Machine Representation of Fundamental Musical Functions

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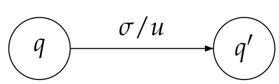
Abstract: *We consider a sequential machine model with output a cancellative monoid in order to describe fundamental music functions (transposition, inversion, retrogate, change of durations, pitch class distribution, move function). The minimal such machine of a prefix preserving function is provided. Musical functions are classified according the complexity of the minimal sequential transducers representing them. Functions coming from contour situations are shown to be sequential and their minimal machines are constructed. A machine simulation based hierarchy of musical contours and the corresponding classification of musical languages are exhibited.*

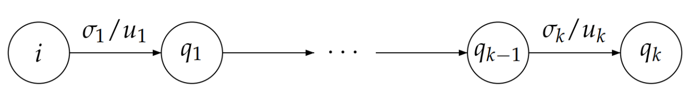
Keywords: *Musical morphism. Musical Contour. Hierarchy of Musical Objects. Sequential Transducer.*

I. Introduction

Mathematical machine models, such as automata, were already used to analyse, interpret and represent musical processes, [1], [5, 6, 7], [4], [17], [2]. Sequential transducers constitute the most general algorithm that can be executed in real time by a finite device. Non-deterministic transducers and weighted transducers have already been used in speech recognition [14, 15], natural language processing [13], [19], image generation [9] and music identification [21], [16]. A sequential transducer is a *deterministic automaton* with transitions labelled with both input and output symbols



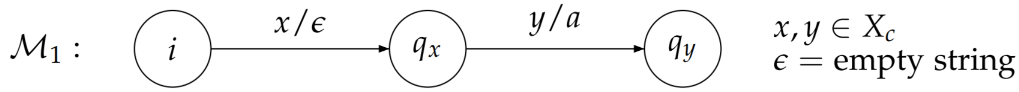
interpreted as follows: if $\mathcal{M}$ is in state q and we input the symbol σ, then $\mathcal{M}$ goes to state q' and outputs the string u. The *behavior* of $\mathcal{M}$ is the function defined in the following way: every input string $\sigma_1 \cdots \sigma_k$, labels a unique path



and the emitted string is $u_1 \cdots u_k$, where ⓘ denotes the single initial state.

Functions computed by such systems are called *sequential* and have the fundamental property to preserve *prefixes*. The preservation of prefixes musically refers to the maintenance of similarities, necessary for outlining the dynamics of the *musical flow*, thus rendering sequential transducers a considerable tool to classify musical strings.

Example I.1. Consider the chromatic alphabet $X_c = \{c, c\sharp, d, d\sharp, e, f, f\sharp, g, g\sharp, a, a\sharp, b\}$ and the sequential transducer



where $a = 1, 0, -1$ whenever y is located in X_c upwards, at the same level or downwards of x respectively. The function f_1 computed by $\mathcal{M}_1$ sends every musical string to its outline. For instance, the musical string

$$w = b\ c\sharp\ e\ b\ c\sharp\ g\sharp\ f\sharp\ e\ g\sharp\ c\sharp\ b$$

resulting from the following melodic line

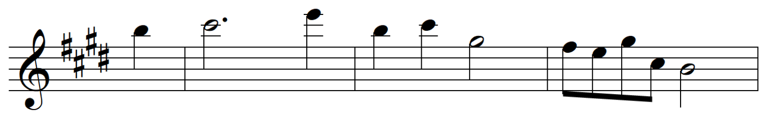
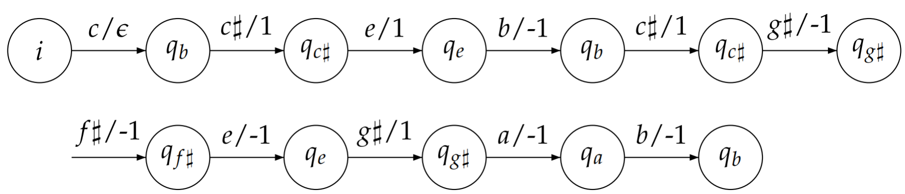


Figure 1: *Opening melodic line of Solon Michaelides' Sappho's Lyre.*

generates the path



and so

$$f_1(w) = 1\ 1\ (-1)\ 1\ (-1)\ (-1)\ (-1)\ 1\ (-1)\ (-1),$$

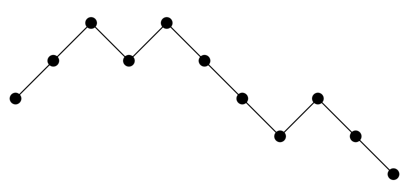


Figure 2: *Outline* $f_1(w)$.

The equivalence relation induced by f_1 classifies two musical strings to the same class if and only if they have the same outline.

The fundamental counterpoint transformations T_t (transposition), I_t (inversion) are sequential but the retrograde function

$$\sigma_1 \cdots \sigma_k \overset{R}{\mapsto} \sigma_k \cdots \sigma_1$$

fails to be sequential, because it does not preserve prefixes. On the other hand, the function counting the numbers of 0's, 1's, ... 11's occuring on a string of pitch classes, as well as the function counting the ascending, horizontal, descending moves on a musical string are not sequential because their destination sets are not free monoids. The same holds for the *move index* function f_0 that informs whether the number of ascending moves in a given musical string is greater than the number of descending moves and vice versa. In order to capture these exceptions as well, we *extend the sequential transducer model* by setting the output to be a cancellative monoid.

Another advantage of this extension is that the minimal sequential transducer $\mathcal{M}_f$ describing a given prefix preserving function $f : \Sigma^* \to M$ (M cancellative monoid) can be effectively constructed by using the residuals of f as in the classical case, [10], [20]. Musical transformations are classified according to the complexity of minimal sequential transducers computing them. If $\mathcal{M}_f$ and $\mathcal{M}_{f'}$ are isomorphic, then f and f' are *syntactically equivalent*, i.e. they represent the same mechanism regardless of the nature of the objects they act upon.

The present paper is divided into four sections. In section 2 we review some basic properties of the structure of cancellative monoids. In section 3 we propose the model of sequential transducer with output a cancellative monoid and construct the minimal such transducer associated with a prefix preserving function. Most of musical viewpoints can be represented by extended sequential transducers. Especially, musical morphisms have simple minimal machines and so they are located at the first level of any hierarchy of musical functions. The framework of the extended sequential transducers is highly appropriate to study musical contour functions (section 4).

A musical contour is a triple (Σ, M, c) consisting of a set Σ of musical elements, a cancellative monoid M of transformations or numbers and a function $c : \Sigma \times \Sigma \to M$ assigning an element $c(s_1, s_2)$ of M to any pair $(s_1, s_2) \in \Sigma^2$. The essence of music contour is an unfolding act of transition between one musical element and the next. This act of transition reflects the true substance of music, an art defined by movement in time, on the stave and the connection between theoretical/analytical significations. [2] studied contours of the form $\Sigma \times \Sigma \to \mathbb{R}$ counting quantitative features of musical strings. We show that any musical contour function with values in a cancellative monoid is sequential and we construct its minimal sequential transducer. Hierarchies of musical contours with respect to transducer simulation, as well as the corresponding musical string hierarchies, are provided.

II. Cancellative Monoids

In order to increase the recognition power of ordinary sequential transducers, we use the structure of cancellative monoid. A *monoid* is a set M equipped with a binary operation

$$\odot : M \times M \to M \quad (m_1, m_2) \mapsto m_1 \odot m_2$$

which is *associative* and admits a *neutral element* $e \in M$. Given operation $\odot : M \times M \to M$, its *opposite* $\odot^{opp} : M \times M \to M$ is defined by $m_1 \odot^{opp} m_2 := m_2 \odot m_1$, for all $m_1, m_2 \in M$. If $(M, \odot, e)$ is a monoid, then $(M, \odot^{opp}, e)$ is again a monoid, called the *opposite monoid* of $(M, \odot, e)$ and is denoted by $(M, \odot, e)^{opp}$.

For a given alphabet Σ, the set Σ^* of all finite strings (finite sequences of letters of Σ) with the concatenation operation and neutral element the empty string ϵ is a monoid, the *free monoid over the alphabet* Σ. A monoid $(M, \odot, e)$ is a *group* if the following additional condition is fulfilled: for every $m \in M$ there exists $m' \in M$ so that $m \odot m' = e = m' \odot m$. The element m' is unique with this property and is called the *symmetric* of m.

The set $\mathbb{Z}_{12} = \{0, 1, 2, \ldots, 10, 11\}$ with the clock addition

$$\begin{aligned} x \oplus y &= x + y \text{ if } x + y < 12 \\ &= x + y - 12 \text{ if } x + y \geq 12 \end{aligned}$$

constitutes a group, the group of modulo12 integers. Given monoids $(M, \odot, e)$ and $(M', \odot', e')$, a function $\phi : M \to M'$ verifying the laws

$$\phi(m_1 \odot m_2) = \phi(m_1) \odot' \phi(m_2) \quad \phi(e) = e', \quad \text{for all } m_1, m_2 \in M$$

is called a *morphism* of monoids. A morphism ϕ is an *epimorphism* (*isomorphism*) whenever ϕ is a surjective (bijective) function.

A monoid $(M, \odot, e)$ is *left cancellative* if it satisfies the condition

$$m \odot m_1 = m \odot m_2 \text{ implies } m_1 = m_2, \quad \text{for all } m_1, m_2, m \in M.$$

For a left cancellative monoid $(M, \odot, e)$ and $\alpha, \beta \in M$ a *left residual* of β by a is an element $\gamma \in M$ such that $a \odot \gamma = \beta$. If $\gamma' \in M$ also satisfies the equation $\alpha \odot \gamma' = \beta$, then $\alpha \odot \gamma = \alpha \odot \gamma'$ and so by left cancellation we obtain $\gamma = \gamma'$. This unique element (if exists) is denoted $\alpha^{-1}\beta$ and is called the *left residual of* β *by* α. In the additive case we adopt the *left difference* notation $\beta - a$. The next properties are immediate:

$$\alpha^{-1}\alpha = e, \; e^{-1}\beta = \beta, \; (\alpha_1 \odot \alpha_2)^{-1}\beta = \alpha_2^{-1}(\alpha_1^{-1}\beta), \quad \text{for all } \alpha, \alpha_1, \alpha_2, \beta \in M.$$

Right cancellative monoids are defined in a dual way. A monoid is said to be *cancellative* whenever it is both left and right cancellative.

Clearly, groups and free monoids are cancellative monoids. The set $\mathbb{N}^k$ of all k-tuples of natural numbers with the pointwise addition constitutes a cancellative monoid.

Every function from a free monoid to a cancellative monoid is said to be a *viewpoint*. A viewpoint $f : \Sigma^* \to M$ is *prefix preserving* whenever for all $s, t \in \Sigma^*$ it holds

$$f(st) = f(s) \odot \alpha_{s,t}, \quad \text{for some } \alpha_{s,t} \in M \text{ and } f(\varepsilon) = e.$$

The *residual*

$$s^{-1}f : \Sigma^* \to M \text{ is given by } (s^{-1}f)(t) = f(s)^{-1}f(st), \quad \text{for all } t \in \Sigma^*.$$

III. Sequential Transducers

We propose a sequential transducer model capable to represent musical functions that ordinary sequential transducers are unable to recognize. The output of that machine is assumed to be a cancellative monoid.

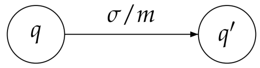
Formally, a *sequential transducer* is a *system*

$$\mathcal{M} = (\Sigma, (M, \odot, e), Q, i, K)$$

where

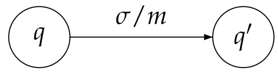
- Σ is the *input alphabet*
- $(M, \odot, e)$ is the *output cancellative monoid*
- Q is the *state set*
- $i \in Q$ is the *initial state* and

- K is a finite set of *transitions* of the form



$q, q' \in Q,\ \sigma \in \Sigma,\ m \in M$

with the property: for every state $q \in Q$ and every input letter $\sigma \in \Sigma$, there exists a unique pair $(m, q') \in M \times Q$ so that



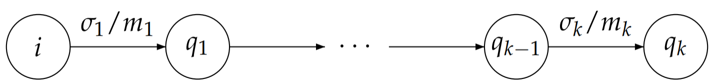
$\in K.$

The trivial transitions

$$q \xrightarrow{\varepsilon/e} q, \quad q \in Q$$

belong to K.

The *function* $f_{\mathcal{M}} : \Sigma^* \to M$ *determined by* $\mathcal{M}$ is obtained as follows: every input string $\sigma_1 \cdots \sigma_k \in \Sigma^*$ labels a unique path

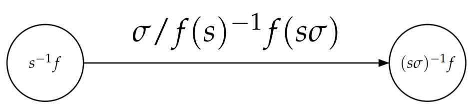


and we put $f_{\mathcal{M}}(\sigma_1 \cdots \sigma_k) = m_1 \odot \cdots \odot m_k$.

A function $f : \Sigma^* \to M$ is said to be *sequential* whenever $f = f_{\mathcal{M}}$ for some sequential transducer $\mathcal{M}$. There is a nice criterion to infer sequentiality.

Theorem III.1. *Let $(M, \odot, e)$ be a cancellative monoid. A prefix preserving function $f : \Sigma^* \to M$ is sequential, if and only if it has finitely many residuals.*

The proof follows the classical one, [10], [20]. In this case, the *minimal* sequential transducer $\mathcal{M}_f$ computing f can be effectively constructed: it has $Q_f = \{s^{-1}f \mid s \in \Sigma^*\}$ as state set, $\epsilon^{-1}f = f$ as initial state and its transitions are of the form



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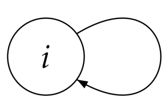
Transformations with simple minimal sequential transducers are monoid morphisms.

Clearly, every monoid morphism $h : \Sigma^* \to M$ is prefix preserving and all its left residuals coincide with h itself: $s^{-1}h = h$ for all $s \in \Sigma^*$. Indeed, for all $t \in \Sigma^*$ we have

$$(s^{-1}h)(t) = h(s)^{-1}h(st) = h(s)^{-1}(h(s) \odot h(t)) = h(t).$$

Thus, the minimal sequential transducer of h has a single state and its graph is

$\mathcal{M}_h :$



$\sigma/h(\sigma), \quad \sigma \in \Sigma.$

The transposition and inversion morphisms $T_t, I_t : \mathbb{Z}_{12}^* \to \mathbb{Z}_{12}^*$ given by $T_t(x) := x \oplus t$, $I_t(x) := -x \oplus t$ are represented by the first two graphs in Figure 3.

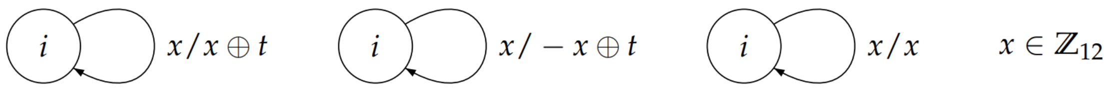


Figure 3: *The minimal sequential transducers of T_t, I_t, R.*

The *retrograde function*

$$R : \mathbb{Z}_{12}^* \to \mathbb{Z}_{12}^*, \quad R(x_k \cdots x_1) = x_1 \cdots x_k$$

satisfies the relation $R(st) = R(t)R(s)$, for all $s, t \in \mathbb{Z}_{12}^*$. Denoting by $\odot$ the opposite of the concatenation operation, $u_1 \odot u_2 = u_2 u_1$, the previous relation is written as $R(st) = R(s) \odot R(t)$, which means that R is a morphism from the monoid $\mathbb{Z}_{12}^*$ to the cancellative monoid $(\mathbb{Z}_{12}^*)^{opp}$.

Hence, its minimal sequential transducer is the third graph in Figure 3. Indeed, the output of the string $x_1 x_2 \cdots x_k$ is $x_1 \odot x_2 \odot \cdots \odot x_k = x_k \cdots x_2 x_1$ as asserted.

According to [8], a *multiple viewpoint* is a function assigning to each melody, a tuple of musical features

$$f : \Sigma^* \to \Sigma_1^* \times \cdots \times \Sigma_k^*.$$

One of the most familiar multiple viewpoints is the function $h : \mathbb{Z}_{12}^* \to \mathbb{N}^{12}$, which to every musical string $w \in \mathbb{Z}_{12}^*$ assigns the vector $(|w|_0, |w|_1, \ldots, |w|_{11})$ of numbers of $0'$s, $1'$s, …, $11'$s occurring in w. This function is a musical morphism and its minimal sequential transducer is depicted in Figure 4.

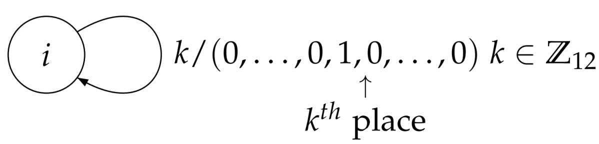


Figure 4: *The minimal machine of h.*

Another remarkable musical morphism is change in durations: we keep the same pitches and change the durations according to a function δ from a specific duration alphabet X into itself, thus imposing a new rhythm.

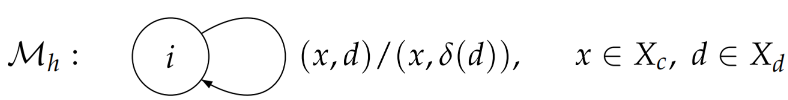


Figure 5: *Change in durations.*

The action of $\mathcal{M}_h$ according to the change of durations

$$3/4 \to 2/4, \quad 1/4 \to 1/8, \quad 2/4 \to 3/4$$

is depicted in the following example.

Figure 6: *A change of durations in Costas Nikitas' Duo for Violin and Piano*

Generally, musical morphisms are located at the first level of any hierarchy of musical functions.

IV. Musical Contours

In this section we show that any contour function with values in a cancellative monoid is sequential and we construct its minimal sequential transducer. Hierarchies with respect to transducer simulation concerning fundamental musical contour functions are provided.

Let Σ be a set of musical elements and $(M, \odot, e)$ be a cancellative monoid. Consider the contour $c : \Sigma \times \Sigma \to M$ and its associated function $f_c : \Sigma^* \to M$ defined by

- $f_c(\sigma_1\sigma_2 \cdots \sigma_k) = c(\sigma_1, \sigma_2) \odot c(\sigma_2, \sigma_3) \odot \cdots \odot c(\sigma_{k-1}, \sigma_k)$, $\sigma_i \in \Sigma$, $k \geq 2$
- $f_c(\sigma) = e = f_c(\epsilon)$, for all $\sigma \in \Sigma$ (ϵ the empty word).

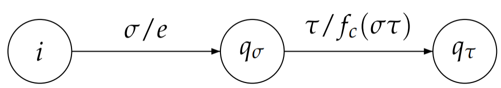
Often, by abusing notation, we say that f_c itself is a contour.

The function f_c is *prefix preserving* since for all $s, t \in \Sigma^*$, it holds

$$f_c(st) = f_c(s) \odot f_c(last(s)t) \text{ and so } s^{-1}f_c = last(s)^{-1}f_c,$$

where $last(s)$ designates the rightmost letter of s. The set of all residuals of f_c is finite, $\sigma^{-1}f_c(\sigma \in \Sigma)$, $\varepsilon^{-1}f_c = f_c$ and so f_c is sequential.

Its minimal sequential transducer has states $q_\sigma(\sigma \in \Sigma)$, $q_\epsilon = i$ (initial state), where for notation simplicity we have put q_σ instead of $\sigma^{-1}f_c$. The transitions are of the form



$\sigma, \tau \in \Sigma$

Figure 7: *Sequential transducer of a contour.*

Let X_c be the chromatic alphabet and consider the classical contour functions

$$f_1 : X_c \times X_c \to \{1, 0, -1\}^* \qquad f_2 : X_c \times X_c \to \{\ell, s, 0, -s, -\ell\}^*$$

$$f_3 : X_c \times X_c \to \{11, \ldots, 1, 0, -1, \ldots, -11\}^*$$

given by

- $f_1(x, y) = 1, 0, -1$ if y is located in X_c upwards, at the same level, downwards of x,
- $f_2(x, y) = s, -s$ (resp. $\ell, -\ell$) if y is located in X_c *one step* (resp. *more than one step*) upwards, downwards of x,
- $f_3(x, y) = k, -k$ if y is located in X_c k *semitones* upwards, downwards of x.

By applying f_2 and f_3 to the musical example of section I, we obtain

$$f_2(w) = s\ell(-\ell)s(-\ell)(-s)(-s)\ell(-\ell)(-s),$$

$$f_3(w) = 23(-5)2(-5)(-2)(-2)4(-7)(-2)$$

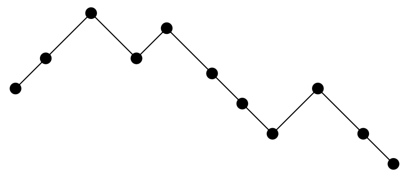


Figure 8: *Outline* $f_2(w)$.

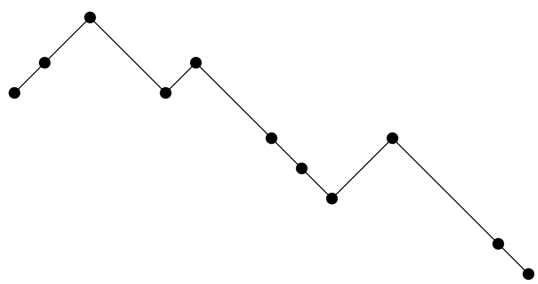


Figure 9: *Outline* $f_3(w)$.

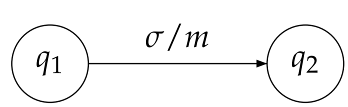
The above outlines simulate to t-norm / t-conorm-union and intersection of fuzzy (musical) sets, [3], [11].

The function $F : X_c^* \to \mathbb{N}^3$, $F(w) = (|f_1(w)|_1, |f_1(w)|_0, |f_1(w)|_{-1})$ gives information about the frequence of the ascending, horizontal, descending moves in a musical string. On the other hand the *move index* function $f_0 : X_c^* \to \mathbb{Z}$, $f_0(w) = |f_1(w)|_1 - |f_1(w)|_{-1}$ tells us whether the number of ascending moves is greater than the number of descending moves and vice versa, providing useful musical statistics for the analyst.

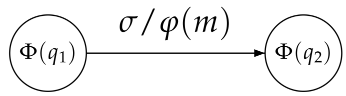
The minimal sequential transducer of f_1 was displayed in section 1. The machines $\mathcal{M}_2$, $\mathcal{M}_3$ of the contours f_2, f_3 are obtained from $\mathcal{M}_1$ by taking $a = \ell, s, 0, -s, -\ell$ and $a = 11, \ldots, 1, 0, -1, \ldots, -11$ respectively. Also, the machine $\mathcal{M}_F$ is obtained from $\mathcal{M}_1$ by taking $a = (1,0,0), (0,1,0), (0,0,1)$, according to $x < y$, $x = y$ or $x > y$ respectively. The above transducers are connected by simulation.

A *morphism* from $\mathcal{M} = (\Sigma, (M, \odot, e), Q, i, K)$ into $\mathcal{M}' = (\Sigma, (M', \odot', e'), Q', i', K')$ is a pair (Φ, φ) consisting of a state function $\Phi : Q \to Q'$ and a monoid morphism $\varphi : M \to M'$, so that

- $\Phi(i) = i'$ (preservation of initial states),
-

If 

is a transition in $\mathcal{M}$, then



is a transition in $\mathcal{M}'$.

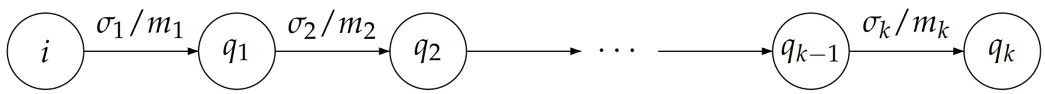
The functions computed by $\mathcal{M}$ and $\mathcal{M}'$ are strongly connected as next statement confirms.

Proposition IV.1. *Keeping the previous notation, we have*

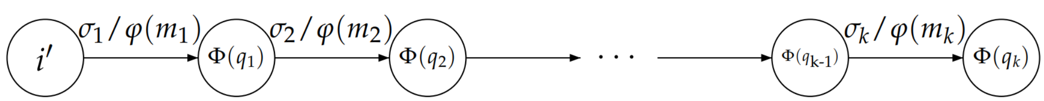
$$f_{\mathcal{M}'} = \varphi \circ f_{\mathcal{M}}$$

where $\circ$ *stands for the composition function performed from right to left.*

Proof. By definition, if



is a path in $\mathcal{M}$, then



is a path in $\mathcal{M}'$. Thus

$$\begin{aligned} f_{\mathcal{M}'}(\sigma_1\sigma_2\cdots\sigma_k) &= \varphi(m_1) \odot \varphi(m_2) \odot \cdots \odot \varphi(m_k) \\ &= \varphi(m_1 \odot m_2 \odot \cdots \odot m_k) \\ &= \varphi(f_{\mathcal{M}}(\sigma_1\sigma_2\cdots\sigma_k)) = (\varphi \circ f_{\mathcal{M}})(\sigma_1\sigma_2\cdots\sigma_k) \end{aligned}$$

hence the announced equality $f_{\mathcal{M}'} = \varphi \circ f_{\mathcal{M}}$. □

A morphism (Φ, φ) is said to be a *simulation* notation $\mathcal{M} \rhd \mathcal{M}'$, whenever both the functions Φ and φ are surjective. According to the above argument, only the states $\Phi(Q)$ and the elements of $\varphi(M)$ participate to the definition of $f_{\mathcal{M}'}$. Therefore, from machine point of view, only simulations are worthy of consideration.

Proposition IV.2. *The former contours are organized in a hierarchy*

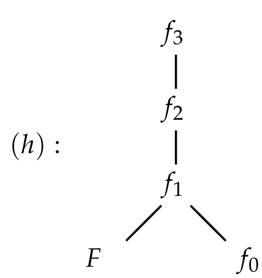


Figure 10: *Contour hierarchy.*

The notation $f\,/\,f'$ means $\mathcal{M}_f$ simulates $\mathcal{M}_{f'}$.

Proof. First observe that all the mashines in question share a common stateset and the state functions in the simulations above are the identity functions. Furthermore, $\mathcal{M}_3 \rhd \mathcal{M}_2 \rhd \mathcal{M}_1$ via the epimorphisms

$$\phi : \{-11, \dots, -1, 0, 1, \dots, 11\}^* \to \{-\ell, -s, 0, s, \ell\}^*, \;\; \phi(x) = \begin{cases} 0, & \text{if } x = 0 \\ \pm 1, & \text{if } x = \pm s \\ -\ell, & \text{if } x < -1 \\ \ell, & \text{if } x > 1 \end{cases}$$

and

$$\psi : \{-\ell, -s, 0, s, \ell\}^* \to \{-1, 0, 1\}^*, \;\; \psi(-\ell) = \psi(-s) = -1, \; \psi(\ell) = \psi(s) = 1, \; \psi(0) = 0.$$

Moreover, $\mathcal{M}_1 \rhd \mathcal{M}_F$ via the Parikh function $w \mapsto (|w|_{-1}, |w|_0, |w|_1)$ whereas $\mathcal{M}_1 \rhd \mathcal{M}_0$ via the epimorphism $\sigma_1 \cdots \sigma_k \mapsto \sigma_1 + \cdots + \sigma_k$.

To complete the proof we have to show that (h) is actually an ordered set. Indeed, no monoid epimorphism ϕ from the additive group $\mathbb{Z}$ of integers to the free monoid $\{-1, 0, 1\}^*$ exists, since

$$\phi(\mathbb{Z}) = \phi(\{n.(-1), n.1 \;/\; n \in \mathbb{N}\}) = \{\phi(-1)^n, \phi(1)^n \;/\; n \in \mathbb{N}\} \subsetneq \{-1, 0, 1\}^*.$$

Hence, $\mathcal{M}_0 \not\rhd \mathcal{M}_1$. Likewise, for every monoid morphism $\phi : \mathbb{N}^3 \to \{-1, 0, 1\}^*$ we obtain

$$\phi(\mathbb{N}^3) = \{\phi(1,0,0)^{n_1}\phi(0,1,0)^{n_2}\phi(0,0,1)^{n_3} \;/\; n_1, n_2, n_3 \in \mathbb{N}\} \subsetneq \{-1, 0, 1\}^*$$

and so $\mathcal{M}_F \not\rhd \mathcal{M}_1$.

The inequalities $\mathcal{M}_1 \not\rhd \mathcal{M}_2 \not\rhd \mathcal{M}_3$ come from the fact that any epimorphism of free monoids $\phi : \{x_1, \dots, x_m\}^* \to \{y_1, \dots, y_n\}^*$ does not increase rank, $m \geq n$. Indeed, by surjectivity, the strings $\phi(x_1), \dots, \phi(x_m)$ generate the free monoid $\{y_1, \dots, y_n\}^*$ and so each letter y_i is a concatenation of these strings. It turns out that $y_1 = \phi(x_{i_1}), \dots, y_n = \phi(x_{i_n})$, with $x_{i_1}, \dots, x_{i_n}$ pairwise distinct, i.e. $m \geq n$. □

Now, let us remind some auxiliary matter. Any function $f : A \to B$ defines an equivalence w_f on the set A by setting

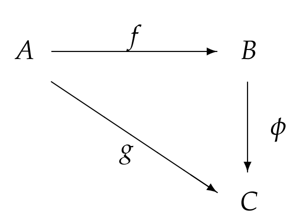
$$a_1 \equiv a_2(w_f) \iff f(a_1) = f(a_2).$$

Given equivalences w, w' on a set A we say that w is *thinner* than w', $w \setminus w'$, whenever

$$a_1 \equiv a_2(w) \;\text{ implies }\; a_1 \equiv a_2(w').$$

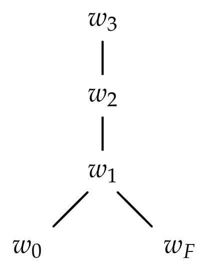
This means that any class of w' is a union of classes of w.

Consider the commutative triangle



$$g = \phi \circ f.$$

If f is a surjection, then $w_f \setminus w_g$. Putting together propositions IV.1, IV.2 and the previous discussion we get the following classification of equivalences on the set X_C^*



where w_i, w_F are the Kernel equivalences of the functions f_i, F.

V. Conclusion

Fundamental musical functions are encoded in sequential transducers. The encoding is realized by assigning to any musical function its minimal sequential transducer. A machine simulation based hierarchy of musical contours and the corresponding classifications of musical languages are provided. Our future intension will be to recognize function / transducer situations in Lewin's Generalized Interval Systems Theory, [12], as well as in self similarity theory [18].

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