Z-Related Hexachords explained by Transpositional Combination and the Complement Union Property

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Abstract: *Two pcsets that have the same interval-class vector (ICV) but are not related by transposition (T_n) and/or Inversion (I) are said to be Z-related. The Z-relation can be extended to pairs of setclasses; the members of the two set-classes have the same ICV, but the members of one are not related to the other under T_n or T_nI. This paper shows how the 15 Z-related hexachordal setclasses can be explained by two functions on pcsets: 1) A hexachord can be described as the transpositional combination of two smaller sets; 2) A hexachord can be described as the complement union of two smaller pcsets. The 15 hexachordal pairs can be constructed by using one or more of five combinations of these two functions. For instance, the set-classes 6-10 [013457] and 6-39 [023458] are constructed by the transpositional combination of two [013] sets at the major 3rd and the complement union of a [048] trichord (augmented chord) and a [013] set.*

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I.

The term *Z-related* applies in the case where two pcsets have the same interval-class vector but are not related by T_n or T_nI. The pcsets {0467} and {2368} share the same interval-class vector (henceforth *ICV*), but T_n or T_nI does not relate the two[1] Since the ICV of a pcset X remains invariant when *X* is subjected to T_n and/or T_nI, all members of a set class (that includes X) have the same ICV. This means that a set-class can be associated with the ICV of its members. Therefore we can say that two set-classes are Z-related if the pcsets included in both set-classes share the same ICV. A familiar example is the two all-interval tetrachordal set-classes, 4-15 and 4-29[2]. There are exactly 23 Z-related pairs of set-classes; that is, 46 out of the 223 set-classes.

Since there can be different set-class systems, the definition of the Z-relation may change in each system. For instance, if set-classes are defined as containing pcsets related by only T_n; then pcsets related by T_nI are not members of the same set-class. But since pcsets related by T_nI share the same ICV, a set-class X that includes the T_nI transformations of pcsets in another set-class Y, implies that set-classes X and Y are Z-related. Similarly, if we expand our definition of set-class membership to include the T_nM and T_nMI[3] transformations, some set-classes that are Z-related in

[1]This should be clear since {0467} includes a major chord {047} (member of set-class 3-11) whereas {2368} does not.

[2]The example of pcsets {0467} and {2368} are all-interval tetrachords; {0467} is a member of set-class 4-29, while {2368} is a member of set-class 4-15

[3]$M(a) = ax5$, mod_{12}; $MI(a) = ax7$, mod_{12}

the ordinary set-class system[4] are joined so they are no longer Z-related in the expanded system. For instance, the Z-related set-classes 4-15 and 4-29 in the ordinary system, are now joined into one set-class in the T_nM and T_nMI set-class system. This is because, for instance, {0467} = T_{10}MI{2368}, so both pcsets are members of the same set-class.

In this expanded set-class system, two of the three pentachordal ordinary Z-related set-class pairs are joined into one-set-class; with regard to hexachords, 3 of the 15 hexachordal Z-related set-class pairs are resolved into a one set-class. So, we see that the expanded system does not resolve all the Z-relations in the ordinary set-class system[5].

There have attempts to find other transformations that included with T_n, T_nI, T_nM, T_nMI will resolve Z-related set-classes into one set; however all these systems merge many ordinary set-classes into the same set-class, sometimes merging Z-related set-classes into a set-class with other non-Z-related set-classes; or not merging the ordinary Z-related set-classes, but combining each of the two ordinary set-classes with other set-classes to form two distinct set-classes.

Another attempt to "resolve" the Z-relation is to define set-class membership by interval-class content. Thus, all the pcsets that share the same ICV are collected into the same set-class. This was done by Howard Hanson and Allen Forte, but it was soon shown that abstract inclusion[6] was incoherent in such systems[7].

II.

The Z-relation was originally defined by Allen Forte[8](and others); subsequently, theorists have been interested in understandingly why Z-relations occur in the first place. What's at stake is that interval class equivalence would seem to be an important way to group pcsets into classes such that all members of the class sounds alike[9]. However, in a transformational context, we know that T_n and T_nI preserve interval class content, so if we have one pcset we can produce another with the same ICV by transposing and/or inverting it But the Z-relation doesn't define transformations that will transform one pcset into its Z-related correspondent. This inquiry has taken theorists in even more abstract territory, asking how the Z-relation works in pitch-class universes, mod n[10]. Recent research has connected into DNA sequencing and X-ray crystallography, where the

[4]That is, "ordinary" set-classes contain pcsets related by T_n and T_nI.

[5]The ordinary Z-related set-classes with higher cardinalities (of 7 and 8 pcs), are similarly joined as in the case of their 4 and 5 pc counterparts due to complementation. So, the Z-relation between 8-15 and 8-29 is resolved in the expanded system.

[6]A set-class X may be defined to be abstractly included in another set-class Y, if for each member pcset x in set-class X, x is literally included under T_n and/or T_nI in each member of the pcsets in set-class Y. For example, if the two set-classes are X = 3-11 and Y = 4-14 and x is {148}, then going through all the members of set-class Y: {0237} includes T_B{148} $=_n$ {037}; 1348 includes T_0{148} = {691}; {2459} includes T_1{148} = {259}; etc.; {0457} includes T_8I{148} = {740}; {1568} includes T_9I{148} = {851}; etc.

[7]The definition of abstract set-class inclusion does not work within the set-class system defined by interval class content. For instance, let X be the set-class whose members share ICV[001110] and Y will be the set-class whose members share ICV [111111]. (X is equivalent to the ordinary set-class 3-11; Y is the merger of ordinary set-classes 4-15 and 4-29.) x = {148} is a member of set-class X; y = {3479} is a member of set-class Y. But there is no transformation of x under T_n and/or T_nI (for all n) that is included in y. Thus, abstract inclusion is incoherent in this set-class system.

[8]The Z stands for zygote.

[9]Of course, the different articulations of pcs in different registers, spacings, and orderings does differentiate the "sound" of a pcset. Yet, all things being equal, interval class content ought to make pcsets that share the same ICV sound (very) similar.

[10]Are there Z-relations in an eight pc-universe? Yes. David Lewin showed that there are Z-triples in mod-16. Z-relations also occur in non-modular pitch-space. For instance, the pitch-sets {2,3,4,8,10,13} and {2,3,8,9,11,13} are Z-related. The two pitch sets regarded in pc-space are members of 6-41 and 6-12, respectively. See Lemke, Skiena, and Smith, *Reconstructing Sets from Interpoint Distances*.

Z-relation is generalized and has been understood for some time[11]. Such research provides a complete and general explanation of the Z-relation[12].

In this paper I take a different tack on understanding the Z-relation in the familiar mod-12 pitch-class universe. My method focuses of what I call the *ZC-relation*; that is, Z-relations that are caused by pcset complementation.

An early result in atonal music theory was the *complement theorem*, which provides a way to calculate the ICV of a pcset from the ICV of its complement. A corollary of the complement theorem is the *hexachord theorem*, which states that two complementary 6-pc pcsets have the same ICV. This implies that the Z-relation can apply to two different hexachordal set-classes, if one class contains the complements of the other class. What the theorem does not imply is that the two complementary sets (or set classes) are related by T_n, T_nI, or any other operation. And in fact, T_n and T_nI do not relate many complementary pairs of hexachordal pcsets. The hexachord theorem often forces the Z-relation on complementary hexachords. I call this kind of Z-relation, a *ZC-relation*.

III.

I will now show that every hexachordal ZC-relation in the mod-12 pcset universe can be explained by using two functions on pcsets. These functions are 1) *transpositional combination* and 2) *the complement union property*.

Transpositional Combination is often written w = TC(x, y) where x and y are different pcsets and w is the result. When we transpose each pc of pcset x, by all the pcs in y, the resultant set is w, the TC(x, y). So if $x = \{025\}$ and $y = \{1356\}$, w = the union of $T_1(0)$, $T_3(0)$, $T_5 5(0)$, $T_6(0)$, $T_1(2)$, $T_3(2)$, $T_5(2)$, $T_6(2)$, $T_1(5)$, $T_3(5)$, $T_5(5)$, and $T_6(5)$, or the union of 1, 3, 5, 6, 3, 5, 7, 8, 6, 8, A, and B, which is the pcset $z = \{135678AB\}$.

When X and Y are set-classes, then TC$(X, Y) = W_1/W_2$[13]. The two results, W_1 and W_2, are produced by doing TC$(x, y) = w_1$ and TC$(x, T_nIy) = w_2$ where x is a pcset included in set-class X, y is a pcset included in set-class Y, w_1 is a pcset included in set-class $W1$, and w_2 is a pcset included in set-class W_2. However, when either X or Y is invariant under T_nI, then set-class $W_1 = W_2$.

We simplify the notation by writing $X@Y = W_1/W_2$ or $X@Y = W$ when $W_1 = W_2$[14].

The Complement Union Property is written CUP$(X, Y) = W$, where X, Y, and W are set-classes. If it is the case that any pcset member of set-class X and any pcset member of set-class Y always produce a member of W in non-intersecting union, we assert CUP$(X, Y) = W$. We say that set-class W has CUP via set-classes X and Y.

Example: CUP(2-2,4-9) = 6-12. This means any member of 2-2 and any member of 4-9 will produce in non-intersecting union a member of set-class 6-12. Let us fix a member of 4-9: pcset $\{0167\}$. What are members of set-class 2-2 that produce a non-intersecting union with $\{0167\}$? They are $\{24\}$, $\{35\}$, $\{8A\}$, $\{9B\}$ and only these. Therefore:

$\{0167\}$ and $\{24\}$ produce $\{012467\}$, a member of 6-12;
$\{0167\}$ and $\{35\}$ produce $\{013567\}$, a member of 6-12;
$\{0167\}$ and $\{8A\}$ produce $\{01678A\}$, a member of 6-12;

[11] Such sets are said to be *holomorphic*.

[12] See Table 4 in *Reconstructing Sets* for a tally of the number of Z-pairs, triples, quartets, etc. in pc-universes from 8 to 23 pcs.

[13] The slash in "W_1/W_2" means "or".

[14] In all the cases where we use TC to explain the ZC-relation, one or both of the set-classes X and Y in TC(X, Y) has pcsets invariant under T_nI, so we can always write TC$(X, Y) = W$ or $X@Y = W$.

$\{0167\}$ and $\{9B\}$ produce $\{01679B\}$, a member of 6-12.

We can simplify the notation of $\text{CUP}(X, Y) = W$ by writing $X\&Y = W$. The previous example can be notated 2-2 & 4-9 = 6-12.

We also define an extension of CUP, the ZCUP function. $\text{ZCUP}(X, Y) = W_1$ or W_2. W_1 and W_2 are Z-related hexachords. If it is the case that any pcset member of set-class X, and any pcset member of set-class Y always produce a member of W_1 or of W_2 in non-intersecting union, we assert $\text{ZCUP}(X, Y) = W_2/W_2$. We say that set-classes W1 and its Z-related partner W_2 have ZCUP via set-classes X and Y.

Example. The Z-related hexachordal set-classes 6-10 and 6-39 have a ZCUP relation. ZCUP(2-6,4-4) = 6-10/6-39. We take a member of 4-4, $\{0125\}$ and members of 2-6 that produce non-intersecting union.

$\{0125\}$ and $\{39\} = \{012359\}$, which is a member of 6-39
$\{0125\}$ and $\{4A\} = \{01245A\}$, which is a member of 6-10

We can simply the notation of $\text{CUP}(X, Y) = W_1/W_2$ by writing $X\&Y = W_1/W_2$. The previous example can be notated 2-6 & 4-4 = 6-10/6-39.

IV.

The main result of this paper is that every ZC-related hexachordal set-class pair is related by TC and CUP, or by ZCUP.

There are five cases involving these relations (W_1 is the ZC-related set-class of W_2 and vice versa.)

Case 1: 2-4 $@X = W_1$ and 3-12 & $X = W_2$. X varies for each different pair. X is a trichord. (TC and CUP)

Case 2: 3-10 $@X = W_1$ and 4-28 & $X = W_2$. X varies for each different pair. X is a dyad. (TC and CUP)

Case 3: 3-5 $@X = W_1$ and 4-9 $\&X = W_2$. X varies for each different pair. X is a dyad. (TC and CUP)

Case 4: 2-6 & $X = W_1/W_2$. X varies for each different pair. X is a tetrachord. (ZCUP)

Case 5: 4-9 & $X = W_1/W_2$. X varies for each different pair. X is a dyad. (ZCUP)

V.

Case 1. Partition the aggregate into four trichords. One is a member of 3-12 (the augmented chord), and the other three are not augmented chords related by T_4 and T_8. The partition is $\{\{048\}$, S, T_4S, and $T_8S\}$. Partitioning the four sets into pairs will produce complementary hexachords, one of which includes an augmented chord while the other does not. Therefore, any two hexachords of two trichords from the partition are Z-related.

Example: Let the partition be $\{\{048\}\{127\}\{56B\}\{9A3\}\}$. S, T_4S, and T_8S are all members of set-class 3-5. Pairs of trichords from the partition yield either one of the two Z-related hexachords, 6-17 or 6-43.

2-4 @ 3-5 = 6-43 and 3-12 & 3-5 = 6-17.

Case 2. Partition the aggregate into five pcsets; one is a member of 4-28 (diminished-seventh chord), and other four are dyads related by T_3, T_6, and T_9. The partition is $\{\{0369\}$, S, T_3S, T_6S, and $T_9S\}$. Make a hexachord out of $\{0369\}$ and any one of the dyads. The other three dyads form a different, but Z-related hexachord. These three dyads can be decomposed into the union of two members of 3-10 (diminished chords).

Example: Let the partition be $\{\{0369\}\{12\}\{45\}\{78\}\{AB\}\}$; S, T_3S, T_6S, and T_9S are members of set-class 2-1. A pairing of one dyad with the tetrachord will produce one of two Z-related hexachords; and the other three dyads in union produce the other Z-related hexachord: either one of the two Z-related hexachords, 6-13 or 6-42.

3-10 @ 2-1 = 6-13 and 4-28 & 2-1 = 6-42.

Case 3. Partition the aggregate into five pcsets; one is a member of 4-9, and other four are dyads related by T_1, T_6, and T_7. The partition is $\{\{0167\}$, S, T_1S, T_6S, and $T_7S\}$. Make a hexachord out of $\{0167\}$ and one of the dyads. The other three dyads form a different Z-related hexachord. These three dyads can be decomposed into the union of two members of 3-5 (since every three-pc subset of 4-9 is a member of 3-5).

Example: Let the partition be $\{\{0167\}\{13\}\{24\}\{79\}\{8A\}\}$; S, T_1S, T_6S, and T_7S are members of set-class 2-2. A pairing of one dyad with the tetrachord will produce one of two Z-related hexachords; and the other three dyads in union produce the other Z-related hexachord: either one of the two Z-related hexachords, 6-12 or 6-41.

3-5 @ 2-2 = 6-41 and 4-9 & 2-2 = 6-12

Case 4. Partition the aggregate into two tetrachords and two dyads; the tetrachords are related by T_6 and the dyads are members of 2-6 (the tritone) related by T_n and T_{n+6}. The combination of one tetrachord and one dyad will yield a member of one of two Z-related hexachords.

Example: Let the partition be $\{\{0134\}\{679A\}\{28\}\{5B\}\}$. So $\{0134\}$ and $\{28\} = \{012348\}$, which is a member of 6-37 and $\{0134\}$ and $\{5B\} = \{01345B\}$, which is a member of 6-4.

2-6 & 4-3 = 6-4/6-37

Case 5. There is only one Z-pair in case 5. Partition the aggregate into two members of 4-9 that form in union 8-28 and two members of 2-3. The combination of one tetrachord and one dyad will yield a member of one of two Z-related hexachords.

For example, $\{\{0167\}\{349A\}\{25\}\{8B\}\}$. Taking a tetrachord and a dyad always results in a member of 6-6. However, re-configure the previous partition to produce $\{\{0167\}\{349A\}\{58\}\{B2\}$. Now, any combination of a tetrachord and a dyad yields a member of 6-38.

4-9 & 2-3 = 6-6/6-38.

VI.

The following chart shows all the ways ZC-related hexachords can be related using TC, CUP, and ZCUP (Table 1). Vertical double lines separate the five cases. The two set-classes within the horizontal double lines are ZC-related. The second column shows the three cases of hexachordal Z-relations resolved under T_nM/T_nMI transformations. Note that some of the ZC-related hexachords can be generated in more than one way.

Table 1: *The fifteen Z-related pairs of Hexachords related by TC and CUP.*

	M/MI	TC and CUP		TC and CUP		TC and CUP		ZCUP	ZCUP
		Case 1		Case 2		Case 3		Case 4	Case 5
		2-4 @ X	3-12 & X	3-10 @ X	4-28 & X	3-5 @ X	4-9 & X	2-6 & X	4-9 & X
		X =	X =	X =	X =	X =	X =	X =	X =
6-3								4-2	
6-36								4-2	
6-4		3-1						4-3	
6-37			3-1					4-3	
6-6	6-38								2-3
6-38	6-6								2-3
6-10		3-2						4-4	
6-39			3-2					4-4	
6-11	6-40							4-11	
6-40	6-11							4-11	
6-12							2-2		
6-41						2-2			
6-13				2- 1					
6-42					2-1				
6-17			3-5				2-5		
6-43		3-5				2-5			
6-19	6-44							4-19	
6-44	6-19							4-19	
6-23				2-2					
6-45					2-2				
6-24			3-7					4-14	
6-46		3-7						4-14	
6-25								4-22	
6-47								4-22	
6-26		3-9						4-26	
6-48			3-9					4-26	
6-28			3-10		2-4				
6-49		3-10		2-4					
6-29					2-5				
6-50				2-5					

Non-hexachordal and/or non-ZC-related pcsets and their set-classes also may enjoy relations under TC, CUP, and/or ZCUP.

For instance:
TC(4-6,2-4) = 8-6
TC(3-2,3-3) = 7-2/7-4
ZCUP(2-3,2-6) = 4-15/4-29
CUP(3-6,3-12) = 6-14
CUP(4-15,4-29) = 8-28

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