages.

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A Systemic Model for Debussy's Prelude No.1

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Abstract: *This paper examines the analytical-compositional methodology called Systemic Modeling, proposed by Pitombeira as an intertextual tool that grasps deep parametric relationships within musical works. A discussion of the Theory of Compositional Systems and the Theory of Intertextuality, with examples, is provided in order to pave the way to the understanding of Systemic Modeling. Short examples of systemic modeling are given for clarification of its methodological phases. The entire Prélude No.1, by Claude Debussy, is modeled and a new piece is composed from this systemic model.*

Keywords*: Systemic Modeling. Compositional Systems. Parametric Generalization. Intertextuality. Claude Debussy.*

I. Introduction

A model, for the purpose of this article, is a representation of a musical structure on specific perspectives. Such perspectives, in terms of musical modeling, are directly related to the identification of parametric relationships amongst musical objects (pitches, chords, intervals, contours, etc.). We introduce in this article an analytical-compositional methodology called Systemic Modeling, shaped through the fusion of the Theory of Compositional Systems and the Theory of Intertextuality. Concomitantly, we propose the expansion of the concept of musical parameter, in order to observe a musical work on several levels of abstraction. As a case study, we propose the systemic modeling of Debussy's *Prélude No.1*, which will be modeled in terms of structure and voice-leading operations. Finally, a new work will be planned and composed using the systemic model of Debussy's piece as a starting point.

II. Systemic Modeling

This research started around 2010 as a means to understand the working principles and the coherence behind ten *Ponteios* by Camargo Guarnieri (1907-1993).[1] Through the application of different analytical tools (harmonic analysis, pitch-class set analysis, motivic analysis, reductive analysis, etc.) we have arrived at ten models presenting a series of general statements that explained the internal connections of those pieces with respect to specific parametric standpoints. These statements were organized into frameworks designated systemic models (or hypothetical compositional systems), as a reference to Systemic Modeling in engineering and computer sciences,

[1] Camargo Guarnieri was a Brazilian composer, pianist, and conductor, from São Paulo, who composed a vast catalogue of works inspired by Brazilian national elements. He wrote fifty *Ponteios* organized into five books. We studied the second book in 2010-11 and the first book in 2014-18.

for which "a model is a set of statements about some system under study" [21, p.27], or "an abstraction of a (real or language-based) system allowing predictions or inferences to be made" [13, p.370].

Systemic Modeling emerges as an epistemological convergence of the Theory of Compositional Systems and the Theory of Intertextuality. Therefore, before we discuss in detail the methodological aspects of Systemic Modeling, it is necessary to examine those two theories (i and iii). Additionally, we propose the expansion of the concept of musical parameter (ii). Such an expansion is fundamental since it will allow us to consider deeper representations of musical structures.

i. Compositional Systems

A compositional system is a set of relations amongst generic musical objects[2] in the scope of specific musical parameters. Compositional systems inherit their main formalization from Klir [9][3], for whom a system is a set of things and relations (Eq.1). Thus, a compositional system may be defined, for example, as a set of three things (τ_1, τ_2, τ_3) and two relations (ρ_1, ρ_2), as shown in (Eq.2), in which relation 1 (ρ_1) applied to τ_1 yields τ_2, and relation 2 (ρ_2) applied to τ_2 yields τ_3. If those things are three trichords (A, B, and C) and the relations are transpositions (T_1 and T_4), we may arrive in a very incipient compositional system such as the one showed in (Eq.3).

$$S = (\tau, \rho) \tag{1}$$

$$S_1 = ((\tau 1, \tau 2, \tau 3), (\rho 1, \rho 2)) | \rho 1(\tau 1) \rightarrow \tau 2 \wedge \rho 2(\tau 2) \rightarrow \tau 3 \tag{2}$$

$$S_1 = ((A, B, C), (T_1, T_4)) | T_1(A) \rightarrow B \wedge T_4(B) \rightarrow C \tag{3}$$

The process of assigning values to the generic objects of a system is called compositional planning (or design)[4] , which also includes: [1] the proposal of temporal articulations for these objects, and [2] the inclusion of additional parametric values (rhythm, dynamics, etc.) not considered in the system's declaration. Hence, let us assign the following value for trichord A, indicated in normal form[5] : $A = (125)$. We only need the value of A, since the other values will be determined by the relations (T_1 and T_4), i.e., $B = T1(125) = (236)$, and $C = T4(236) = (67A)$. Figure 1 shows a possible temporal articulation for the trichords, which will be distributed into three instruments (flute, oboe, and bassoon) as shown in the musical realization (Figure 2). One should notice that rhythm, dynamics, articulation, register, and timbre were freely applied to create the score.

A compositional system can be presented as [1] a set of declarations (the previous example), [2] a computational algorithm, or [3] a set of tables and diagrams. We may also think in a typology[6] consisting of three types of compositional systems: open, semi-open, and feedback (Figure 3). An open system works like a function that modifies inputs. A semi-open system has only output, which is generated by some kind of internal rule, deterministic or stochastic (the previous example is semi-open deterministic). A feedback system is found in chaotic implementations. We will provide below examples of open and feedback systems.

[2]The term object is inspired by Castrén [4]).

[3]This Theory of Compositional Systems, especially as it is defined in the master dissertation of Flávio Lima (2011)[14], written under my supervision, is also in debt to the General Systems Theory [1].

[4]For a comprehensive study of compositional design with pitch-classes see Morris [17].

[5]In this paper, normal forms are indicated inside parenthesis and prime forms are indicated inside brackets. Also, pitch classes 10 and 11 are represented by letters A and B, respectively.

[6]Our typology is loosely inspired by Durand [5].

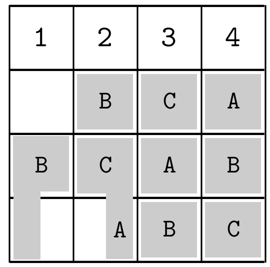


Figure 1: *A possible temporal articulation for the trichords of S_1.*

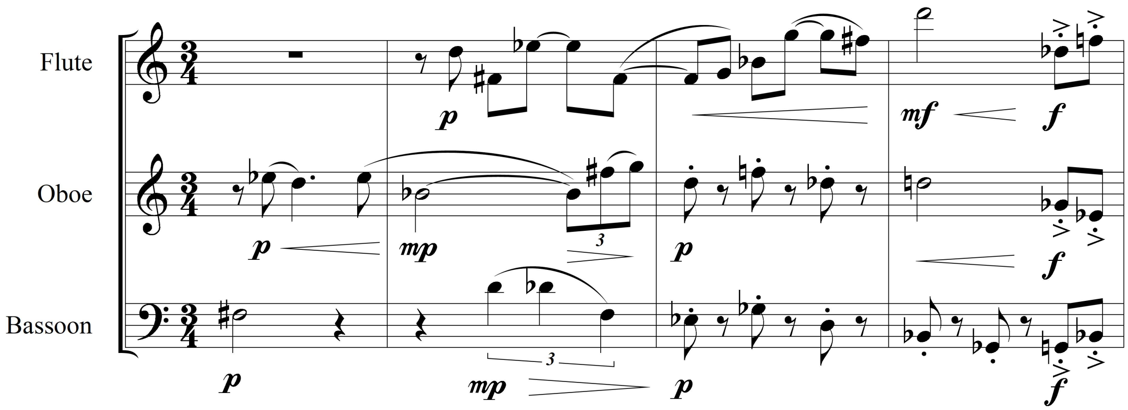


Figure 2: *A possible musical realization of the design shown in Figure 1. Trichords A = (125), B = (236), and C = (67A).*

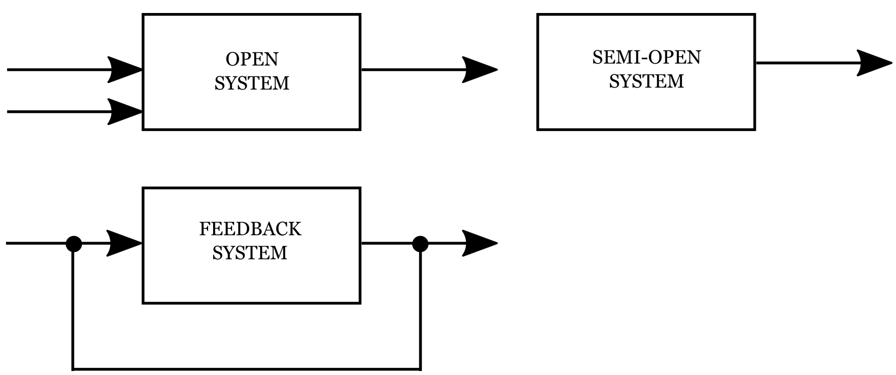


Figure 3: *Types of compositional systems.*

In order to exemplify an open compositional system let us consider the diagram of Figure 4, in which the open system S_2 receives the fragment A as input. This fragment is separated into two parametric components: pitch (B) and rhythm (C). Inputs k and w are functions defined by a composer during the compositional planning. Let us choose $k \to p = 2.p_0 - 60$, which means that the original pitch (p_0) multiplied by 2 and subtracted from 60 yields the new pitch p. Pitches, here, are measured in MIDI values, in which middle C equals 60. Therefore, the MIDI values in B (60 64 62 69 67), with the application of the rule for pitch (k), are transformed into the MIDI values in D (60 68 64 78 74). We may say that C maps onto D through operation k. Likewise, let us choose $w \to r = 2 + r_0$, i.e., each rhythmic value is added by two units. Each unit in this example equals a sixteenth note. Thus, the rhythmic values (or, more specifically, the durations) in C (6 2 7 1 8) map onto E (8 4 9 3 10) through operation w. The integration of D and E produce a new music fragment (F). Systems S_2, represented as a diagram in Figure 4, can be easily translated into a computational algorithm, as it is shown in Figure 5, using the application Octave with the MidiToolBox package [23].

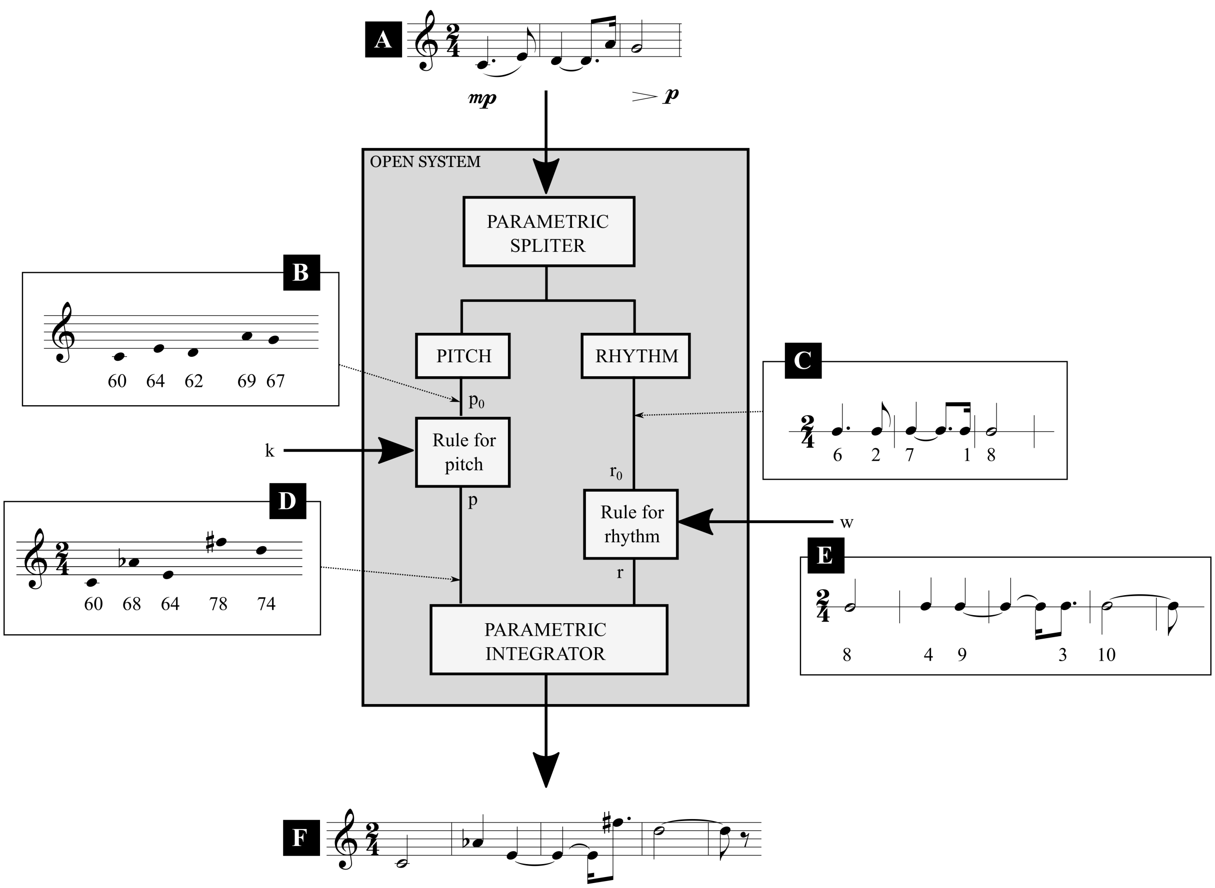


Figure 4: *An example of open system S2, in which A is the original line, B is pitch component of A, C is the rhythmic component of A, D and E are the result of operations on B and C, and F is the integration of D and E.*

```
midi2nm;
pitch = nm(:,4)'; % select only pitches and convert to row format
duration = nm(:, 2)'; % select only durations and convert to row format

newpitch = pitch*2-60; % modify pitches
newdur = duration + 0.5; % modify durations

lines = length (nm) % number of lines

Onset_ = [0]; % iniatilize the new onsets based on the new durations

for z=1:lines-1
    newvalue = Onset_(z) + newdur(z);
    Onset_ = [Onset_ newvalue];
end

Track = repmat (1,1,lines); %make a matrix for tracks with only 1s
Vel = repmat (60,1,lines); %make a matrix for velocities with only 60s

% Send the results to a new midifile

Matrizmidi_1 = Onset_';
Matrizmidi_2 = newdur';
Matrizmidi_3 = Track';
Matrizmidi_4 = newpitch';
Matrizmidi_5 = Vel';
Matrizmidi_6 = Matrizmidi_1*.5;
Matrizmidi_7 = Matrizmidi_2*.5;

Matrizmidi = [Matrizmidi_1 Matrizmidi_2 Matrizmidi_3 Matrizmidi_4 Matrizmidi_5
              Matrizmidi_6 Matrizmidi_7];

nm2midi(Matrizmidi, 'newpiece.mid');
```

Figure 5: *A simple computational rendition of S_2 using Octave and MidiToolBox.*

As an example of feedback systems we can propose the one-dimensional chaotic system called logistic map, formally defined as $x_{n+1} = k.x_n.(1 - x_n)$.[7] If k assumes a value around 4, the system becomes chaotic producing values apparently aleatoric. Figure 6 shows a very simple implementation of system S_3 as a computational algorithm written in Octave. This algorithm assigns the results of the logistic map to pitches and durations following rules that constrain the pitch space into two octaves and quantize the durations as multiples of sixteenth notes. The first six measures of a musical fragment generated with system S_3, for k equals 4 and the initial value of x equals 0.01, is shown in Figure 7. We have added articulations, tempo, and dynamics.

The systems aforementioned were created from scratch. Thus, they are original systems. As we will see later, a compositional system can also be derived from another piece, another work of art, or even from anything else that can be understood as a set of objects and relations. Before we proceed to the Theory of Intertextuality—the other theory that gave rise to Systemic Modeling—we will propose an expansion of the concept of musical parameter.

[7] A comprehensive study on chaotic systems of various dimensions applied to music can be found in Bidlack ([2]). Additional applications of one- and two-dimension chaotic systems can also be found in Pitombeira and Barbosa [18] and [19].

```
% pitch and duration matrices as well as k and value are initialized

k=4;
value=0.01;
pitches = [];
durations = [];

%calculations for durations and pitches

for n=1:100
  value=value*k*(1-value);
  pitch = mod(round(value*100),24)+60; %pitches are restricted to two octaves
  duration = ((mod(pitch,4)+1)*0.25); % durations are quantized to 16th notes
  pitches = [pitches pitch];
  durations = [durations duration];
end

music = [pitches' durations']


lines = length (music) % number of lines

Onset_ = [0]; % iniatilize the new onsets based on the new durations

for z=1:lines-1
    newvalue = Onset_(z) + durations(z);
    Onset_ = [Onset_ newvalue];
end

Track = repmat (1,1,lines); %make a matrix for tracks with only 1s
Vel = repmat (60,1,lines); %make a matrix for velocities with only 60s

% Send the results to a new midifile

Matrizmidi_1 = Onset_';
Matrizmidi_2 = durations';
Matrizmidi_3 = Track';
Matrizmidi_4 = pitches';
Matrizmidi_5 = Vel';
Matrizmidi_6 = Matrizmidi_1*.5;
Matrizmidi_7 = Matrizmidi_2*.5;

Matrizmidi = [Matrizmidi_1 Matrizmidi_2 Matrizmidi_3 Matrizmidi_4 Matrizmidi_5
              Matrizmidi_6 Matrizmidi_7];

nm2midi(Matrizmidi, 'newpiecechaos.mid');
```

Figure 6: *A simple implementation of S_3 in Octave .*

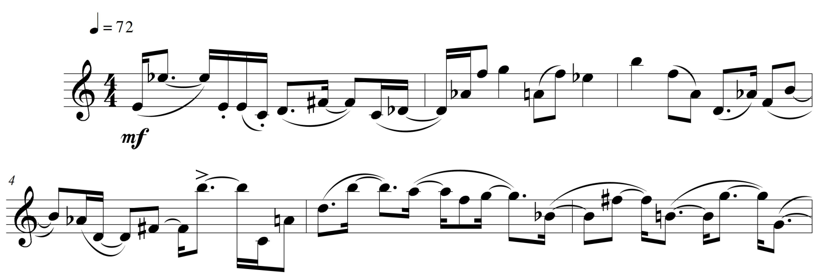


Figure 7: *A fragment generated with S_3.*

ii. The expanded concept of parameter

Musical parameters are formal descriptions of particular layers of a musical structure. Traditionally, parameters have acoustic counterparts. For example, the pitch parameter is isomorphically related to the frequency of a sound[8] , the dynamic parameter is related to the intensity of a sound, and so on. It is possible to organize parameters in terms of three internal categories (Table 1): dimension, type, and value. Thus, the pitch parameter can be observed in a musical score as an absolute entity (a note), horizontally (melodic line) and vertically (chords). A melodic line is built upon the concatenation of intervals, which can be measured considering four types: ordered pitch intervals, unordered pitch intervals, ordered pitch-class intervals, and unordered pitch-class intervals, i.e., interval classes. Moreover, melodic segments can be labeled in terms of normal and prime forms. Chords can be labeled as harmonic functions as well as normal and prime forms.

Table 1: *Categories of parameters*

Parameter	Dimension	Type	Value
Pitch	Neutral	Pitch	64 (MIDI)
		Pitch-class	4
	Horizontal	Ordered Pitch interval	19
		Unordered Pitch interval	19
		Ordered Pitch-class interval	7
		Interval class	5
	Vertical and Horizontal	Normal form	(1379)
		Set class (prime form)	[0268]
	Vertical	Harmonic function	Ger6+
Rhythm	Durational	Duration	♪
	Positional	Attack-point	3rd ♪, 3rd bar
Dynamics			*mf*

Furthermore, we can also classify parameters in terms of their level of salience as surface and abstract. Surface parameters are pitch (in absolute values, e.g., E4), rhythm (in durational and positional dimensions), dynamics, tempo, and articulation. Timbre represented as an encapsulated object can also be a surface parameter (for example: flute, oboe, sinewave, etc.). On the other hand, abstract parameters are related to surface parameters through some kind of non-bijective function, which means that there is no isomorphic relation between them. In other words, one can move from a surface parameter to an abstract parameter but not the other way. Consequently, there is loss of information in the mapping operation. Prime forms are good examples of abstract parameters. Let us take for example set class 0127 (shown in Figure 8). There is a function that maps each member of this class to its prime form. However, the prime form has no information about the original set (5670), which means that knowing its prime form means only to know its interval classes but not its actual pitch-classes. The same paradigm applies to contour, inversional axis, rhythmic partitions, degree of harmony endogeny, etc. When we say that a pitch segment is symmetrical around axis 0-6 we have no information about the actual pitches. As we will see later in this paper, this non-isomorphic property between surface and abstract parameters, including

[8]There is a bijective correspondence between pitches and their frequencies: $A_4 = 440Hz$, $B\flat4$ = 466.16Hz, and so on (for an equal tempered scale, with ratio $\sqrt[12]{2}$).

the loss of information, is a key concept for our modeling methodology.

Pitch-class set	{0 7 6 5}	Surface
Normal form	(5670)	
Prime form	[0127]	Abstract
Pitch contour	<0321>	
Durational contour	<2013>	
Inversional axis	0-6	

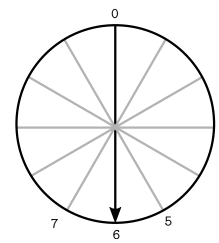


Figure 8: *Example of abstract parameters.*

iii. Intertextuality

As we have mentioned previously, the Theory of Intertextuality is also fundamental to the understanding of Systemic Modeling. The compositional use of intertextuality is not a 20th century innovation. The literature is replete with examples. The Theory of Intertextuality originated in the field of Literature. Julia Kristeva created the concept in a theoretical environment concomitantly connected with Bahktin's Dialogism and Saussurre's Semiotics. She said: "any text is constructed as a mosaic of quotations; any text is the absorption of another" [12]. Intertextuality in music has received profound investigations from Korsyn[9] [10], Straus [22], and Klein [8]. More recently, a master dissertation developed under my supervision in the Graduate Program in Music at Federal University of Rio de Janeiro as well as in the MusMat Research Group, proposed a new taxonomy for intertextual procedures in music [16]. In Mesquita's work, Systemic Modeling is a type of intertextuality that has an implicit presence and a subverted intentionality. In other words, it is an abstract intertextuality that focus on deep musical relationships, as we will examine in the next section.

iv. Systemic Modeling

Systemic Modeling aims at revealing a hypothetical compositional system (i.e., a systemic model) of a musical work. The modeled system is hypothetical since it does not necessarily express the composer's original intention. A model is achieved by focusing exclusively on relationships amongst musical parameters (surface and abstract), disregarding their particular values. Therefore, we will have a set of relationships and a set of generic objects. The borrowing of relationships (to be used in a new work) and the concomitant mention of particular parametric values from an original work is, therefore, an epistemological inheritance of both Theory of Intertextuality and Theory of Compositional Systems.

This methodology applied to music consists of three phases. The first phase is called parametric selection, which is basically achieved through a prospective analysis of a piece in order to determine the parameters that will be examined as well as the best analytical techniques to accomplish the task. The second phase is the analysis itself. It produces a structure called compositional profile, which consists of a set of objects and relationships. A systemic model is reached in the third

[9]Korsyn translates Bloom's revisionary ratios [3] to the musical domain; Straus proposes a series of eight intertextual tools for composition. Both procedures received practical applications in Flávio Lima's Master Dissertation [14].

phase—called parametric generalization—when the objects are removed from the scene and only the relationships amongst them remain.

The diagram in Figure 9 illustrates the methodological cycle of systemic modeling. In this example (a fragment from J. S. Bach's *Minuet in G major*), one starts by selecting the parameter to be examined. The pitch parameter was selected, exclusively in its horizontal dimension. Therefore, the excerpt is rewritten considering only the chosen parameter, i.e., only the pitches are examined. Then, an analysis takes place producing a compositional profile. For this analysis we considered the following criteria[10]:

1. There are three types of melodic movements or operations: [i] Conjunct (C), [ii] Leap (L), and [iii] Repetition (R). Each operation is labeled with a letter and number that indicates the number of occurrences of such operation. Operations can be combined generating an operation complex, for example L^1C^4.

2. A dashed line means that a pitch was just prolonged.

3. A full line with arrow indicates that a pitch moves onto another by means of some operation.

4. The circle means an overlap and it is used when a pitch is both the end and the start of an operational group.

Finally, the particular values are disregard and only the relationships remain. This is a possible systemic model or a hypothetical compositional system for Bach's fragment. Let us call it S_4.

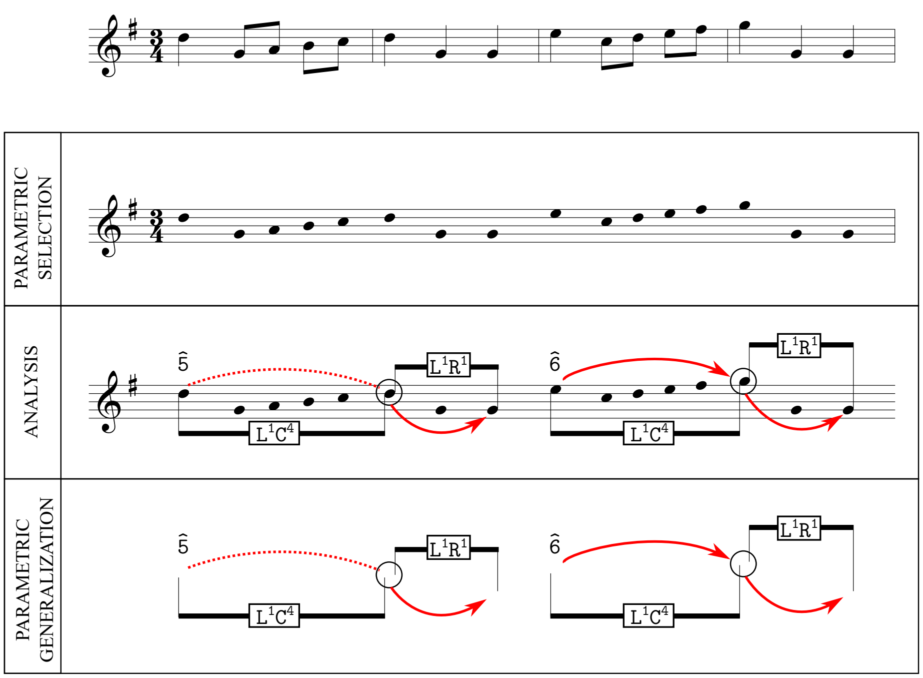


Figure 9: *The methodological cycle of systemic modeling. In the upper part we have the excerpt (J. S. Bach's Minuet in G major) that will be modeled with focus on the pitch parameter in its horizontal dimension. The systemic model appears in the parametric generalization line.*

[10]This analytical method is inspired by Kraft [11, pp.15–19]. A more interesting methodology for melodic analysis is found in Gentil-Nunes [7]. I have composed a piece for saxophone and piano (entitled *Linhas*) using this methodology.

From this model (system S_4) one can plan and compose a new fragment, which employs the same relationships of S_4 but different surface objects. We call this process Compositional Planing, which also has three phases. The first phase is called Particularization and consists of applying objects that satisfy the model relationships. Since our only starting point to compose the new fragment is the model, we have to define in advance the scale, i.e., the pitch space in which the piece will be designed. Messiaen's third Mode of Limited Transposition ($C, D, E\flat, E, G\flat, G, A\flat, B\flat, B$) will be the chosen scale. The fifth and sixth scale degrees of this mode in this transposition are respectively $G\flat$ and G. We can observe in Figure 10 that, once the systemic model does not specify direction of leap (L), the contour of the new fragment differs from Bach's contour.

The second phase of Compositional Planing is called Application. It is only necessary when the objects are abstract, i.e., when they are not surface parameters (an example will be given below).

The third phase is called Complementation and it consists of adding the parameters not declared in the system. The metric will be 5/8, the rhythmic structure and the articulations will be freely added. For the tempo we will choose quarter note equals 72, and the instrumentation will be solo clarinet.

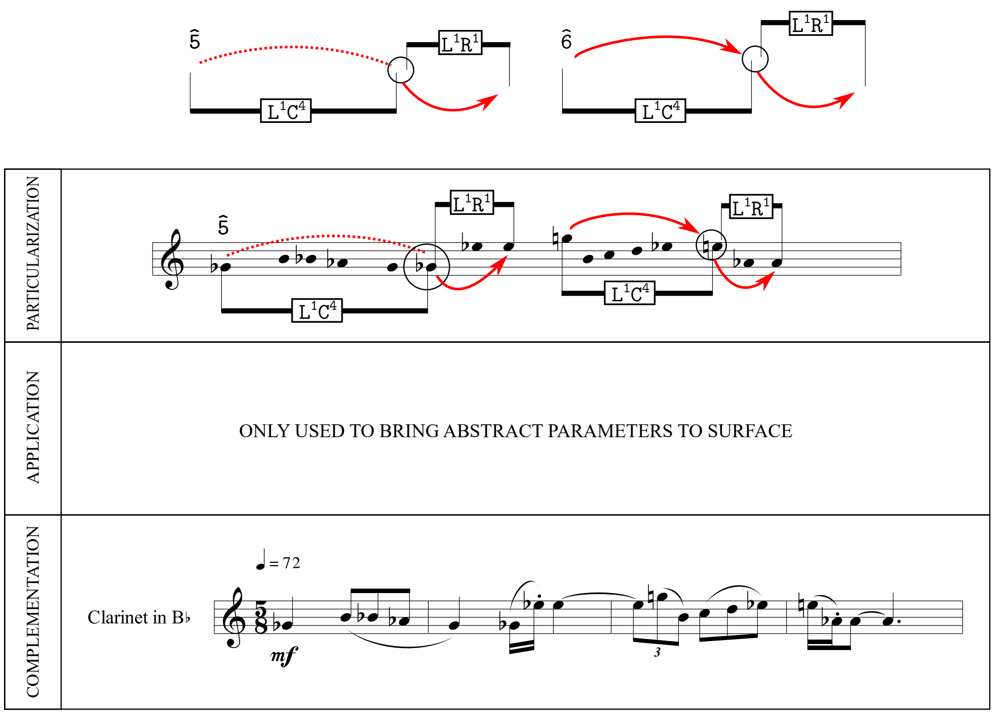
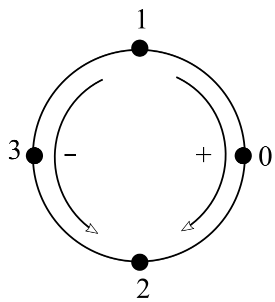


Figure 10: *Compositional planning of a new fragment from system S_4.*

The next example is from Webern's *Cello Sonata* (see Figure 13). Let us consider only the melodic contours of the first three measures of the cello part. Therefore, in the parametric selection phase, we disregard pitch, rhythm, dynamics, articulation, and timbre in order to focus only in the melodic contour. In the analytical phase, we find out that there are three segments of contours: <1023>, <2310>, and <0312>. We can search for relationships between the first contour and the others. The second contour is the inversion of the first contour.[11] To determine the relationship between first and last contours we have to define two operations: rotation (r) and subrotation (sr).

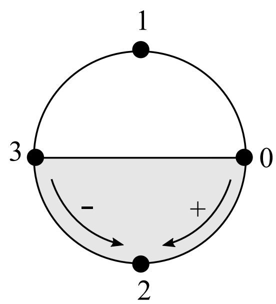
[11]The inversion of a contour C is calculated by the formula: $I(C_i) = (n-1) - C_i$, in which n is the cardinality of the contour and i varies from 1 to n. So, for the contour <1023>, with cardinality 4, C_1, i.e., the first contour point, is

If we locate contour points in a circle, the rotational operation, clockwise (r+) or counterclockwise (r-), is accomplished by starting a contour in different points.[12] For example, the contour <1023> has eight rotations, shown in Figure 11. Subrotation (sr+ and sr-) consists of freezing the first contour point and rotate the remaining points. Figure 12 shows the six subrotations of contour <1023>. Therefore, the relationship between C1 and C3 is a compound operation: rotation followed by subrotation. Figure 13 shows the systemic modeling of the first three measures of the cello part of Webern's *Cello Sonata*. The result is system S_5.



		r^+	r^-		
X	r_0^+	1 0 2 3	1 3 2 0	r_0^-	Y
R(Y)	r_1^+	0 2 3 1	3 2 0 1	r_1^-	R(X)
I(X)	r_2^+	2 3 1 0	2 0 1 3	r_2^-	I(Y)
RI(Y)	r_3^+	3 1 0 2	0 1 3 2	r_3^-	RI(X)

Figure 11: *Rotations of contour <1023>.*



sr_0^+ 1 0 2 3 sr_0^- 1 0 3 2
sr_1^+ 1 ...... 2 3 0 sr_1^- 1 ...... 3 2 0
sr_2^+ 1 .... 3 0 2 sr_2^- 1 .... 2 0 3

Figure 12: *Subrotations of contour <1023>.*

From the systemic model shown in Figure 13, let us initiate a compositional plan. In the first phase—particularization—we reinsert objects to satisfy the model relationships. In fact, we just have to decide the first contour segment, since the others will be produced through model operations. We have chosen <0123> for the first contour segment (C1). Therefore, C2 equals <3210> and C3 equals <1302>. Those results have to be applied to surface parameters. Let us apply to the pitch parameter. This is shown in the results for the second phase—application (see Figure 14).

In the last phase—complementation—we introduce parameters not present in the systemic model. So, we have chosen tempo (72), metric (6/8), timbre (clarinet), and rhythm, articulations and dynamics shown in Figure 14.

1. Therefore, applying the formula, we will have $I(1) = (4-1) - 1 = 2$. For the second contour point we will have $I(0) = (4-1) - 0 = 3$, and so on. For a more comprehensive study on contour operations see [15], [17], and [20].

[12] Contour rotations are defined in a different fashion in [6]. One can observe that rotations produce also the canonic operations (original, inversion, retrograde, and inversion of the retrograde).

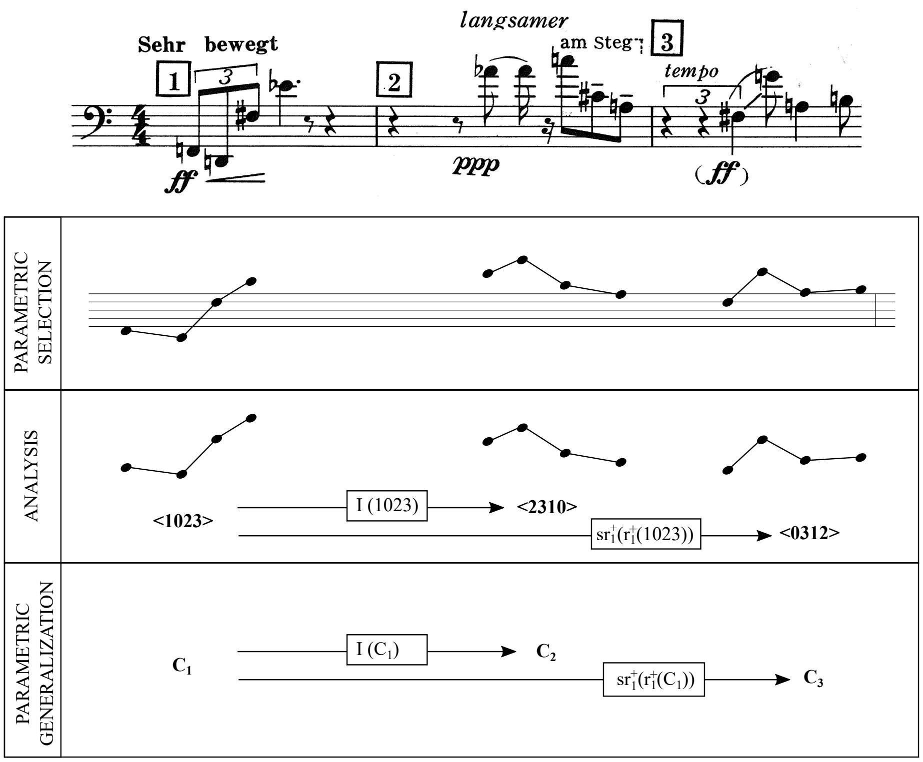


Figure 13: *S_5: systemic modeling of the first gestures of Webern's Cello Sonata.*

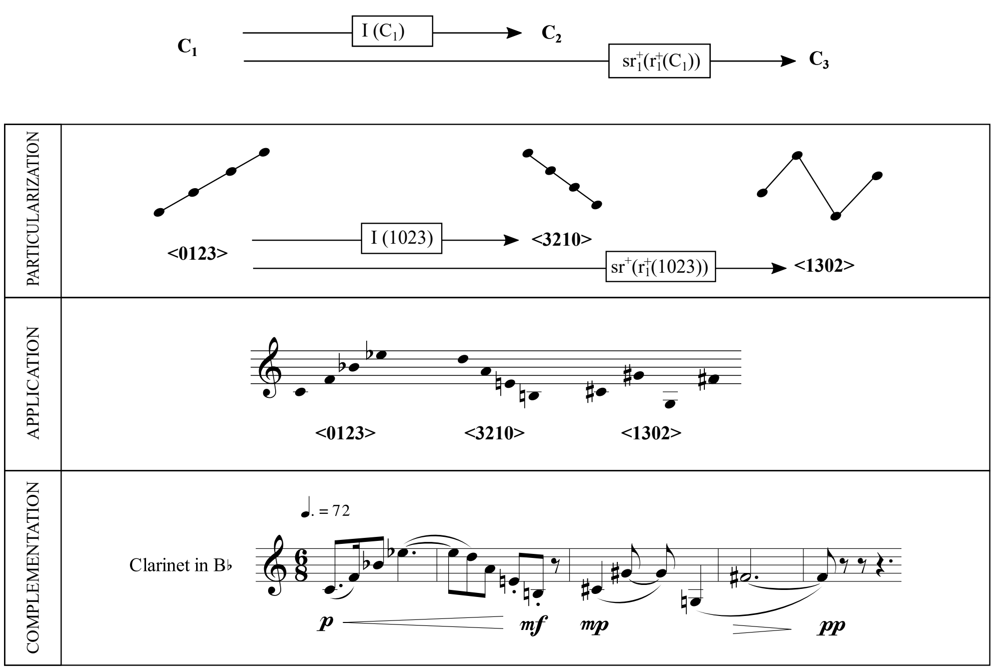


Figure 14: *Compositional planning of a new fragment from system S_5.*

III. Systemic Modeling of Debussy's *Prélude No.1*

Debussy's Prélude No.1, from the first book of Préludes, has loosely a form AA'BA"+ coda. Figure 15 shows section A (its first five measures). Through this figure we demonstrate our analytical strategy, which consists of focusing on the form as well as on the pitch parameter, in both its vertical and horizontal dimensions. Therefore, the passage, was divided into four melodic segments (a1, a1, a2, and a3) and 21 chords (labeled inside gray circles). The segment a3 is particularly idiomatic in terms of Debussy's aesthetical language since it is built with a voice-leading technique called *planning*. The melodic lines of segments a1 and a2 are built from a combination of chord notes (in the downbeats) with ornamental notes (passing and neighbor notes and an interpolation at the end of segment a2). The 21 chords of section A were modeled in terms of voice-leading operations and the relations amongst them, shown in Table 3, were translated into an algorithm in Python (show in Figure 16). Thus, the first chord (labeled X1), with cardinality 3, is transformed into the second one (X2), with cardinality 4, through the application of two operations: pitch-class A is split into 9 and B, and pitch-classes 2 and 5 move to pitch-classes 3 and 7 through parsimonious movement (+1 and +2, respectively). The same procedure goes for the remaining chords. The algorithm allows us, by changing only the first trichord, to build a new set of chords having the same relationships of the original piece. Such a possibility is of fundamental importance during the compositional planning of a new work. Through the application of the same analytical paradigm for the remaining sections we arrive at the following conclusions:

1. Section A' (mm.6-10) differs from section A only in the chords onset and register: chords are shifted by an eight note forward and filled up with repeated pitch-classes in different registers in a planning-like style; segment a3 is basically the same (except for a metric change to 4/4);

2. Section B (mm.11–24) is formed by five segments:

   (a) b1 and b1' (mm.11–14) are basically formed by the juxtaposition of a pedal note (same lowest note of the last chord of segment a3), an ascending planning, and a descending melodic line built exclusively on $B\flat$ pentatonic (b1) and $E\flat$ pentatonic (b1').

   (b) b2, b3, and b4 (right hand of the piano) are built on the chromatic scale, except for one pitch-class ($B\flat$). Segments b2 and b3 have a definite pitch centricity: the first pitch-class of b1 for the segment b2, and the first pitch-class of b1' for b3. The latter is formed by two statements of a motive in two measures, ending with the centric note. Chords in b3 are treated as a separate layer. Therefore, the melodic motive does not collaborate to chord formation in this section. This decision has the purpose of analytical simplification, since the chords are clearly triadic material, which would be interpreted otherwise if we had considered the notes of the melodic line. The last nine pitches of b4 are formed by arpeggios of chord X_1: $T_5I(X_1)$, $T_7I(X_1)$, and $T_6I(X_1)$.

3. Section A" is formed by two segments a1', that differs from the original (a1) by an eight-note shift of the first pitch and by the use of *planning*.

4. Coda is built with four chords (the two last ones are the very first chord) against a bass alternation of two pitch-classes: the lowest note of the first chord and the lowest note of the last chord of section A.

The generalization of these analytical conclusions yields the systemic modeling for Debussy's Prélude, shown in Table 2.

Table 2: *One possible systemic modeling for Debussy's Prélude No.1*

Section	Measures	Segment	Comments
A	1	a1	Chord notes in the downbeat + nonchord tones
	2	a1	
	3–4.2.1	a2	Same as a1
	4.2.2–5	a3	Planning
A′	6	a1′	a1 shifted by an eight note forward and filled up with repeated pitch-classes in different registers
	7	a1′	
	8–9.2.1	a2′	Same as a1′
	9.2.2–10	a3	
B	11–12	b1	Juxtaposition of a pedal note (same lowest note of the last chord of segment a3), a planning, and a melodic line built exclusively on pentatonic scales.
	13–14	b1′	
	15–17	b2	b2, b3, and b4 are built on the chromatic scale, except for one pitch-class. Segments b2 and b3 have a definite pitch centricity: the first pitch-class of b1 for the segment b2, and the first pitch-class of b1′ for b3. The latter is formed by two statements of a motive in two measures, ending with the centric note. Chords in b3 are treated as a separate layer. Therefore, the melodic motive does not collaborate to chord formation in this section. The last nine pitches of b4 are formed by arpeggios of chord X_1: $T_5I(X_1), T_7I(X_1)$, and $T_6I(X_1)$.
	18–20	b3	
	21–24	b4	
A″	25	a1″	Shift of the first pitch and use of planning
	26	a1″	
Coda	27–31		Built with four chords (the two last ones are the very first chord) against a bass alternation of two pitch-classes: the lowest note of the first chord and the lowest note of the last chord of section A.

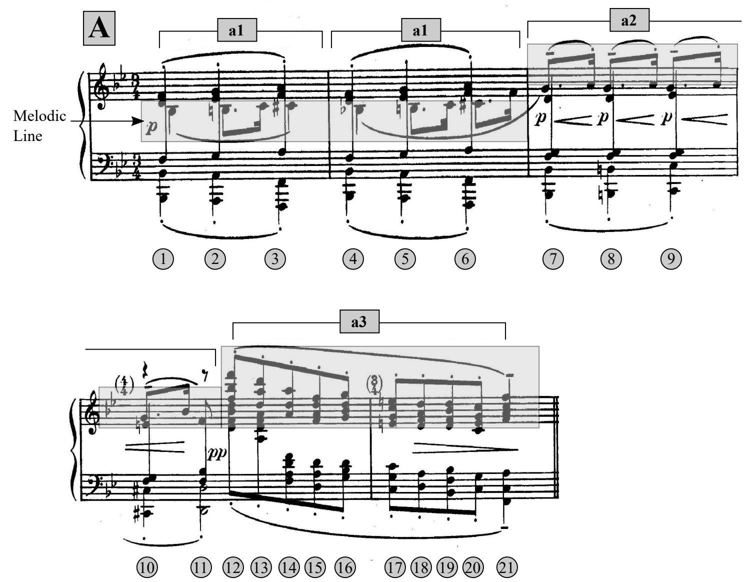


Figure 15: *Debussy's Prélude No.1, section A (mm.1–5) showing chords and melodic segments.*

Table 3: *Chords of section A of Debussy's Prélude No.1, and their relations.*

Measure	Beat	Chord	pcs			
1	1	**X1**	**A**		**2**	**5**
			-1	+1	+1	+2
	2	**X2**	**9**	**B**	**3**	**7**
			(-4)	+2	+2	+2
	3	**X3**	**5**	**1**	**5**	**9**
			—	—	—	—
2	1	**X4**	**A**		**2**	**5**
			-1	+1	+1	+2
	2	**X5**	**9**	**B**	**3**	**7**
			(-4)	+2	+2	+2
	3	**X6**	**5**	**1**	**5**	**9**
			+5	+1	+0	-2
3	1	**X7**	**A**	**2**	**5**	**7**
			+1	+0	+0	+0
	2	**X8**	**B**	**2**	**5**	**7**
			+1	+1	+0	+0
	3	**X9**	**0**	**3**	**5**	**7**
			+1	+1	+0	+0
4	1	**X10**	**1**	**4**	**5**	**7**
			+1	+1	+0	+3
	2	**X11**	**2**	**5**	**5**	**A**
			+0	+0	—	+0
		X12	**2**	**5**		**A**
			+0	+0		-1
	3	**X13**	**2**	**5**		**9**
			+0	+0		+0
		X14	**2**	**5**		**9**
			+0	+0		+0
	4	**X15**	**2**	**5**		**9**
			+0	+2		+1
		X16	**2**	**7**		**A**
			+2	+0		+2
5	1	**X17**	**4**	**7**		**0**
			+1	+2		+2
		X18	**5**	**9**		**2**
			+0	+1		+0
	2	**X19**	**5**	**A**		**2**
			-1	-3		-2
		X20	**4**	**7**		**0**
			+1	+2		+0
	3	**X21**	**5**	**9**		**0**

```
"""
Created on Tue Oct  9 21:18:17 2018
Debussy Prelude 1 -- model for chords
@author: Liduino Pitombeira
"""

Entrada = input('Enter with a trichord in the format xyz using a for 10 and b for 11:')
print('')

Gen=[]

# Convert Entrada to a list of integers base 12
for i in range(3):
    x = int(Entrada[i],12)
    Gen.append(x)

#Calculates the chords contents

#SECTIONS A A'
X1 = Gen
X2 = [(Gen[0]-1)%12,(Gen[0]+1)%12,(Gen[1]+1)%12, (Gen[2]+2)%12]
X3 = [(X2[2]+2)%12, (X2[1]+2)%12, (X2[2]+2)%12, (X2[3]+2)%12]
X4 = Gen
X5 = X2
X6 = X3
X7 = [(X6[0]+5)%12, (X6[1]+1)%12, (X6[2])%12, (X6[3]-2)%12]
X8 = [(X7[0]+1)%12, (X7[1])%12, (X7[2])%12, (X7[3])%12]
X9 = [(X8[0]+1)%12, (X8[1]+1)%12, (X8[2])%12, (X8[3])%12]
X10 = [(X9[0]+1)%12, (X9[1]+1)%12, (X9[2])%12, (X9[3])%12]
X11 = [(X10[0]+1)%12, (X10[1]+1)%12, (X10[2])%12, (X10[3]+3)%12]
X12 = [(X11[0])%12, (X11[1])%12, (X11[3])%12]
X13 = [(X12[0])%12, (X12[1])%12, (X12[2]-1)%12]
X14 = X13
X15 = X14
X16 = [(X15[0])%12, (X15[1]+2)%12, (X15[2]+1)%12]
X17 = [(X16[0]+2)%12, (X16[1])%12, (X16[2]+2)%12]
X18 = [(X17[0]+1)%12, (X17[1]+2)%12, (X17[2]+2)%12]
X19 = [(X18[0])%12, (X18[1]+1)%12, (X18[2])%12]
X20 = [(X19[0]-1)%12, (X19[1]-3)%12, (X19[2]-2)%12]
X21 = [(X20[0]+1)%12, (X20[1]+2)%12, (X20[2])%12]
```

Figure 16: *Algorithm for the chords of section A of Debussy's Prélude No.1.*

The systemic model for Debussy's *Prélude No.1* consists of a generalization of the analytical conclusions (show in Table 2). The chords' grammar is encapsulated into a Python algorithm (part of it is shown in Figure 16). If we generalize the pentatonic scales of subsection b1, Table 2 is already in the format of a generalized compositional system. From this system, we will describe the compositional planning of a new piece, in the next section of this article.

IV. Compositional Planning of *Le sphinx des Naxiens*

Based on the algorithm for Debussy's *Prélude No.1* (Figure 16), using a first chord (037) different from the original (A25), we have determined a new set of chords (Table 4) that will be the basis for a new piece: the first movement of *Trois Monuments*, for piano. This movement is entitled *Les sphinx des Naxiens*. The title is a reference to monuments of the Delphi Sanctuary, similarly how it was applied in Debussy's *Prélude No.*1, which is entitled *Danseuses de Delphes*, a fragment of sculpture from Apolo's temple in Delphi. The other two movements of *Trois Monuments* were also composed with the use of intertextuality. *Le pillier des Rhodiens* employed a type of stylistic intertextuality by the use a modal melody in a planning set, pedal points, and ostinatos. *Le trésor des Athéniens* is built by the juxtaposition of one-measure quotations (and developments) from all the twelve Préludes of the first Book.

The new piece has the same number of measures of Debussy's piece (31) but different time signatures (2/4 + 3/8 and two occurrences of 3/4 + 3/8). The first task was to distribute the chords of Table 4 through those 31 measures. After that we built the melodic segments following the rules declared in Table 2, with respect to the use of nonchord tones, *planning*, scalar material, centricity, and initial pitch-class. In Figure 17 one can see that the segment a1 uses chords X01, X02, and X03. The upper pitches form the melodic line for this segment (in green) to which is added the nonchord tone E (in red). Segment a3 (with a square frame) is built with the use of *planning*. In Figure 17, important observations taken from Debussy's systemic model (Table 2) are highlighted, in order to make it easy to compare the musical realization with the system itself. Information regarding other parameters not declared in the systemic model was freely determined during the compositional process.

Table 4: *Chords of Pitombeira's Le sphinx des Naxiens, generated by the Python algorithm of Debussy's Prélude No.1.*

Sections A and A'	Section B	Section A'' + Coda
X01= [0, 3, 7]	X22= [5, 9, 2]	X60= [0, 3, 7]
X02= [11, 1, 4, 9]	X23= [6, 11, 4]	X61= [1, 4, 9]
X03= [6, 3, 6, 11]	X24= [8, 0, 5]	X62= [2, 5, 10]
X04= [0, 3, 7]	X25= [10, 2, 7]	X63= [3, 6, 11]
X05= [11, 1, 4, 9]	X26= [11, 4, 9]	X64= [0, 3, 7]
X06= [6, 3, 6, 11]	X27= [1, 6, 11]	X65= [1, 4, 9]
X07= [11, 4, 6, 9]	X28= [3, 7, 0]	X66= [2, 5, 10]
X08= [0, 4, 6, 9]	X29= [4, 9, 2]	X67= [3, 6, 11]
X09= [1, 5, 6, 9]	X30= [4, 10, 2]	X68= [3, 6, 8, 11]
X10= [2, 6, 6, 9]	X31= [6, 0, 4]	X69= [3, 6, 8, 11]
X11= [3, 7, 6, 0]	X32= [8, 2, 5]	X70= [0, 3, 7]
X12= [3, 7, 0]	X33= [9, 3, 7]	X71= [0, 3, 7]
X13= [3, 7, 11]	X34= [11, 5, 9]	
X14= [3, 7, 11]	X35= [1, 7, 11]	
X15= [3, 7, 11]	X36= [3, 9, 1]	
X16= [3, 9, 0]	X37= [6, 0, 3]	
X17= [5, 9, 2]	X38= [8, 2, 6]	
X18= [6, 11, 4]	X39= [10, 3, 6]	
X19= [6, 0, 4]	X40= [0, 5, 8]	
X20= [5, 9, 2]	X41= [1, 6, 9]	
X21= [6, 11, 2]	X42= [1, 3, 7, 9, 11]	
	X43= [1, 3, 6, 9, 11]	
	X44= [1, 3, 7, 9, 11]	
	X45= [1, 3, 6, 9, 11]	
	X46= [1, 4, 10]	
	X47= [11, 3, 9]	
	X48= [10, 1, 7]	
	X49= [11, 3, 7]	
	X50= [9, 2, 7]	
	X51= [9, 1, 5]	
	X52= [7, 0, 5]	
	X53= [8, 11, 5]	
	X54= [6, 11, 4]	
	X55= [6, 9, 5]	
	X56= [6, 10, 2]	
	X57= [5, 10, 3]	
	X58= [5, 8, 2]	
	X59= [5, 9, 1]	

We have discussed in this paper the theoretical aspects of Systemic Modeling, including its genealogical connection with the Theory of Compositional Systems and the Theory of Intertextuality. Examples of compositional systems (open, semi-open, and feedback) were given for methodological clarification. We also have proposed an expansion of the concept of musical parameter in order to include abstract characteristics of a musical text. The cycles of Systemic Modeling and Compositional Planning were examined in detail with the aid of two hypothetical examples. Finally, we have modeled the first *Prélude* of Debussy with the purpose of composing a new piece from its systemic model. This methodology has been studied by the author of this paper since 2010. Presently, it has been the main topic of two research projects developed in the Graduate Program of Music of the Federal University of Rio de Janeiro, as well as in the MusMat Research Group[13]. Approximately 20 new pieces were produced from systemic models by Pitombeira and also by his Graduate and Undergraduate students.

[13]http://musmat.org

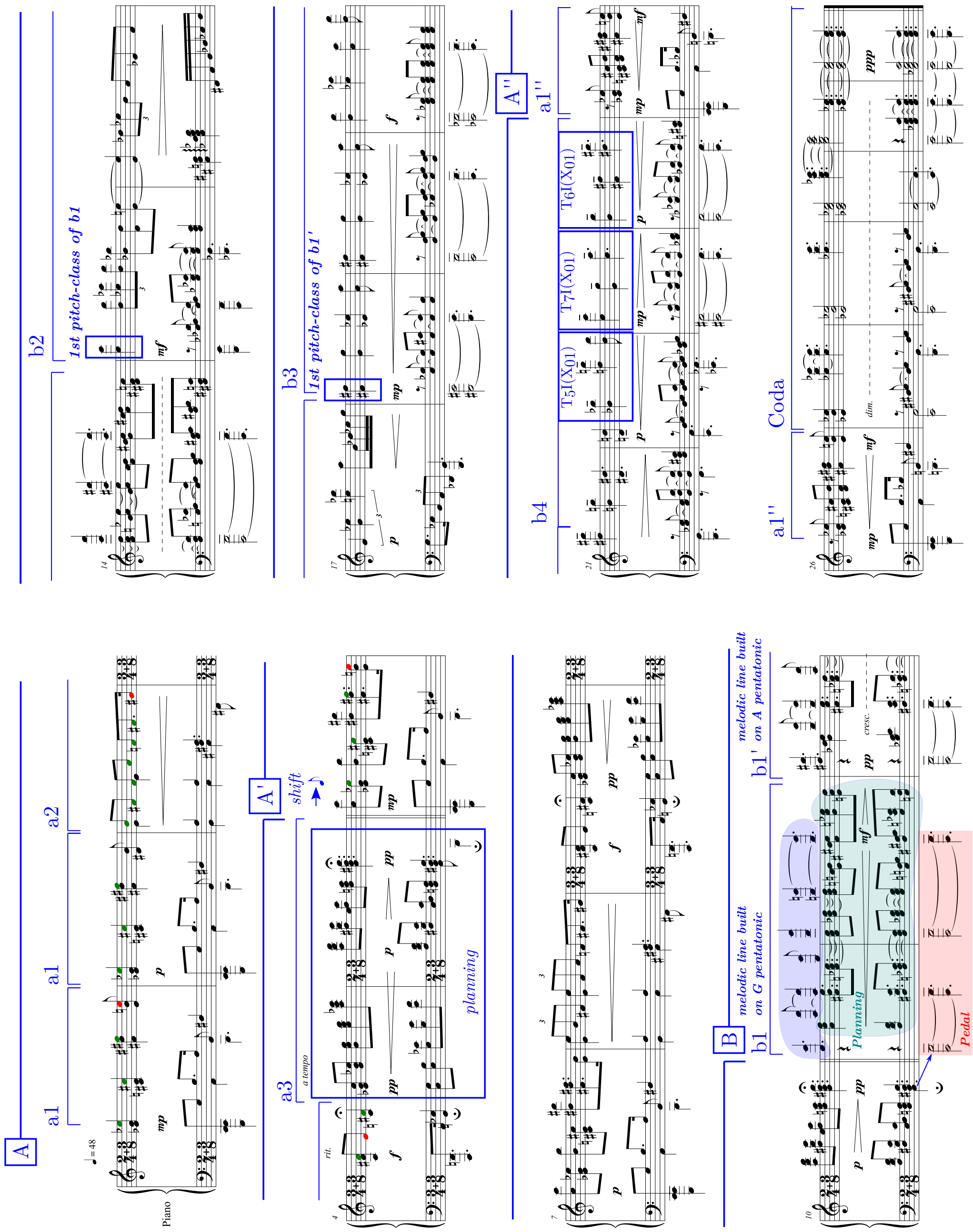


Figure 17: *Pitombeira's Le sphinx des Naxiens, first movement ofTrois Monuments.*

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Contour Similarity Algorithms

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Abstract: *The Musical Contour literature provides multiple algorithms for melodic contour similarity. However, most of them are limited in use by the melody length of input data. In this paper I review these algorithms, propose two new algorithms, compare them in three experiments with contours from the Bach Chorales, from a Schumann song and automated generated, and present a brief review of the contour and similarity literature.*
Keywords*: Music Contour Theory. Melodic Similarity. Algorithms. Music Analysis. Computational Musicology.*

I. Introduction

Similarity is a paramount concept in Music. It is important because it supports the recognition of music structures such as motives, themes, and chords. The measurement of music similarity is in the heart of the Music Analysis, as the base for identity tests and comparison of musical units. Music similarity has been discussed by many researchers of Music Theory, Ethnomusicology, Music Psychology and Cognition, and Music Information Retrieval fields.

The music contour similarity is one of the three kinds of melodic similarity, together with the pattern and global similarity [8]. There are many algorithms to measure the similarity between melodic contours in the literature, such as the Oscillation spectrum correlation [26, 27], the Embedded contour patterns [17], and the Rigid and Fuzzy comparison [22]. Despite the abundance of contour similarity algorithms, there is none that performs well with both small and large contours. Algorithms such as Rigid and Fuzzy comparison are only able to compare contours with the same size, the Embedded contour algorithm has high computational complexity (factorial), and Oscillation spectrum algorithm doesn't perform well with small contours with less than six points—See [7] for further information concerning computational complexity.

In this paper, I present two new contour similarity algorithms to fill this gap, a comparison with the available contour similarity algorithms and a brief review of the similarity literature.

II. Similarity

Many researchers have addressed concepts and measures concerning melodic similarity [8, 21, 28, 27, 5, 9, 15, 19, 4]. According to Eerola [8], two melodies are considered similar if they contain similar short patterns of pitches or rhythms, or resemblance shape. He classifies the similarity in three types: pattern, contour and global similarity—a combination of various representations of melodies, however, the nature of melodic similarity is not clear neither in the Psychology nor in the Music Information Retrieval research fields [16, p. 1].

*Thanks to MCTI/CNPQ for the research support.

Melodic similarity is a relative concept and it depends on the chosen music properties set used to compare the melodies [13, p. 1]. In a way, this relativeness helps to understand why of so many measures of similarity available in the literature. These measurements are based on multiple approaches such as the string matching of individual notes and geometric representation of melodies. Technically, the basic techniques for measuring the similarity are edit distance, n-grams, correlation and difference coefficients, and hidden Markov models [19].

The music contour similarity has been measured by multiple algorithms, based on the correlation of contour oscillations spectra [26, 27], the patterns embedded in the compared contours, and on the relations among pairwise contour points [17]. These algorithms return real values from zero to one to represent similarity between contour pairs. Other contour similarity measures returns another kind of information, such as complexity order [2, 6], or compare one contour with an abstract average contour of a collection [22]. In Section IV, I review the algorithms that return real values for comparisons of contour pairs.

III. Music Contour basic concepts

The understanding of a minimum set of contour concepts is necessary to follow this paper's premise[1]. A contour is an abstraction of musical parameters. Technically, a music contour is defined as "a set of points in one sequential dimension ordered by any other sequential dimension" [18, p. 283]. The melodic contour, for instance, is the pitch set (abstracted as contour points) ordered in time. A sequential dimension is an attribute dimension where the values can be ordered. Note pitch and note duration, for instance, are sequential dimensions, because the pitches can be ordered from lower to higher and duration from shorter to longer. Thus, despite the focus on the melody in the algorithm's analysis in Section VI, they can be used to compare contours of any musical dimension.

The central aspect of the contour study is the observation of the relationships among its points. The relation of any pair of contour points (CP) is ternary: one CP is lower, equal to, or higher than the other (See the Comparison Function [18], in the Equation (1)). The contour can be analyzed and represented in a linear or combinatorial[2] way. The linear representation—also known as Contour Adjacent Series (CAS) [11]—regards only the relations between adjacent CPs; the combinatorial contour considers relations between adjacent and non-adjacent CPs. The linear representation is a sequence of "+" and "-" signs to represent the relations between adjacent CPs and the combinatorial representation is a sequence of positive integers, where is the CP with the lower value.

$$CMP(a,b) = \begin{cases} - & : \text{if } b < a \\ 0 & : \text{if } b = a \\ + & : \text{if } b > a \end{cases} \tag{1}$$

For instance, let the contours M and N (Figures 1c and 1d), both from the Mozart's *Eine Kleine Nachtmusik* antecedent and consequent melodies (Figures 1a and 1b). Their Contour Adjacent Series are $M <$ - + - + - + + + $>$ and $N <$ + - + - + - - - $>$, and their combinatorial representation are $M <$ 1 0 1 0 1 0 1 2 3 $>$ and $N <$ 3 2 3 2 3 2 1 2 0 $>$.

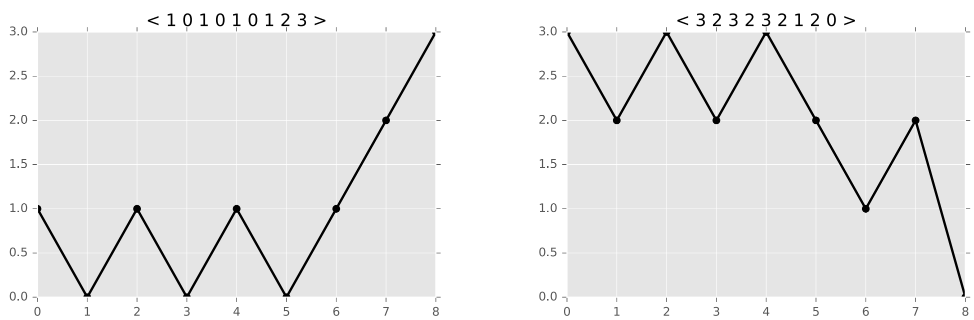
The relations among the contour points are represented in a self-comparison matrix. Two contours are equivalents if they or one of their reflexions—retrograded, inverted, and retrograded-inverted versions—share the same comparison matrix. For instance, the M and N comparison

[1]See [2] and [3] for further information concerning Music Contour Theory.
[2]See [20] and [23] for further information concerning combinatorial contours.

matrices are represented in Tables 1a and 1b. They are not equivalent, once their matrices are different.

(a) *Antecedent melody*

(b) *Consequent melody*



(c) *Antecedent contour*

(d) *Consequent contour*

Figure 1: *Mozart's Eine Kleine Nachtmusik K525 antecedent and consequent.*

	1	0	1	0	1	0	1	2	3
1	0	-	0	-	0	-	0	+	+
0	+	0	+	0	+	0	+	+	+
1	0	-	0	-	0	-	0	+	+
0	+	0	+	0	+	0	+	+	+
1	0	-	0	-	0	-	0	+	+
0	+	0	+	0	+	0	+	+	+
1	0	-	0	-	0	-	0	+	+
2	-	-	-	-	-	-	-	0	+
3	-	-	-	-	-	-	-	-	0

(a) *Antecedent contour matrix*

	3	2	3	2	3	2	1	2	0
3	0	-	0	-	0	-	-	-	-
2	+	0	+	0	+	0	-	0	-
3	0	-	0	-	0	-	-	-	-
2	+	0	+	0	+	0	-	0	-
3	0	-	0	-	0	-	-	-	-
2	+	0	+	0	+	0	-	0	-
1	+	+	+	+	+	+	0	+	-
2	+	0	+	0	+	0	-	0	-
0	+	+	+	+	+	+	+	+	0

(b) *Consequent contour matrix*

Table 1: *Mozart's Eine Kleine Nachtmusik K525 antecedent and consequent contour self-matrices.*

IV. Available contour similarity algorithms

Most of the contour similarity algorithms available in the Contour Theory returns a real number between 0 and 1, other algorithms such as Trace [6] and Multiple Linear Regression [2] returns a polynomial degree to express the contour complexity; and the Fuzzy similarity [22] returns an index value in the comparison of a contour with a set of contours that it belongs. Only the algorithms that return 0 to 1 similarity index between pairs of contours are presented in this paper. For illustration purposes, the melodic contours of Mozart's *Eine Kleine Nachtmusik* antecedent and consequent (Figure 1) are used for similarity calculus with all algorithms. The formalization of these algorithms is available in the Appendix.

	1	0	1	0	1	0	1	2	3
1		-	0	-	0	-	0	+	+
0			+	0	+	0	+	+	+
1				-	0	-	0	+	+
0					+	0	+	+	+
1						-	0	+	+
0							+	+	+
1								+	+
2									+
3									

(a) *Antecedent contour matrix*

	3	2	3	2	3	2	1	2	0
3		-	0	-	0	-	-	-	-
2			+	0	+	0	-	0	-
3				-	0	-	-	-	-
2					+	0	-	0	-
3						-	-	-	-
2							-	0	-
1								+	-
2									-
0									

(b) *Consequent contour matrix*

Table 2: *Comparison between upper triangle of the Mozart's Eine Kleine Nachtmusik K525 antecedent and consequent contour self-matrices.*

i. Rigid Matrix (CSIM)

The Contour Similarity (CSIM) [17] is based on the equivalence of all pairwise contour points of both compared contours. Equal values are summed and divided by the number of relations (See Algorithm 1). This calculus can be made also using the upper right-hand triangle of their comparison matrix.

For instance, the equivalent values of the Mozart's contours M and N matrices (Figure 1) are represented in gray color in Table 2. The similarity between these matrices is 0.44 (16/36). However, the similarity value must be the highest of the comparison among all the reflexions of the two given contours (the original, inverted, retrograded and retrograded-inverted versions). The similarity values for these forms of M and N contours are 0.33, 0.44, 0.5 and 0.55. Thus, the algorithm returns 0.55 as the similarity between the contours M and N. This similarity value is obtained by more than one combination of these reflexions versions, such as between the contour $M < 1\ 0\ 1\ 0\ 1\ 0\ 1\ 2\ 3 >$ and $I(N) < 0\ 1\ 0\ 1\ 0\ 1\ 2\ 1\ 3 >$.

I prefer to call this algorithm as Rigid Similarity Algorithm to differ to the Fuzzy one. In Section VI this algorithm is abbreviated as CMS (Contour Matrix Similarity).

ii. Embedded contours (ACMEMB)

The All Mutually Embedded Contours similarity index (ACMEMB) is presented by [17] as a solution to compare contours with different sizes. It is a type of global/local pattern similarity algorithm. All the contour subsequences embedded in both given contours are calculated and the number of subsequences embedded in both given contours is divided by the total number of subsequences.

The calculus of this index demands the subroutines $TRANSLATION$ (Algorithm 2), $CEMB$ (Algorithm 3), $ALLCEMB$ (Algorithm 6), and $COUNT$ (Algorithm 5). The $TRANSLATION$ algorithm returns a normalized version of the contour. For instance, a subsequence $< 3\ 6\ 1 >$ is normalized to $< 1\ 2\ 0 >$. The $CEMB$ algorithm is a modified Combination algorithm[3]. It returns all the given contour m-sized subsequences combinations. The only change to the combination algorithm is the contour translation of the subsequences. For instance, the function $CEMB(3, A)$

[3] See https://rosettacode.org/wiki/Combinations for multiple implementations of combinations algorithms in multiple languages.

returns the contour 3-sized subsequences $< 0\ 2\ 1 >$, $< 0\ 1\ 2 >$, $< 0\ 1\ 2 >$, and $< 1\ 0\ 2 >$, for the given contour $A < 0\ 2\ 1\ 3 >$.

The $ALLCEMB$ algorithm returns all the possible subsequence combinations of a given contour, with sizes 2 to n, where n is the contour size. These subsequences are used in the $ACMEMB$ (Algorithm 6) to return the similarity value between the given contours. For instance, the Embedded contour similarity value for the Mozart's contours M and N (Figure 1) is 0.28.

In Section VI this algorithm is abbreviated as EMB.

iii. Contour correspondence

Ellie Hisama [12] proposed the contour deviance measure, the number of deviations between the adjacency series of two given contours. In order to return a similarity index, this operation could be inverted and return the correspondences between the adjacency series of the two given contours. This value could be divided by the series size, to return a real number between 0 and 1, such as in the algorithm 7. Thus, the similarity index between the Mozart's contours M and N is 0.75.

In Section VI this algorithm is abbreviated as COR.

iv. Correlation of contour oscillation spectrum amplitude (OSC)

In the contour oscillation spectrum approach, the contour profile is considered as a sample of a complex wave. The similarity between the given contours is the correlation value between the amplitude spectra of both contours. This calculus can be divided into two parts: an adapted inverse (Fast) Fourier Transform to obtain the wave partials amplitudes[4] (Algorithm 8), and a Pearson's correlation between these amplitudes.

For instance, the oscillation spectra of the Mozart's contours M and N are $[(0.46 - 0.97j), (0.36 - 1.89j), (0.19 - 2.62j), (0.25 - 1.5j)]$ and $[(0.4 + 1.8j), (0.14 + 0.69j), (0.19 + 1.57j), (0.48 - 0.02j)]$, respectively. The amplitude is given by the real part of these complex numbers, in polar coordinates. The correlation between the amplitude of these spectra is 0.14, the similarity value between their respective contours.

In cases where the contour spectra have different sizes, the last partials of the bigger contour are discharged. For instance, let the contours $A < 1\ 0\ 3\ 2\ 1 >$ and $B < 2\ 0\ 2\ 1\ 3\ 2\ 4 >$, and their spectra $[(0.55 - 3.00j), (0.46 - 0.72j)]$ and $[(0.57 - 1.57j), (0.33 - 1.57j), (0.52 - 1.57j)]$, the last partial of B—$(0.52 - 1.57j)$—is not considered for the correlation. In this case, the correlation is 0.19.

I am using the first version of the algorithm, proposed in 1999, with unweighted contours, to follow Friedmann [11] initial definitions in a more strict way: the repeated adjacent pitches are not taken into account and the real intervals between pitches are discarded. Thus, I don't use time weighted pitches, as proposed by Schmuckler [27].

In this paper, we use the acronym OSC to refer to the Correlation of contour oscillation spectrum amplitude.

V. Proposed algorithms

I propose two new contour similarity algorithms: the Adjacent Global Pattern Contour Similarity Algorithm (AGP) and the Adjacent Edit Distance Contour Similarity Algorithm (AED). Both

[4] See `https://rosettacode.org/wiki/Fast_Fourier_transform` for multiple implementations of the Fast Fourier Transform (FFT) algorithm. In the Algorithm 8, I present an Inverted Discrete Fourier Transform, instead of FFT, just for a better comprehension.

contours are based on established string similarity algorithms.

i. Adjacent Global Pattern Similarity Algorithm (AGP)

The Adjacent Global Pattern Contour Similarity Algorithm (AGP) is an application of the standard Ratcliff/Obershelp Pattern Recognition algorithm [24] to handle the contour similarity problem. This algorithm is available in the Dictionary of Algorithms and Data Structures [1] and is implemented in Difflib Python library[5] [10].

Originally, the Ratcliff/Obershelp algorithm returns an index for the similarity between two given strings. The algorithm finds the longest common substrings of the both given strings, counts the remaining characters and divides this number by the sum of the two strings sizes.

For instance, let the strings "Pennsylvania" and "Pencilvaneya" (author's example), the substrings "Pen", "lvan" and "a" are common to both strings. The remaining characters are "nsy" and "ci", "i" and "ey". There are 8 remaining characters and the strings have 12 characters, each. Thus, the similarity index of the strings "Pennsylvania" and "Pencilvaneya" is 0.66 ($\frac{(12+12)-8}{12+12}$).

Our proposition is to use this algorithm to obtain similarity index between two given contour linear representations strings. For instance, the linear representations of the Mozart's contours M and N are $<$ - + - + - + + + $>$ and $<$ + - + - + - - - $>$, expressed as the strings "-+-+-+++" and "+-+-+—", respectively. The AGP similarity value is 0.75.

This algorithm favors both local and global contour features. In the global level, the longest common sequences are previously aligned, keeping possible large-scale contour structure. In the local level, the adjacent relations are compared and only the differences between common sequences decrease the similarity.

ii. Adjacent Edit Distance (AED)

The Adjacent Edit Distance Contour Similarity Algorithm (AED) is an application of the standard Levenshtein Edit Distance algorithm [14] to handle the contour similarity problem. This algorithm is available in the Dictionary of Algorithms and Data Structures [1] and is implemented in many programming languages[6].

Originally, the Levenshtein algorithm returns an integer as the distance value between two given strings. The distance is given by the number of insertions, deletions or substitutions to transform one string into the other. For instance, let the strings m "Pencilvaneya" and n "Pennsylvania", there are three substitutions ("c" with "n" at position 4, "i" with "s" at position 5 and "e" with "i" at position 10), one insertion ("y" at position 5) and one deletion ("y" at position 11) to transform m into n. Thus, the distance between m and n is 5.

My proposition is to use the Levenshtein algorithm as the base to obtain the similarity index between two given contour linear representations strings. To calculate the similarity index, the ratio between the distance and the size of the bigger string is subtracted from 1 (Equation 2).

$$AED(m,n) = 1 - \frac{LEV(m,n)}{MAX(m.length, n.length)} \tag{2}$$

For instance, the linear representations of the contours M and N are $<$ - + - + - + + + $>$ and $<$ + - + - + - - - $>$, expressed as the strings "-+-+-+++" and "+-+-+—". The distance between these strings is divided by the size of the bigger contour. Thus, the AED similarity value is 0.5 (4/8).

[5] The Difflib source code is available at https://github.com/python/cpython/blob/master/Lib/difflib.py.

[6] See https://rosettacode.org/wiki/Levenshtein_distance for implementations of the Levenshtein Distance algorithm.

	OSC	AGP	AED	CMS	EMB	COR
Mean	0.83	0.71	0.59	0.63	0.58	0.50
Std	0.29	0.15	0.20	0.13	0.11	0.27

Table 3: *Statistical summary of data of comparisons among contours of size from 2 to 6 of the various algorithms.*

VI. Algorithms comparison

I used three contour collections to test the algorithms. The first is a sample of contours with sizes from two to six generated automatically. This sample was obtained with confidence level 95% and confidence interval 5%. The second is a sample of contours from the Bach Chorales collection, and the third is composed by the contours of the phrases of the Robert Schumann's Op. 15, n. 7 (*Träumerei*). I run a statistical analysis with the data from the three collections.

I didn't use the CMS, COR and EMB algorithms to analyze Bach's Chorales and Schumann's *Träumerei* Analysis because they are able only for contours with equal size—CMS and COR—or are computational high complex—EMB (See Section VII for further information).

i. Contours with 2 to 6 points generated automatically

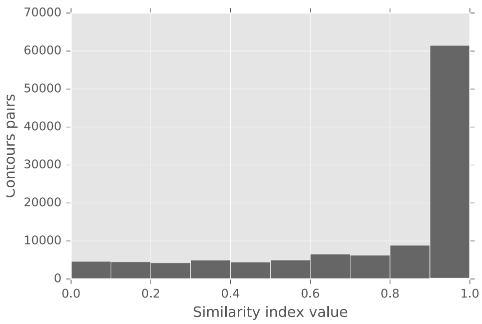
The first experiment dataset is a matrix of the similarity values given by OSC, AGP, AED, CMS, EMB and COR algorithms to 142.845 equal sized contour pairs. These contours—535 in total—are a random sample of contours with size between 2 and 6. The similarity values of each algorithm differ from each other. The data mean ranges from 0.50 to 0.83, and the standard deviation, from 0.11 to 0.29 (See the Table 3). In general, these values are irregularly distributed, except by the EMB algorithm data, normally distributed (See the Figure 2). There is a huge concentration of values in the range between 0.9 and 1.0 in the OSC algorithm data (See the Figure 2a). The causes for this concentration will be discussed in Section VII.

In general, there is a low correlation among these algorithms data (0.21 on average). The highest correlation, 0.86, occurs between AED and COR algorithms data—both algorithms are based on differences in the CAS elements—, and the lowest, 0, occurs between OSC and COR algorithms (See Table 4). The relation among these algorithms data can be viewed in Figure 3. The distribution that involves CMS, AGP and AED have aligned points for specific values—For instance, in Figure 3d, there is a horizontal line in the AGP similarity 0.6 and 0.8. It shows that these algorithms return a small set of values with this collection. This situation doesn't occur in the same way with OSC and OMB algorithms, despite the concentration of OSC similarity value in the 1.0 (See Figure 3b).

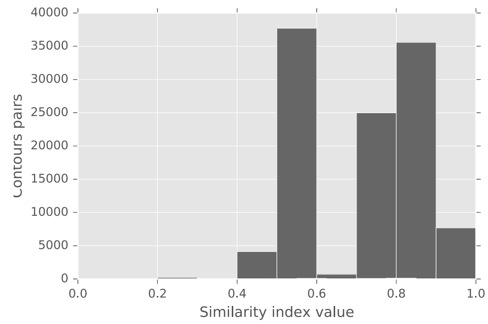
The reason of this small set of values is the nature of the algorithm: an integer from zero to the size of the contour divided by the size of the contour. Once the data has operations among equal sized contours, the algorithm returns only a few different values. For instance, contours with four points have CAS with three elements. The results will be only 0/3, 1/3, 2/3 and 3/3.

ii. Bach Chorales phrases analysis

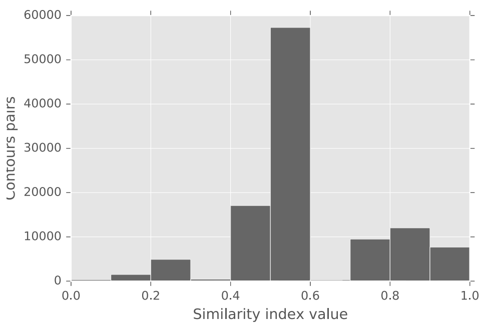
The second experiment dataset is a matrix of the similarity values given by the OSC, AGP and AED algorithms to 70.500 contour pairs. These contours—376 in total—are a random sample of contours from all four voices of the Bach Chorales phrases. This sample has phrases with contours with sizes from one to 28 points, with mean and standard deviation 8.27 and 3.41, respectively.



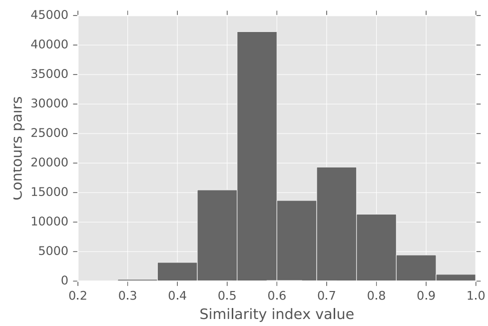
(a) *OSC*



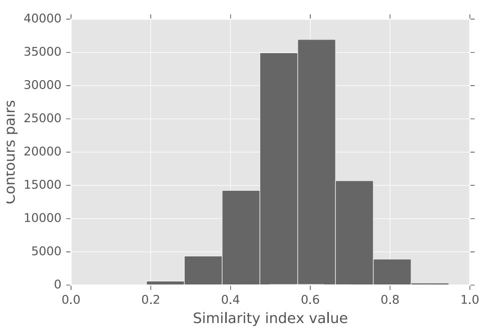
(b) *AGP*



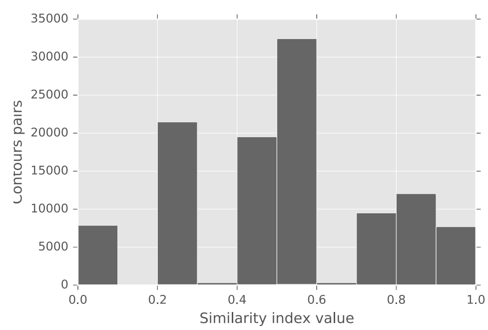
(c) *AED*



(d) *CMS*



(e) *EMB*

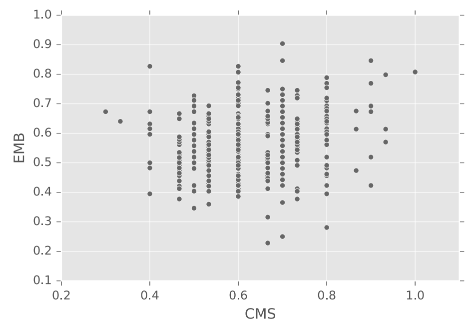


(f) *COR*

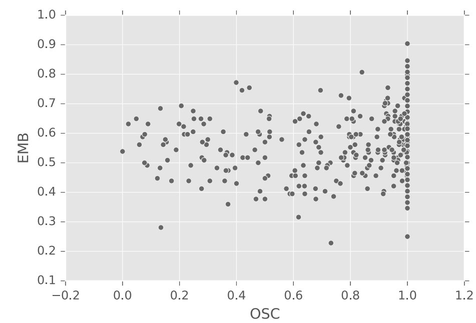
Figure 2: *Algorithm values distribution in comparisons among contours with size from 2 to 6*

	OSC	AGP	AED	CMS	EMB	COR
OSC	1.00	-0.01	-0.04	0.10	0.11	0.00
AGP	-0.01	1.00	0.76	0.13	0.31	0.51
AED	-0.04	0.76	1.00	0.10	0.17	0.86
CMS	0.10	0.13	0.10	1.00	0.16	-0.02
EMB	0.11	0.31	0.17	0.16	1.00	0.12
COR	0.00	0.51	0.86	-0.02	0.12	1.00

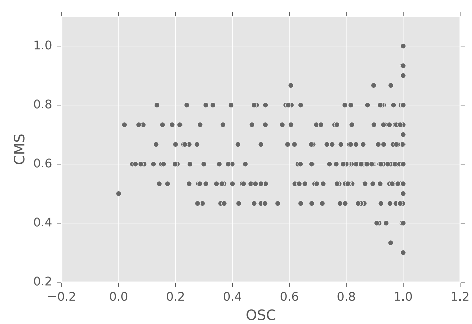
Table 4: *Correlation of algorithm measures of similarity among contours with size from 2 to 6.*



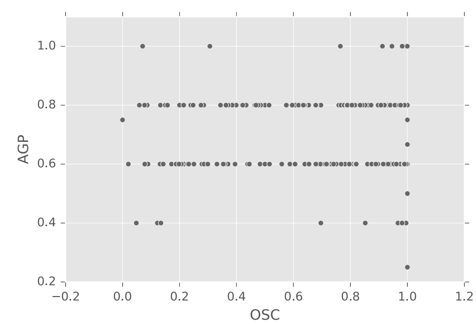
(a) *CMS and EMB*



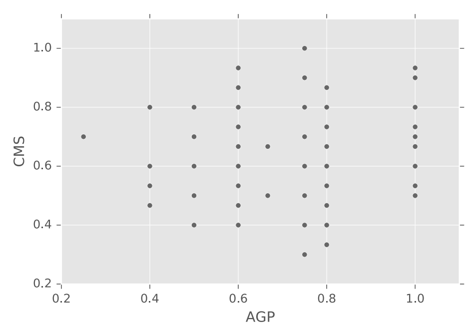
(b) *OSC and EMB*



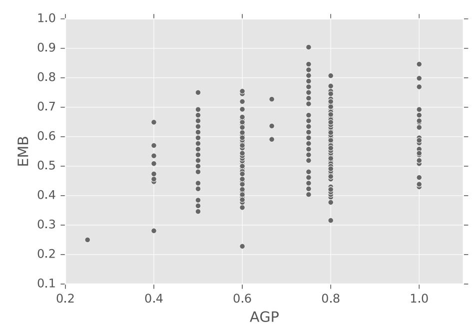
(c) *OSC and CMS*



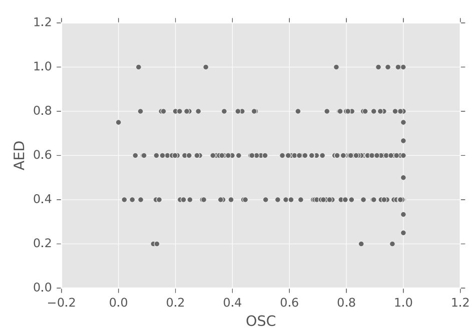
(d) *OSC and AGP*



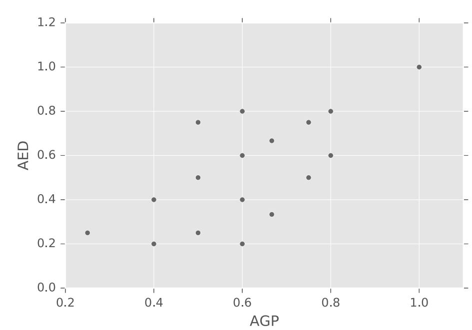
(e) *AGP and CMS*



(f) *AGP and EMB*



(g) *OSC and AED*



(h) *AGP and AED*

Figure 3: *Comparison of algorithms values in contours with size from 2 to 6*

	OSC	AGP	AED
Mean	0.60	0.62	0.52
Std	0.29	0.15	0.14

Table 5: *Statistical summary of data of comparing contours of a random sample of 376 Bach Chorales phrases by the various algorithms.*

	OSC	AGP	AED
OSC	1.00	0.03	-0.01
AGP	0.03	1.00	0.87
AED	-0.01	0.87	1.00

Table 6: *Correlation among algorithms datasets in Bach Chorales*

The similarity values of each algorithm differ from each other, but not as much as in the previous experiment. The data mean ranges from 0.52 to 0.62 and the standard deviation, from 0.14 to 0.29 (See Table 5). These values are normally distributed (AGP and AED algorithms) and fairly uniformly distributed (OSC) (See the Figure 4). There is an expressive concentration of values in the range between 0.9 and 1.0 in the OSC algorithm data (Figure 4a), slightly lower in comparison to the first experiment (Figure 2a).

Table 6 contains the correlation values between the algorithms datasets. Figure 5 contains similarity comparison for pairs of algorithms. There is no correlation between OSC dataset and AGP and AED datasets—with correlation values are 0.03 and -0.01, respectively. The data figures 5a and 5b confirm this lack of correlation. However, there is a strong correlation between AGP and AED datasets (0.87), confirmed by the data in the figure 5c.

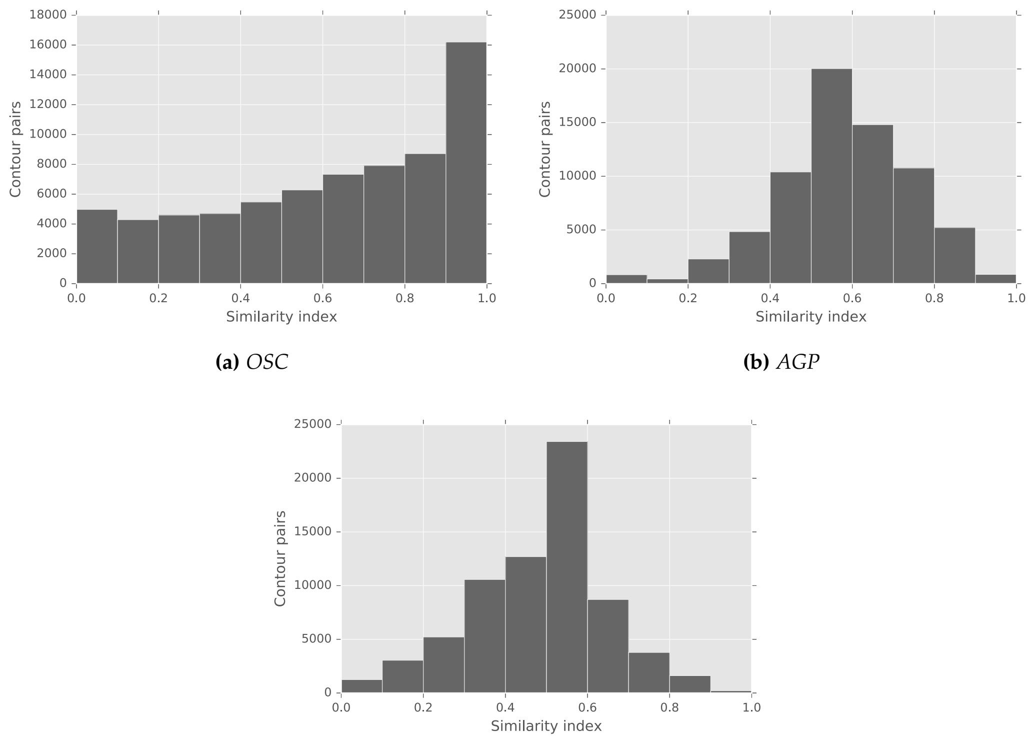
iii. Analysis of Schumann's Op. 15, n. 7—Träumerei

The third experiment dataset is a matrix of the similarity values given by OSC, AGP and AED to 15 contour pairs. These contours—6 in total—were obtained from Schumann's *Träumerei* phrases (See the melody in Figure 6 and the contours in Table 7). This piece's six phrases are equal in size, but with a different number of contour points.

All piece phrases begin with anacrusis and similar contour, but end in different ways. The OSC, AGP, and AED similarity values are very close (See Table 8). The values mean is between 0.81 and 0.86, and the standard deviation, between 0.07 and 0.09 (See Table 9). The correlation between AGP and AED similarity values is high, still slightly higher than in the other experiments (0.96), and the correlation among OSC and both AGP and AED similarity values is expressive (0.53 and 0.50, respectively), much higher than in the other experiments, where there was no correlation

Phrase	Contour
1	< 0 2 1 2 4 6 9 8 7 6 9 3 4 5 7 2 3 4 6 3 >
2	< 0 2 1 2 3 4 10 9 8 7 8 10 5 8 7 6 5 7 4 >
3	< 0 3 2 3 5 7 9 8 7 6 8 4 5 6 5 4 1 >
4	< 0 3 2 3 4 6 9 8 7 6 8 4 5 6 5 4 2 1 >
5	< 0 2 1 2 4 6 9 8 7 6 9 3 4 5 7 2 3 4 6 3 >
6	< 0 3 2 3 5 7 11 10 9 8 7 9 4 5 6 8 4 5 6 8 1 2 3 >

Table 7: *Robert Schumann's Träumerei contours*

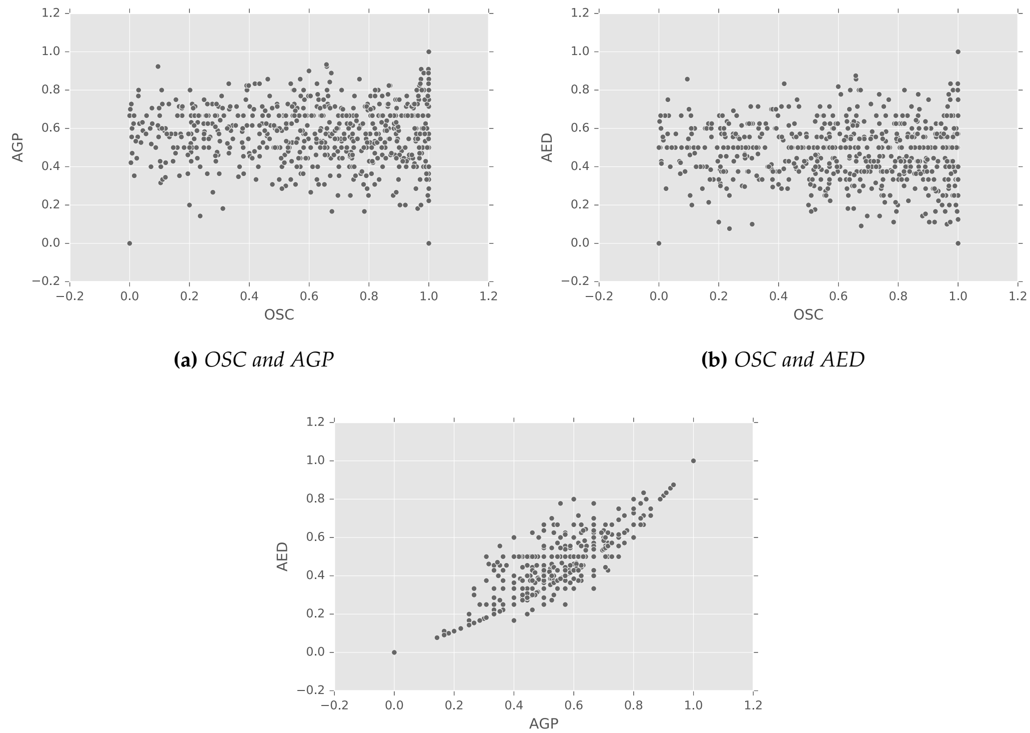


(a) *OSC*

(b) *AGP*

(c) *AED*

Figure 4: *Contour similarity distributions of a random sample of 376 Bach Chorales phrases.*



(a) *OSC and AGP*

(b) *OSC and AED*

(c) *AGP and AED*

Figure 5: *Comparison similarity values for a random sample of 376 Bach chorales contours.*

Figure 6: *Robert Schumann's Träumerei melody (Op. 15, n. 7).*

	1	2	3	4	5	6
1	1.00	0.81	0.70	0.74	1.00	0.85
2	0.81	1.00	0.92	0.97	0.81	0.87
3	0.70	0.92	1.00	0.95	0.70	0.86
4	0.74	0.97	0.95	1.00	0.74	0.83
5	1.00	0.81	0.70	0.74	1.00	0.85
6	0.85	0.87	0.86	0.83	0.85	1.00

(a) *OSC*

	1	2	3	4	5	6
1	1.00	0.81	0.86	0.83	1.00	0.93
2	0.81	1.00	0.88	0.91	0.81	0.75
3	0.86	0.88	1.00	0.97	0.86	0.79
4	0.83	0.91	0.97	1.00	0.83	0.77
5	1.00	0.81	0.86	0.83	1.00	0.93
6	0.93	0.75	0.79	0.77	0.93	1.00

(b) *AGP*

	1	2	3	4	5	6
1	1.00	0.79	0.79	0.79	1.00	0.86
2	0.79	1.00	0.83	0.83	0.79	0.68
3	0.79	0.83	1.00	0.94	0.79	0.68
4	0.79	0.83	0.94	1.00	0.79	0.68
5	1.00	0.79	0.79	0.79	1.00	0.86
6	0.86	0.68	0.68	0.68	0.86	1.00

(c) *AED*

Table 8: *Contour similarity values among Schumann's Träumerei phrases. The columns and rows refer to the phrase numbers.*

	OSC	AGP	AED
Mean	0.84	0.86	0.81
Std	0.09	0.07	0.09

Table 9: *Statistical summary of data of comparing contours of a Schumann's Träumerei phrases by the various algorithms.*

(See Table 10).

Finally, the similarity values for the adjacent contours perform a similar shape with the data of the three algorithms (See Figure 7). There is an ascendent similarity until the phrase 4, the lowest similarity between the phrases 4 and 5, and a return to a median similarity value between the phrases 5 and 6. This observation of the similarity progress between adjacent phrases helps to understand how approximation and withdraws—and consequently contrast and continuity—occur throughout a music piece.

VII. Discussion

As seen in Section IV, the Embedded Contour Similarity (EMB) and Oscillation Correlation Similarity (OSC) are the only algorithms provided by the contour literature that are able to

	OSC	AGP	AED
OSC	1.00	0.53	0.50
AGP	0.53	1.00	0.96
AED	0.50	0.96	1.00

Table 10: *Correlation of algorithms measures of similarity among the Schumann's Träumerei phrases contours.*

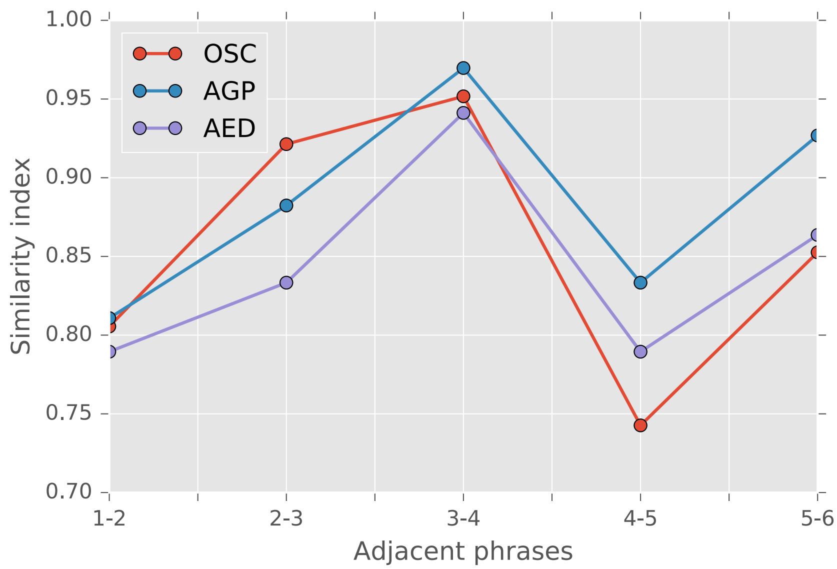


Figure 7: *OSC, AGP and AED similarity values among adjacent phrases in Schumann's Träumerei.*

Algorithm	Complexity	Different sizes
CMS	Quadratic	No
EMB	Factorial	Yes
COR	Linear	No
OSC	Linear-logarithmic	Yes
AGP	Quadratic	Yes
AED	Linear-logarithmic	Yes

Table 11: *Algorithms computational complexity and ability to handle comparisons among contours with different sizes*

compare contours with different sizes. Both proposed algorithms AGP, and AED are also able to compare contours with different sizes.

In terms of computational complexity, the OSC, AGP, and AED have acceptable complexity growth, but EMB not (See the Table 11). The EMB has a high computational complexity, of a factorial order, and is therefore limited to small contours comparisons. The factorial order complexity means that, given a contour with m points, it will take about $m!$ calculations to obtain the similarity value between it and a smaller contour. For this reason, it was not used in the experiments 2 and 3, in Section VI, where there were big contours.

The computational time complexity of the EMB algorithm is factorial in function of the CEMB. The number of combinations of a sequence is given by the binomial coefficient[7] (Equation (3)). Once the number of combinations of a given sequence is increased on a factorial basis, the complexity of the CEMB algorithm will be of a factorial order of computational time complexity. For instance, the number of calculations to obtain the similarity between the contours M and N is

[7]See [25] for further information concerning combinations and their calculus.

in the order of 10^6.

$$C(n,r) = \frac{n!}{r!(n-r)!} \tag{3}$$

Despite this problem of computational complexity, the algorithm returns the most normal curve of similarity values for contours with size from 2 to 6 (See the Figure 2e). It performs better than OSC, AGP and AED algorithms.

Unlike the EMB, the OSC algorithm is able to compare high length contours because has an acceptable computational complexity order, linear logarithmic. It means that given a contour with m points, it will take about $m \log(m)$ calculations to obtain the similarity value between it and a smaller contour.

However, this algorithm has some weaknesses such as theoretical and practical concerns about the use of Fourier analysis with discrete and short series as input—in contrast to the expected sample of an infinitely periodic signal [26, p. 304]. The small size of the input series will cause "edge effects", decreasing the tool's predictive power. Schmuckler defends the use of the tool arguing that the goal "is not the typical predictive forecasting commonly associated with time-series analyses" and that "Fundamentally, Fourier analysis is simply a mathematical decomposition procedure that is applicable to any numerical series." [26, p. 304], but admits that the applicability of Fourier analysis to melodic contour is an open question.

According to Beard, another weakness is the way Schmuckler conducted his experiments: the subjects are not asked to identify their perception of contour similarity, but to classify contour complexity in a scale. Thus, the similarity is derived indirectly from contour complexity [2, p. 186]. He concludes the similarity is achieved by the subjects' responses, not by Fourier model. However, Schmuckler argues "multidimensional scaling literature suggests that derived similarity data produces scaling metrics comparable to more direct similarity measures" [26, p. 317].

Besides the Schmuckler information about "edge effects" in Fourier analysis, the correlation calculus with datasets with only two points is not very useful, once it returns only the values -1, 0 or 1. The Fourier transform of small contours, with less than six points, results in only two partials. Thus, this correlation results will be an "all or nothing" type.

In this way, the Contour Theory provides an algorithm that can be used only with small contours (EMB) and another that can not be used with small contours (OSC). It's difficult to compare two contours with 5 and 12 points, for instance.

For these reasons I proposed the AGP and AED contour similarity algorithms, that can be used with small and large contours. Once Fast Fourier Transform, Levenshtein and Ratcliff/Obershelp algorithms have a linear-logarithmic order of complexity, both OSC, AGP[8] and AED have a similar computational complexity.

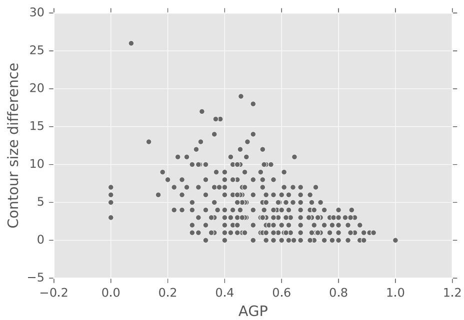
The use of CAS as input ensures the local level similarity features in both AGP and AED algorithms. However, only AGP has a focus on the global aspect, because the longest common strings principle.

The pitfalls of these algorithms are the maximum similarity value for some different contours pairs and the dependency on the contour size difference for the final value. Different contours like $< 0\ 2\ 1\ 3 >$ and $< 2\ 3\ 0\ 1 >$ have AGP similarity index 1, because both are linearly represented as $< + - + >$. And the size difference between contours decreases the similarity values. For instance, contours like $< 0\ 1 >$ and $< 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0 >$ have AGP similarity index 0.22, despite the pattern repetition.

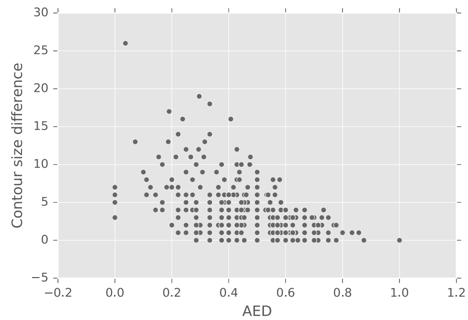
[8] The AGP algorithm complexity is cubic in the worst case and quadratic in the expected case, but Difflib's implementation is quadratic in the worst case and linear in the best case [10].

	OSC	AGP	AED
Correlation	-0.08	-0.48	-0.55

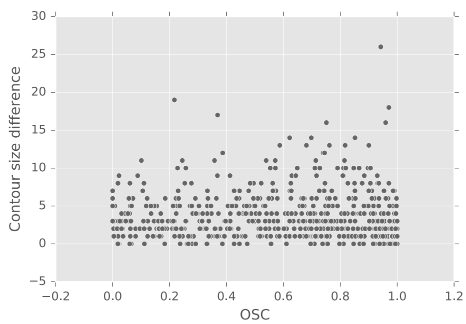
Table 12: *Correlation between contour similarity value and contour size difference.*



(a) *AGP*



(b) *AED*



(c) *OSC*

Figure 8: *Relation between contour similarity values and contour size difference*

The size difference between the contour pairs has a negative correlation to the similarity values given by both AGP and AED algorithms—in the Bach Chorales dataset. This correlation is -0.48 and -0.55, respectively. There is no correlation between the size difference and OSC similarity values (See the Table 12 and Figure 8).

Finally, the experiments of the Section VI reveal there are contours with high OSC similarity index values and low AGP ones, and vice-versa. For instance, OSC similarity index between the contours $< 0\ 1\ 2\ 3 >$ and $< 3\ 2\ 1\ 0 >$, $< 0\ 1\ 2\ 3 >$ and $< 4\ 3\ 2\ 1\ 0 >$ is 1.0, and the AGP similarity index, 0.0. In the opposite corner, the contours $< 3\ 2\ 1\ 2\ 3\ 2\ 1\ 0 >$ and $< 4\ 1\ 0\ 2\ 5\ 4\ 3\ 4\ 1 >$ has OSC similarity index 0.01 and AGP similarity index 0.93. This difference occurs in the algorithms pitfals: OSC processing small contours and AGP processing contours with great size difference.

For the pitfalls in the algorithms, I consider the contour similarity measurement as an open problem. I propose AGP and AED to fill the gap of different size contours similarity, but this proposition doesn't close the contour similarity problem.

VIII. Conclusion

In this paper, I presented contour similarity algorithms provided by the literature and introduced two new algorithms. I compared the algorithm's results in three experiments—with generated contours with 2 to 6 points, contours from the Bach Chorales phrases and with contours from a single piece, by Schumann.

The OSC [26] and EMB [17] algorithms are sensitive to the contour sizes. The OSC algorithm is good for contours with more than 5 points, and EMB, for contours with less than 7 or 8 points. Neither is efficient in comparing small with large contours. I introduce two new algorithms to fill this gap, but the difference between contour sizes is a pitfall in both of them. Thus, the contour similarity problem has not yet been solved and demands further research.

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Appendix

Algorithm 1 CSIM algorithm

CSIM(A,B), where A and B are two contours with the same size n.

Let $v = 0$
for i from 0 to $n - 1$ **do**
  for j from $i + 1$ to $n - 1$ **do**
    if $cmp(A_i, A_j) = cmp(B_i, B_j)$ **then**
      $v = v + 1$
    end if
  end for
end for
return $\frac{2v}{n^2 - n}$

Algorithm 2 Translation algorithm

TRANSLATION(A), where A is a contour sequence with n elements and H is a hash table with records in $\{k, v\}$ format.

Let H
Let $B = SORT(A)$
LET $T = []$
for i from 0 to $n - 1$ **do**
  add $\{B[i], i\}$ to H
end for
for i from 0 to $n - 1$ **do**
  add $H[A[i]]$ to T
end for
return T

Algorithm 3 CEMB algorithm

CEMB(m, A), where A is a contour sequence with n elements and m is the subsequence size.

```
if m = 0 then
  return [[]]
end if
Let combinations = []
for i from 0 to n − 1 do
  x = A[i]
  S = CEMB(m − 1, A[i + 1])
  for j from 0 to S.length − 1 do
    add TRANSLATION([x] + S[j]) to combinations
  end for
end for
return combinations
```

Algorithm 4 ALLCEMB algorithm

ALLCEMB(A), where A is a contour sequence with n elements.

```
Let combinations = []
for i from 2 to n do
  add CEMB(i, A) to combinations
end for
return combinations
```

Algorithm 5 COUNT algorithm

COUNT(ARR), where ARR is an array of contours and H is a hash table with records in $\{k, v\}$ format.

```
Let H
for i from 0 to ARR.length do
  if ARR[i] ∈ H.keys then
    H[ARR[i]] = 0
  end if
  H[ARR[i]] = H[ARR[i]] + 1
end for
return H
```

Algorithm 6 ACMEMB algorithm

ACMEMB(A, B), where A and B are two contour sequences and $MERGE$ is a function that merges two arrays in one and $UNIQUE$ returns only the distinct elements of a given array.

Let $Acomb = ALLCEMB(A)$
Let $Bcomb = ALLCEMB(B)$
Let $ALLcomb = MERGE(Acomb, Bcomb)$
Let $Z = UNIQUE(ALLcomb)$
Let $Acount = COUNT(Acomb)$
Let $Bcount = COUNT(Bcomb)$
Let $v = 0$
for i from 0 to $Z.length - 1$ **do**
  if $Z[i] \in Acomb \wedge Z[i] \in Bcomb$ **then**
    $v = v + Acount[Z[i]]$
    $v = v + Bcount[Z[i]]$
  end if
  add $CEMB(i, A)$ to *combinations*
end for
return $\frac{v}{Allcomb.length}$

Algorithm 7 Correspondence algorithm

CORRSIM(A,B), where A and B are two contours with the same size n.

Let $v = 0$
for i from 0 to $n - 2$ **do**
  if $cmp(A_i, A_{i+1}) = cmp(B_i, B_{i+1})$ **then**
    $v = v + 1$
  end if
end for
return $\frac{v}{n}$

Algorithm 8 Adapted Inverse Discrete Fourier Transform algorithm

AIFFT(A), where A is a sequence of numbers, and *real* returns the real part of a given complex number.

Let $N = A.length$
Let $S = []$
for k from 0 to $N - 1$ **do**
  Let $x = 0$
  for n from 0 to $N - 1$ **do**
    $x = x + e^{-2i\pi kn/N} A[k]$
  end for
  $S[k] = real(x/N)$
end for
return S

Z-Related Hexachords explained by Transpositional Combination and the Complement Union Property

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Abstract: *Two pcsets that have the same interval-class vector (ICV) but are not related by transposition (T_n) and/or Inversion (I) are said to be Z-related. The Z-relation can be extended to pairs of setclasses; the members of the two set-classes have the same ICV, but the members of one are not related to the other under T_n or T_nI. This paper shows how the 15 Z-related hexachordal setclasses can be explained by two functions on pcsets: 1) A hexachord can be described as the transpositional combination of two smaller sets; 2) A hexachord can be described as the icomplement union of two smaller pcsets. The 15 hexachordal pairs can be constructed by using one or more of five combinations of these two functions. For instance, the set-classes 6-10 [013457] and 6-39 [023458] are constructed by the transpositional combination of two [013] sets at the major 3rd and the complement union of a [048] trichord (augmented chord) and a [013] set.*

Keywords: *Atonal Set-theory. Z-relation. Transpositional Combination. Complement Union. Holomorphic Sets. Twelve-tone Theory. Serial Theory.*

I.

The term *Z-related* applies in the case where two pcsets have the same interval-class vector but are not related by T_n or T_nI. The pcsets {0467} and {2368} share the same interval-class vector (henceforth *ICV*), but T_n or T_nI does not relate the two[1] Since the ICV of a pcset X remains invariant when *X* is subjected to T_n and/or T_nI, all members of a set class (that includes X) have the same ICV. This means that a set-class can be associated with the ICV of its members. Therefore we can say that two set-classes are Z-related if the pcsets included in both set-classes share the same ICV. A familiar example is the two all-interval tetrachordal set-classes, 4-15 and 4-29[2]. There are exactly 23 Z-related pairs of set-classes; that is, 46 out of the 223 set-classes.

Since there can be different set-class systems, the definition of the Z-relation may change in each system. For instance, if set-classes are defined as containing pcsets related by only T_n; then pcsets related by T_nI are not members of the same set-class. But since pcsets related by T_nI share the same ICV, a set-class X that includes the T_nI transformations of pcsets in another set-class Y, implies that set-classes X and Y are Z-related. Similarly, if we expand our definition of set-class membership to include the T_nM and T_nMI[3] transformations, some set-classes that are Z-related in

[1] This should be clear since {0467} includes a major chord {047} (member of set-class 3-11) whereas {2368} does not.

[2] The example of pcsets {0467} and {2368} are all-interval tetrachords; {0467} is a member of set-class 4-29, while {2368} is a member of set-class 4-15

[3] M(a) = $ax5$, mod_{12}; MI(a) = $ax7$, mod_{12}

the ordinary set-class system[4] are joined so they are no longer Z-related in the expanded system. For instance, the Z-related set-classes 4-15 and 4-29 in the ordinary system, are now joined into one set-class in the T_nM and T_nMI set-class system. This is because, for instance, {0467} = T_{10}MI{2368}, so both pcsets are members of the same set-class.

In this expanded set-class system, two of the three pentachordal ordinary Z-related set-class pairs are joined into one-set-class; with regard to hexachords, 3 of the 15 hexachordal Z-related set-class pairs are resolved into a one set-class. So, we see that the expanded system does not resolve all the Z-relations in the ordinary set-class system[5].

There have attempts to find other transformations that included with T_n, T_nI, T_nM, T_nMI will resolve Z-related set-classes into one set; however all these systems merge many ordinary set-classes into the same set-class, sometimes merging Z-related set-classes into a set-class with other non-Z-related set-classes; or not merging the ordinary Z-related set-classes, but combining each of the two ordinary set-classes with other set-classes to form two distinct set-classes.

Another attempt to "resolve" the Z-relation is to define set-class membership by interval-class content. Thus, all the pcsets that share the same ICV are collected into the same set-class. This was done by Howard Hanson and Allen Forte, but it was soon shown that abstract inclusion[6] was incoherent in such systems[7]

II.

The Z-relation was originally defined by Allen Forte[8](and others); subsequently, theorists have been interested in understandingly why Z-relations occur in the first place. What's at stake is that interval class equivalence would seem to be an important way to group pcsets into classes such that all members of the class sounds alike[9]. However, in a transformational context, we know that T_n and T_nI preserve interval class content, so if we have one pcset we can produce another with the same ICV by transposing and/or inverting it But the Z-relation doesnâĂŹt define transformations that will transform one pcset into its Z-related correspondent. This inquiry has taken theorists in even more abstract territory, asking how the Z-relation works in pitch-class universes, mod n[10]. Recent research has connected into DNA sequencing and X-ray crystallography, where the

[4] That is, "ordinary" set-classes contain pcsets related by T_n and T_nI.

[5] The ordinary Z-related set-classes with higher cardinalities (of 7 and 8 pcs), are similarly joined as in the case of their 4 and 5 pc counterparts due to complementation. So, the Z-relation between 8-15 and 8-29 is resolved in the expanded system.

[6] A set-class X may be defined to be abstractly included in another set-class Y, if for each member pcset x in set-class X, x is literally included under T_n and/or T_nI in each member of the pcsets in set-class Y. For example, if the two set-classes are X = 3-11 and Y = 4-14 and x is {148}, then going through all the members of set-class Y: {0237} includes T_B{148} $=_n$ {037}; 1348 includes T_0{148} = {691}; {2459} includes T_1{148} = {259}; etc.; {0457} includes T_8I{148} = {740}; {1568} includes T_9I{148} = {851}; etc.

[7] The definition of abstract set-class inclusion does not work within the set-class system defined by interval class content. For instance, let X be the set-class whose members share ICV[001110] and Y will be the set-class whose members share ICV [111111]. (X is equivalent to the ordinary set-class 3-11; Y is the merger of ordinary set-classes 4-15 and 4-29.) x = {148} is a member of set-class X; y = {3479} is a member of set-class Y. But there is no transformation of x under T_n and/or T_nI (for all n) that is included in y. Thus, abstract inclusion is incoherent in this set-class system.

[8] The Z stands for zygote.

[9] Of course, the different articulations of pcs in different registers, spacings, and orderings does differentiate the "sound" of a pcset. Yet, all things being equal, interval class content ought to make pcsets that share the same ICV sound (very) similar.

[10] Are there Z-relations in an eight pc-universe? Yes. David Lewin showed that there are Z-triples in mod-16. Z-relations also occur in non-modular pitch-space. For instance, the pitch-sets {2,3,4,8,10,13} and {2,3,8,9,11,13} are Z-related. The two pitch sets regarded in pc-space are members of 6-41 and 6-12, respectively. See Lemke, Skiena, and Smith, *Reconstructing Sets from Interpoint Distances*.

Z-relation is generalized and has been understood for some time[11]. Such research provides a complete and general explanation of the Z-relation[12].

In this paper I take a different tack on understanding the Z-relation in the familiar mod-12 pitch-class universe. My method focuses of what I call the *ZC-relation*; that is, Z-relations that are caused by pcset complementation.

An early result in atonal music theory was the *complement theorem*, which provides a way to calculate the ICV of a pcset from the ICV of its complement. A corollary of the complement theorem is the *hexachord theorem*, which states that two complementary 6-pc pcsets have the same ICV. This implies that the Z-relation can apply to two different hexachordal set-classes, if one class contains the complements of the other class. What the theorem does not imply is that the two complementary sets (or set classes) are related by T_n, T_nI, or any other operation. And in fact, T_n and T_nI do not relate many complementary pairs of hexachordal pcsets. The hexachord theorem often forces the Z-relation on complementary hexachords. I call this kind of Z-relation, a *ZC-relation*.

III.

I will now show that every hexachordal ZC-relation in the mod-12 pcset universe can be explained by using two functions on pcsets. These functions are 1) *transpositional combination* and 2) *the complement union property*.

Transpositional Combination is often written w = TC(x, y) where x and y are different pcsets and w is the result. When we transpose each pc of pcset x, by all the pcs in y, the resultant set is w, the TC(x, y). So if $x = \{025\}$ and $y = \{1356\}$, w = the union of $T_1(0)$, $T_3(0)$, $T_55(0)$, $T_6(0)$, $T_1(2)$, $T_3(2)$, $T_5(2)$, $T_6(2)$, $T_1(5)$, $T_3(5)$, $T_5(5)$, and $T_6(5)$, or the union of 1, 3, 5, 6, 3, 5, 7, 8, 6, 8, A, and B, which is the pcset $z = \{135678AB\}$.

When X and Y are set-classes, then TC$(X, Y) = W_1/W_2$[13]. The two results, W_1 and W_2, are produced by doing TC$(x, y) = w_1$ and TC$(x, T_nIy) = w_2$ where x is a pcset included in set-class X, y is a pcset included in set-class Y, w_1 is a pcset included in set-class $W1$, and w_2 is a pcset included in set-class W_2. However, when either X or Y is invariant under T_nI, then set-class $W_1 = W_2$.

We simplify the notation by writing $X@Y = W_1/W_2$ or $X@Y = W$ when $W_1 = W_2$[14].

The Complement Union Property is written CUP$(X, Y) = W$, where X, Y, and W are set-classes. If it is the case that any pcset member of set-class X and any pcset member of set-class Y always produce a member of W in non-intersecting union, we assert CUP$(X, Y) = W$. We say that set-class W has CUP via set-classes X and Y.

Example: CUP(2-2,4-9) = 6-12. This means any member of 2-2 and any member of 4-9 will produce in non-intersecting union a member of set-class 6-12. Let us fix a member of 4-9: pcset $\{0167\}$. What are members of set-class 2-2 that produce a non-intersecting union with $\{0167\}$? They are $\{24\}$, $\{35\}$, $\{8A\}$, $\{9B\}$ and only these. Therefore:

$\{0167\}$ and $\{24\}$ produce $\{012467\}$, a member of 6-12;
$\{0167\}$ and $\{35\}$ produce $\{013567\}$, a member of 6-12;
$\{0167\}$ and $\{8A\}$ produce $\{01678A\}$, a member of 6-12;

[11]Such sets are said to be *holomorphic*.
[12]See Table 4 in *Reconstructing Sets* for a tally of the number of Z-pairs, triples, quartets, etc. in pc-universes from 8 to 23 pcs.
[13]The slash in "W_1/W_2" means "or".
[14]In all the cases where we use TC to explain the ZC-relation, one or both of the set-classes X and Y in TC(X, Y) has pcsets invariant under T_nI, so we can always write TC$(X, Y) = W$ or $X@Y = W$.

$\{0167\}$ and $\{9B\}$ produce $\{01679B\}$, a member of 6-12.

We can simplify the notation of CUP$(X,Y) = W$ by writing $X\&Y = W$. The previous example can be notated 2-2 & 4-9 = 6-12.

We also define an extension of CUP, the ZCUP function. ZCUP(X,Y) = W_1 or W_2. W_1 and W_2 are Z-related hexachords. If it is the case that any pcset member of set-class X, and any pcset member of set-class Y always produce a member of W_1 or of W_2 in non-intersecting union, we assert ZCUP$(X,Y) = W_2/W_2$. We say that set-classes W1 and its Z-related partner W_2 have ZCUP via set-classes X and Y.

Example. The Z-related hexachordal set-classes 6-10 and 6-39 have a ZCUP relation. ZCUP(2-6,4-4) = 6-10/6-39. We take a member of 4-4, $\{0125\}$ and members of 2-6 that produce non-intersecting union.

$\{0125\}$ and $\{39\} = \{012359\}$, which is a member of 6-39
$\{0125\}$ and $\{4A\} = \{01245A\}$, which is a member of 6-10

We can simply the notation of CUP$(X,Y) = W_1/W_2$ by writing $X\&Y = W_1/W_2$. The previous example can be notated 2-6 & 4-4 = 6-10/6-39.

IV.

The main result of this paper is that every ZC-related hexachordal set-class pair is related by TC and CUP, or by ZCUP.

There are five cases involving these relations (W_1 is the ZC-related set-class of W_2 and vice versa.)

Case 1: 2-4 @$X = W_1$ and 3-12 & $X = W_2$. X varies for each different pair. X is a trichord. (TC and CUP)

Case 2: 3-10 @$X = W_1$ and 4-28 & $X = W_2$. X varies for each different pair. X is a dyad. (TC and CUP)

Case 3: 3-5 @$X = W_1$ and 4-9 &$X = W_2$. X varies for each different pair. X is a dyad. (TC and CUP)

Case 4: 2-6 & $X = W_1/W_2$. X varies for each different pair. X is a tetrachord. (ZCUP)

Case 5: 4-9 & $X = W_1/W_2$. X varies for each different pair. X is a dyad. (ZCUP)

V.

Case 1. Partition the aggregate into four trichords. One is a member of 3-12 (the augmented chord), and the other three are not augmented chords related by T_4 and T_8. The partition is $\{\{048\}, S, T_4S,$ and $T_8S\}$. Partitioning the four sets into pairs will produce complementary hexachords, one of which includes an augmented chord while the other does not. Therefore, any two hexachords of two trichords from the partition are Z-related.

Example: Let the partition be $\{\{048\}\{127\}\{56B\}\{9A3\}\}$. S, T_4S, and T_8S are all members of set-class 3-5. Pairs of trichords from the partition yield either one of the two Z-related hexachords, 6-17 or 6-43.

2-4 @ 3-5 = 6-43 and 3-12 & 3-5 = 6-17.

Case 2. Partition the aggregate into five pcsets; one is a member of 4-28 (diminished-seventh chord), and other four are dyads related by T_3, T_6, and T_9. The partition is $\{\{0369\}$, S, T_3S, T_6S, and $T_9S\}$. Make a hexachord out of $\{0369\}$ and any one of the dyads. The other three dyads form a different, but Z-related hexachord. These three dyads can be decomposed into the union of two members of 3-10 (diminished chords).

Example: Let the partition be $\{\{0369\}\{12\}\{45\}\{78\}\{AB\}\}$; S, T_3S, T_6S, and T_9S are members of set-class 2-1. A pairing of one dyad with the tetrachord will produce one of two Z-related hexachords; and the other three dyads in union produce the other Z-related hexachord: either one of the two Z-related hexachords, 6-13 or 6-42.

3-10 @ 2-1 = 6-13 and 4-28 & 2-1 = 6-42.

Case 3. Partition the aggregate into five pcsets; one is a member of 4-9, and other four are dyads related by T_1, T_6, and T_7. The partition is $\{\{0167\}$, S, T_1S, T_6S, and $T_7S\}$. Make a hexachord out of $\{0167\}$ and one of the dyads. The other three dyads form a different Z-related hexachord. These three dyads can be decomposed into the union of two members of 3-5 (since every three-pc subset of 4-9 is a member of 3-5).

Example: Let the partition be $\{\{0167\}\{13\}\{24\}\{79\}\{8A\}\}$; S, T_1S, T_6S, and T_7S are members of set-class 2-2. A pairing of one dyad with the tetrachord will produce one of two Z-related hexachords; and the other three dyads in union produce the other Z-related hexachord: either one of the two Z-related hexachords, 6-12 or 6-41.

3-5 @ 2-2 = 6-41 and 4-9 & 2-2 = 6-12

Case 4. Partition the aggregate into two tetrachords and two dyads; the tetrachords are related by T_6 and the dyads are members of 2-6 (the tritone) related by T_n and T_{n+6}. The combination of one tetrachord and one dyad will yield a member of one of two Z-related hexachords.

Example: Let the partition be $\{\{0134\}\{679A\}\{28\}\{5B\}\}$. So $\{0134\}$ and $\{28\} = \{012348\}$, which is a member of 6-37 and $\{0134\}$ and $\{5B\} = \{01345B\}$, which is a member of 6-4.

2-6 & 4-3 = 6-4/6-37

Case 5. There is only one Z-pair in case 5. Partition the aggregate into two members of 4-9 that form in union 8-28 and two members of 2-3. The combination of one tetrachord and one dyad will yield a member of one of two Z-related hexachords.

For example, $\{\{0167\}\{349A\}\{25\}\{8B\}\}$. Taking a tetrachord and a dyad always results in a member of 6-6. However, re-configure the previous partition to produce $\{\{0167\}\{349A\}\{58\}\{B2\}$. Now, any combination of a tetrachord and a dyad yields a member of 6-38.

4-9 & 2-3 = 6-6/6-38.

VI.

The following chart shows all the ways ZC-related hexachords can be related using TC, CUP, and ZCUP (Table **??**). Vertical double lines separate the five cases. The two set-classes within the horizontal double lines are ZC-related. The second column shows the three cases of hexachordal Z-relations resolved under T_nM/T_nMI transformations. Note that some of the ZC-related hexachords can be generated in more than one way.

Table 1: *The fifteen Z-related pairs of Hexachords related by TC and CUP.*

	M/MI	TC and CUP		TC and CUP		TC and CUP		ZCUP	ZCUP
		Case 1		Case 2		Case 3		Case 4	Case 5
		2-4 @ X	3-12 & X	3-10 @ X	4-28 & X	3-5 @ X	4-9 & X	2-6 & X	4-9 & X
		X =	X =	X =	X =	X =	X =	X =	X =
6-3								4-2	
6-36								4-2	
6-4		3-1						4-3	
6-37			3-1					4-3	
6-6	6-38								2-3
6-38	6-6								2-3
6-10		3-2						4-4	
6-39			3-2					4-4	
6-11	6-40							4-11	
6-40	6-11							4-11	
6-12							2-2		
6-41						2-2			
6-13				2- 1					
6-42					2-1				
6-17			3-5				2-5		
6-43		3-5				2-5			
6-19	6-44							4-19	
6-44	6-19							4-19	
6-23				2-2					
6-45					2-2				
6-24			3-7					4-14	
6-46		3-7						4-14	
6-25								4-22	
6-47								4-22	
6-26		3-9						4-26	
6-48			3-9					4-26	
6-28			3-10		2-4				
6-49		3-10		2-4					
6-29					2-5				
6-50				2-5					

Non-hexachordal and/or non-ZC-related pcsets and their set-classes also may enjoy relations under TC, CUP, and/or ZCUP.

For instance:
TC(4-6,2-4) = 8-6
TC(3-2,3-3) = 7-2/7-4
ZCUP(2-3,2-6) = 4-15/4-29
CUP(3-6,3-12) = 6-14
CUP(4-15,4-29) = 8-28

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