An Overview of Scale Theory via Word Theory: Notes and Words, Commutativity and Non-Commutativity

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Abstract: *About a decade ago, scale theory in music was invigorated by methods and results in the mathematical subfield of combinatorics on words. This paper reviews some of the insights stemming from this application. In particular, the interplay between notes and words is explored: the* words *of the theory capture musical intervals, but they conveniently "forget" the notes that constitute the intervals. For the purposes of mathematical music theory, this level of abstraction is usually convenient indeed, but reinstating the notes that give life to the musical interpretation in turn enriches the formal theory. Consideration of this dichotomy leads to the mathematical opposition of commutativity vs. non-commutativity, in terms of the relations between the non-abelian free group F_2 and the additive abelian group $\mathbb{Z}^2$, and between the free non-commutative and commutative monoids embedded, respectively, in those groups.*

Keywords: *Christoffel words, conjugation, Sturmian morphisms, well-formed scales*

I. Introduction

Word theory studies sequences of elements drawn from a set or *alphabet*; such sequences, finite or infinite, are what are meant by *words*. From around the 1980s, the subfield called algebraic combinatorics on words brought various mathematical disciplines to bear on these objects, with a great deal of growth in the past three decades. This happens to coincide with the flourishing of formal studies of musical scales, which proceeded largely in ignorance of the relevant work in mathematics and computer science. While the joining of word theory to scale theory within mathematical music theory is a fairly recent development, there were some applications to music from the beginning, principally in studies of rhythm and in certain compositional models. An historical survey may be found in [1], an editorial by the three guest editors of a recent special issue of *Journal of Mathematics and Music* devoted to music and combinatorics on words. Four articles ([2], [3], [4], [5]) in this special issue address the present state of scale theory in light of word theory (which in this article refers to algebraic combinatorics on words, given an exposition in the pseudonymous Lothaire series, [6], [7], [8]).

*Thank you to my colleagues in these studies, Norman Carey and Thomas Noll.

In the scale theory of the 1980s, the focus was typically on properties of certain distinguished pitch-class sets, drawn from the usual 12-tone equal-tempered universe (or, generalizing, from some equal-tempered universe of n pitch classes, modeled by $\mathbb{Z}_n$). As such, although the canonical examples qualified in some ill-defined sense as musical "scales," formally the objects studied were simply pitch-class sets, that is, subsets of $\mathbb{Z}_{12}$, or $\mathbb{Z}_n$. Although scale order was typically adduced, it was a cyclic order: no tonic or modal final was posited. By convention, the representative set is its prime form, which for the usual diatonic (Forte's set-class 7-35) is $\{0, 1, 3, 5, 6, 8, 10\}$.[1]

For the purposes of the present article the germinal 1985 work was [9], in which Clough and Myerson studied generalizations of the usual diatonic pitch-class set, which they called *diatonic systems*. To define diatonic systems they generalized the property that generic non-zero diatonic pitch-class intervals come in two specific sizes (generic diatonic seconds are specifically major or minor; similarly, generic thirds; generic fourths are specifically perfect or augmented, etc.), calling this Myhill's Property (MP). They proved that all diatonic systems, i.e., those with MP, of cardinality N, possess the apparently stronger but in fact equivalent property Cardinality equals Variety (CV), wherein ordered sets of cardinality $1 \leq k \leq N$ of a given generic description come in k specific varieties (e.g., diatonic triads, formed from 3 notes separated by two generic thirds, come in the 3 species major, minor, and diminished); for details, see [9]. Among other properties, they showed that diatonic systems admit a *generating interval* or generalized fifth, such that all of the pitch classes of the system may be ordered in a chain in which adjacent elements are separated by the generating interval. In the case of the usual diatonic, the generating interval is either the perfect fifth or perfect fourth. In the prime form of 7-35 the chain of perfect-fourth related pitch classes is 0, 5, 10, 3, 8, 1, 6; that is, the set is generated by $5j$ mod 12, $j = 0, 1, \ldots, 6$. Clough and Myerson derive an equivalent algorithm to generate the pitch classes in scale order (ascending in the usual order for integers), by taking floors of multiples of $\frac{12}{7}$: $\lfloor j\frac{12}{7} \rfloor$, $j = 0, 1, \ldots, 6$. In general, for a chromatic universe of cardinality n and an embedded diatonic system of coprime cardinality $N < n$, the algorithm is $\lfloor j\frac{n}{N} \rfloor$, $j = 0, 1, \ldots, N-1$.

Observe that in this presentation, two orderings of the pitch classes, that is, the musical *notes*, are set in relief. The scale order comes into play because the generic interval notion uses the scale itself as a rough measurement, with the number of steps of the scale determining the generic interval, without regard for the quality or specific interval. The other ordering principle is that given by the generating interval. The relationship between these two orderings is essential to the construction, as we will see. For now, note that the generic measure for a cardinality N diatonic system is modeled by the cyclic group $\mathbb{Z}_N$ with addition modulo N. Then the generating-interval ordering is the image of an injective map from the scale ordering into the chromatic universe. In the case of the usual diatonic $f : \mathbb{Z}_7 \to \mathbb{Z}_{12} : z \to f(z) = 5z$ mod 12 is injective, preserving the order of ordinary arithmetic. Since f is injective, and since $5^{-1} = 5$ mod 12, multiplication by 5 modulo 12 applied to the image of f returns the generic scale order, 0, 1, . . . , 6. This relationship is not explicit in [9], but it is implicit. Again, neither ordering confers a privileged status on a particular musical mode. We might consider that mathematically the orderings beginning with 0 should have priority, but musically we might resist that suggestion, since in the case of the usual diatonic the algorithms yield the (ill-favored) Locrian mode

In 1989, Carey and Clampitt in [10] took the relationship between the orderings as a point of departure, and generalized beyond rational generators, to define *well-formed scales*. Instead of taking

[1]The style of this article is rather discursive for a mathematical study, rather formal and algebraic for a music theory work. In order to avoid tedious and awkward formulations, certain musical concepts are taken as known, and similarly mathematical terminology and concepts that are common in mathematics are introduced without comment. No knowledge of word theory is assumed. Because this is in the nature of an overview, quite a bit of the development refers to recent literature.

MP as an axiom, they considered generated sets, and wished to determine which scales had desired properties, again taking the usual diatonic as a model. They allowed the generating interval to be either rational or irrational with respect to the octave, normalized as unity. That is, mathematically the situation is modeled by $\mathbb{R}/\mathbb{Z}$, with addition modulo 1. In this environment, we may generate the twelve-tone equal-tempered usual diatonic by taking fractional parts of multiples of $\frac{7}{12}$, representing the equal-tempered perfect fifth. If we notate the fractional part of a real number x as $\{x\} = x - \lfloor x \rfloor$, then the equal-tempered diatonic is modeled by $\{j\frac{7}{12}\}, j = 0, 1, \ldots, 6$; in scale order $0 < \{2(\frac{7}{12})\} = \frac{1}{6} < \{4(\frac{7}{12})\} = \frac{1}{3} < \{6(\frac{7}{12})\} = \frac{1}{2} < \{1(\frac{7}{12})\} = \frac{7}{12} < \{3(\frac{7}{12})\} = \frac{3}{4} < \{5(\frac{7}{12})\} = \frac{11}{12}$. But we might also generate the Pythagorean diatonic by fractional parts of multiples of the just perfect fifth, represented by the irrational $\log_2\frac{3}{2}$: $\{j(\log_2\frac{3}{2})\}$, $j = 0, 1, \ldots, 6$; in scale order $0 < \{2(\log_2\frac{3}{2})\} < \{4(\log_2\frac{3}{2})\} < \{6(\log_2\frac{3}{2})\} < \{1(\log_2\frac{3}{2})\} < \{3(\log_2\frac{3}{2})\} < \{5(\log_2\frac{3}{2})\}$. What characterizes both scales as well-formed diatonic scales is that the transformation on the index set $\mathbb{Z}_7$ representing the successive perfect fifth multiples to their position in scale order is the automorphism of the additive cyclic group $\mathbb{Z}_7 \to \mathbb{Z}_7$: $z \to 2z \bmod 7$. The definition of a well-formed scale is therefore: a generated set such that the mapping from the set in generation order to the set in scale order is a group automorphism of the cyclic group $\mathbb{Z}_n$ under addition modulo n. It follows that, equivalently, a generated set is well-formed if and only if the mapping of the set in scale order to the set in generation order is a group automorphism of the underlying cyclic group.

Since the n-note equal division of the octave (or whatever the interval of periodicity is) may be generated by any integer g coprime with n, it follows that the scale is well-formed. These cases are the *degenerate* well-formed scales; all other cases are non-degenerate, and unless otherwise specified, well-formed is by default non-degenerate.

The answer to the question of which scales, for a given generator, are well-formed was answered in [10]: all and only those generated scales with cardinalities equal to the denominator of a convergent or semi-convergent in the continued fraction representation of the size of the generator. For the just perfect fifth, $\log_2(\frac{3}{2})$, this sequence is 1, 2, 3, 5, 7, 12, 17, 29, 41, 53, 94, ... Among the musically significant scales in this hierarchy are the *tetractys* or tonic-subdominant-dominant roots, the usual pentatonic, the usual diatonic, and the chromatic, distinguishing between diatonic and chromatic semitones. Given any generating interval θ such that $4/7 < \theta < 3/5$, the continued fraction will begin with denominators 1, 2, 3, 5, 7, 12. In the case where $\theta = 7/12$, the final scale in the hierarchy is the degenerate well-formed 12-note equal division of the octave. The mathematical proof of the equivalence of well-formedness and the continued fraction property, and also the proof that a generalized MP (allowing for irrational specific interval sizes) is equivalent to non-degenerate well-formedness, are found in [11] and [12].

For the usual diatonic, we can establish the formal equivalence of scales in all tunings for generating intervals θ within the limit set above. As we have seen, the scales in this class are characterized by the mapping $\mathbb{Z}_7 \to \mathbb{Z}_7$: $z \to 2z \bmod 7$, so that the scale order in terms of the perfect fifths generation order is the sequence 0 2 4 6 1 3 5 (0). While generic step intervals are all represented as differences modulo 7 between adjacent elements of the sequence, i.e., 2 mod 7, the sequence of specific step intervals is represented by the sequence of differences in ordinary arithmetic between adjacent elements, 2 2 2 -5 2 2 -5. This is a way of encoding the sequence of tones and semitones of the Lydian mode, $TTTSTTS$, but so far in scale theory no particular rotation of this word was privileged over another, so in this first step toward word theory, the word $TTTSTTS$ was construed as a *circular word*, that is, as the equivalence class of all seven distinct rotations of the word. In word theory generally, though, the seven distinct rotations, or *conjugates* as they are referred to in this field, are considered distinct *words*, opening a possible avenue to a study of the diatonic modes, and to the modal varieties of well-formed scales in general. That there are N distinct modes of a (non-degenerate) well-formed scale of cardinality N

is a known consequence of CV, which follows from generalized MP, but it was not obvious that simply encoding the modes in N distinct words would lead to a more refined modal knowledge.

II. A Modal Refinement Through Word Theory

The subject matter of combinatorics on words arose, but only implicitly, in two 1996 articles. In [13], the concept of a *region* was introduced, equivalent to the *central word*, and in [14], an infinite word was constructed, which was equivalent to a *Sturmian word* (in this case of slope $\sqrt{2}$). In the latter article, equivalence classes of well-formed scales were defined as words characterized, up to rotation, by cardinality N and by the multiplicity of one of the step intervals, g. Thus, all possible tunings of the usual diatonic are in the equivalence class of all rotations of the word *aaabaab*, determined by the ordered pair $(N, g) = (7, 2)$. Even a dual class to the diatonic, all rotations of *abababb*, determined by $(N, g^{-1}_{\text{mod } N}) = (7, 4)$, was defined, but there was no acknowledgment yet that these objects had been defined abstractly within mathematics. The applications of word theory to musical scale theory began in 2007, at the First International Conference of the Society for Mathematics and Computation in Music in Berlin in May, and at the Sixth International Conference on Words (WORDS 2007) in Marseille-Luminy in September.

i. The monoid of words over an alphabet

The *monoid of words* over an alphabet A is the set A^* of all finite sequences or strings of letters drawn from a finite set of symbols or *alphabet* A: $A^* = \{w = w_1 \dots w_n | w_i \in A, i = 1, \dots, n, n \in \mathbb{N}\}$. The associative monoid operation is concatenation of words, and it is understood that the empty word ε is in A^* and is an identity. A^* is thus also a semigroup with identity. If $u, v \in A^*$ with $w = uv$, then we say that u and v are *factors* of w; ε is a factor of any word w. A *prefix* of a word is a factor that begins that word, and a *suffix* of a word is a factor that ends that word. The *length* $|w|$ of a word is the number of letters it contains: if $w = w_1 w_2 \dots w_k$, $|w| = k$. Another notation: given a letter ℓ from our alphabet A, $|w|_\ell$ refers to the number of occurrences of the letter ℓ in the word w.

Note that the monoid A^* is free. Two words $u, v \in A^*$ commute if and only if they are powers of the same word: $uv = vu \iff u = t^j, v = t^k$ for some $t \in A^*, j, k \in \mathbb{N}$. For example, if $A = \{a, b\}, u = aab, v = (aab)^2 = aabaab$, then $uv = vu = (aab)^3$.

For example, our alphabet might be $A = \{a, b, c\}$, and it might be convenient to consider A to be ordered, for example, $a < b < c$. If we wished to model the Japanese *hira joshi* mode, E-F-A-B-C-(E′), we might represent its sequence of scale step intervals by the word *acbac*, where $a = 1$ semitone interval, $b = 2$ semitones interval, and $c = 4$ semitones interval. Evidently, the information about the particular notes specifying this mode is lost in the representation as a word; it follows that this word with this interpretation may represent any or all musical transpositions of this mode. But different modes of this scale are represented by different words, different cyclic rotations of *acbac*, or as word theorists prefer to say, different *conjugates* of the word. Two words w and w' in A^* are *conjugate* (or conjugates of each other) if $w = uv$ and $w' = vu$ for some $u, v \in A^*$. The full set of conjugates of a word w is called its *conjugacy class*, and this set defines the *circular word* (w). The conjugacy class of $acbac = \{acbac, cbaca, bacac, acacb, cacba\}$. It should be evident that conjugacy is an equivalence relation on A^*. Note that for any of the conjugates r, we have $|r|_a = 2, |r|_b = 1, |r|_c = 2$.

ii. Sturmian endomorphisms

Our words for well-formed scales will be over two-letter ordered alphabets, $A = \{a < b\}$ or $A = \{x < y\}$. Any mapping $f : A^* \to A^*$ that replaces every occurrence of a with the word $f(a)$ and every occurrence of b with the word $f(b)$ is by definition an endomorphism of the monoid A^*: if $w = w_1 w_2 \ldots w_n \in A^*, f(w) = f(w_1 w_2 \ldots w_n) = f(w_1) f(w_2) \ldots f(w_n)$. It follows that, for any $u, v \in A^*, f(uv) = f(u)f(v)$, i.e., f is an endomorphism.

The distinguished endomorphisms (henceforth *morphisms*) of A^* are the Sturmian morphisms St in [7]. The generators of St are defined as follows:

$G(a) = a$	$\tilde{G}(a) = a$	$D(a) = ba$	$\tilde{D}(a) = ab$
$G(b) = ab$	$\tilde{G}(b) = ba$	$D(b) = b$	$\tilde{D}(b) = b$

$$E(a) = b, E(b) = a.$$

The set of all compositions of these morphisms (together with the identity mapping) forms the monoid St under composition of mappings. The submonoid St_0 of special Sturmian morphisms excludes the exchange morphism E, and is generated by the remaining four morphisms. Neither St nor St_0 are freely generated. For one thing, G and $\tilde{G}$ commute, and D and $\tilde{D}$ also commute. Moreover, the special Sturmian morphisms have the following presentation ([7], [15]):

$$St_0 \cong \left\langle G, \tilde{G}, D, \tilde{D} \quad | \quad GD^k\tilde{G} = \tilde{G}\tilde{D}^k G, DG^k\tilde{D} = \tilde{D}\tilde{G}^k D \text{ for all } k \in \mathbb{N} \right\rangle$$

.

Certain pairs of these morphisms generate distinguished free submonoids of St_0: the standard morphisms, $\langle G, D \rangle$; the anti-standard morphisms, $\langle \tilde{G}, \tilde{D} \rangle$; the Christoffel morphisms, $\langle G, \tilde{D} \rangle$; and the anti-Christoffel morphisms, $\langle \tilde{G}, D \rangle$.

Since the morphisms are completely defined by their actions on the single letters a and b, we must begin by applying them to either the word ab or ba. ab is considered to be a *Christoffel* word, because beginning with a single letter, $\tilde{D}(a) = ab$ and $G(b) = ab$; either way ab is the image under a Christoffel morphism. Similarly, ba is considered to be an *anti-Christoffel* word (and we classify other words as standard or anti-standard in the same way). As a matter of convention, we choose the Christoffel word ab as root word. We apply compositions of Sturmian morphisms to the root word ab, and in order to keep track separately of the images of a and b, we often introduce a divider symbol, writing "$a|b$" or, in this article, "a, b".

St and St_0 enter into mathematical music theory by offering another means of generating all and only well-formed scales, and moreover to address the modal identities. This approach also ratifies the distinction in the history of music theory between authentic and plagal modes. A few examples will illustrate these matters.

In the musical interpretation, we understand the root word a, b as expressing an octave, divided into perfect fifth and perfect fourth, a and b, respectively. This is known in modal theory as, going back to medieval chant, the authentic division of the octave (perfect fifth plus perfect fourth above the modal final). Just as ab is privileged over ba in the construction of the mathematical theory, in the history of modal theory, the authentic division is privileged over the plagal division (perfect fourth below the final, plus perfect fifth above the modal final). The final (*finalis*) is a *note*; as such it is has no role in word theory, but it is essential to the musical interpretation. Similarly, we may regard the divider symbol as standing in for a musical note: in the C-G-(C') authentic division of the octave, we say that note G is the *divider*. If we apply morphism D to a, b, we have $D(a, b) = ba, b$. In the interpretation, the meaning of the letter b continues to be the interval perfect fourth, because D leaves b fixed, but a now represents the diatonic whole step, the difference between a perfect fifth and perfect fourth, as D replaces a by ba, perfect fourth followed by whole step. The word ba, b models, then, the ancient Greek *tetractys*, C-F-G-(C′). Here, C is again the final and G is the divider. Applying $\tilde{D}$, on the other hand, we have $\tilde{D}(a, b) = ab, b$, representing the other mode of the *tetractys*, divided authentically. Remark that ba, b may be considered a standard word and ab, b may be considered a Christoffel word. If we prepend G to compose it with D, we obtain the (now unambiguously) standard word $GD(a, b) = G(ba, b) = aba, ab$. This may be interpreted as the usual pentatonic scale, in the mode C-D-F-G-A-(C′). Under the action of G, a retains its meaning (fixed under G), while now the interpretation of b as perfect fourth is replaced by the (diatonic) minor third, which is a pentatonic step interval. Composing G with $\tilde{D}$, and composing $\tilde{G}$ with both D and $\tilde{D}$ we exhaust the possibilities, and the images of a, b form standard, Christoffel, anti-Christoffel, and anti-standard words, four of the five conjugates in the conjugacy class, representing four of the five modes of the usual pentatonic.

In his 1547 treatise *Dodecachordon*, Glarean expanded the system of diatonic modes, adding authentic and plagal Ionian and Aeolian modes, an increase from eight to twelve modes. Thus, six of the seven octave species are represented, each in authentic and plagal form. For example, authentic Dorian is divided $TSTT, TST$, or in our alphabet A, $abaa, aba$. The remaining modal possibilities (modern Locrian) do not support authentic and plagal divisions and were rejected by Glarean: (*hyperaeolius reiectus* or diminished fifth plus augmented fourth; *hyperphrygius reiectus* or augmented fourth plus diminished fifth).

The twelve Glarean modes are captured by elements of St as images of the root Christoffel word ab: the six authentic modes as images of special Sturmian morphisms, the six plagal modes as images of ba under the same morphisms; in other words, images of ab of morphisms in the complement of St_0 (the six morphisms associated with the authentic modes but preceded by E; see below). The rejected Locrian modes are also excluded in this environment: they are associated with the “bad conjugate" of the word theorists (the divided words in the conjugacy class of a Christoffel word that are not the images of a, b nor of b, a under an element in St_0; [16]). We refer to words such words as *amorphic*; words that may be realized as images of these morphisms are *morphic*. To reiterate, the authentic and plagal divisions are entailed by separating the images of a and b. For example, the authentic Ionian pattern of step intervals is captured transformationally by the standard morphism $GGD(a, b) = GG(ba, b) = G(aba, ab) = aaba, aab$. Word theorists refer to the division as a *standard factorization*, $f(a)f(b)$, for some special Sturmian morphism f (different meaning here of “standard”)[18]. The transformational representations with their traditional modal finals and modern names are presented below (keeping the collection fixed as the “white notes” or C-major set; see [17] for details).

Transformation	Mode
$GGD(a, b) = aaba, aab$	C authentic Ionian
$\tilde{G}GD(a, b) = abaa, aba$	D authentic Dorian
$\tilde{G}\tilde{G}D(a, b) = baaa, baa$	E authentic Phrygian
$GG\tilde{D}(a, b) = aaab, aab$	F authentic Lydian
$\tilde{G}G\tilde{D}(a, b) = aaba, aba$	G authentic Mixolydian
$\tilde{G}\tilde{G}\tilde{D}(a, b) = abaa, baa$	A authentic Aeolian

Transformation	Mode
$GGDE(a,b) = aab, aaba$	C plagal Ionian
$\tilde{G}GDE(a,b) = aba, abaa$	D plagal Dorian
$\tilde{G}\tilde{G}DE(a,b) = baa, baaa$	E plagal Phrygian
$GG\tilde{D}E(a,b) = aab, aaab$	F plagal Lydian
$\tilde{G}G\tilde{D}E(a,b) = aba, aaba$	G plagal Mixolydian
$\tilde{G}\tilde{G}\tilde{D}E(a,b) = baa, abaa$	A plagal Aeolian

One may observe that the musical interpretations of ab, $D(ab)$, $GD(ab)$, and $GGD(ab)$ yield modes of well-formed scales in the hierarchy of such scales generated by the perfect fifth (to within some reasonable size) modulo the octave, those of cardinalities 2, 3, 5, 7. If one continues with $DGGD(ab)$, the well-formed 12-note chromatic follows. Indeed, we may identify modes of well-formed scales with the conjugacy classes of words generated in this fashion, as discussed in [17]. Where well-formed scales are defined by considering the distribution of integer multiples of a real number modulo 1, the same scales appear in the guise of words, which model the filling in of larger step intervals by smaller ones at each step. The procedures appear different, but they are equivalent.

So far the words representing modes of well-formed scales have been defined as the images of the minimal root word ab under Sturmian morphisms. One might well ask how to characterize such words more directly. One may also ask if there is a mathematically privileged member of the conjugacy class, to serve as an identifier for the class. A geometric definition, point of departure in [18], provides affirmative answers. In Figure 1 we see a construction which yields what Berstel et al. call lower and upper Christoffel words [18]; the terminology is adjusted here to be consistent within this article. Note that the word encoded by the lower path is *aaabaab*, the word representing Lydian mode, which was the mode suggested by the well-formed scale construction of the diatonic set generated by perfect fifths; see Section I. The word encoded by the upper path is *baabaaa*, the word representing Locrian mode, which was the mode suggested by the Clough-Myerson algorithm (or by the well-formed scale construction, taking perfect fourth as generator).

To integrate this geometric construction with the transformational perspective, consider a refinement of Figure 1 which considers the points on the paths, lower and upper, that approach closest to the line segment of slope 2/5. Make that point the divider of the respective words, and the lower word is $aaab, aab$, or the result of $GG\tilde{D}(a,b)$, a Christoffel word, representing authentic Lydian; the upper word is $baa, baaa$, or the result of $\tilde{G}\tilde{G}DE(a,b)$, an anti-Christoffel word (anti-Christoffel morphism applied to anti-Christoffel word b, a), representing plagal Phrygian.

Note that the upper and lower words are reversals of each other. The general Christoffel/anti-Christoffel word construction follows the example: given positive coprime integers p, q, one constructs the line segment of slope q/p from the origin to point (p, q), and the lower and upper polygonal paths through the nearest lattice points to the line segment, such that no lattice points are included within the region enclosed by the paths. The words are encoded over a two-letter alphabet; by convention, the lexicographically first letter is assigned to the horizontal line segments, the second letter to the vertical line segments. It follows that lower Christoffel words are lexicographically least of their conjugacy class ([18]). The words w are of length $|w| = N = p + q$, with $|w|_a = p$, $|w|_b = q$. The shortest non-trivial Christoffel/anti-Christoffel pair are those of slope 1, ab and ba, respectively. The trivial cases are the word of slope 0, a, and the word of slope ∞, b. Words defined by irrational slopes are infinite words of minimal complexity, called *Sturmian*. We won't discuss infinite words in this article (members of A^* are finite).

Foreshadowing the focus of this article, that is, commutativity vs. non-commutativity, we can propose that the preference in word theory for the lower, Christoffel word over the upper,

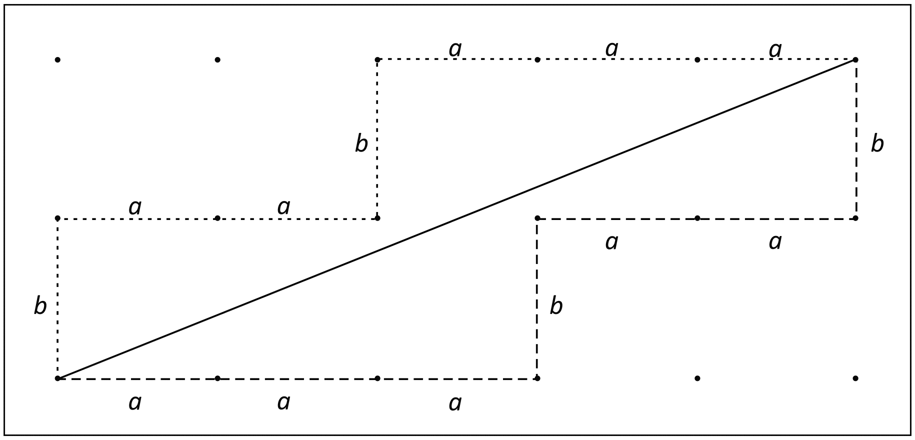

Figure 1: *Lower/(upper) Christoffel/(anti-Christoffel) words of slope $q/p = 2/5$, defined by paths (dashed and dotted, respectively) from the origin $(0,0)$ to $(5,2)$ that connect points of the lattice such that no points of the lattice lie in the region of the plane between the respective paths and the line segment of slope $2/5$. The words are encoded by labeling horizontal unit line segments a and vertical unit line segments b. These are also sometimes referred to as* digitized line segments.

anti-Christoffel word, has a basis in mathematics, part and parcel of a preference for special Sturmian morphisms over those in the complement of St_0 in St. This latter preference mirrors the music-theoretical bias towards the authentic modes over the plagal modes, as reflected in the very terminology. We will introduce a matrix associated with an element f of St, its *incidence matrix*, M_f, defined in terms of $f(a,b)$.

$$M_f = \begin{pmatrix} f(a)_a & f(b)_a \\ f(a)_b & f(b)_b \end{pmatrix}$$

Details of this construction will be given below, but for now consider $M_{GG\tilde{D}}$ and $M_{\tilde{G}\tilde{G}DE}$. Since $GG\tilde{D}(a,b) = aaab, aab$ and $\tilde{G}\tilde{G}DE(a,b) = baa, baaa$ we have:

$$M_{GG\tilde{D}} = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$$

and

$$M_{\tilde{G}\tilde{G}DE} = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}.$$

The mathematical distinction is that $det(M_{GG\tilde{D}}) = 1$, whereas $det(M_{\tilde{G}\tilde{G}DE}) = -1$. In other words, $M_{GG\tilde{D}}$ is in the special linear group $SL(Z)_2$. This is the motivation for the qualification *special* for St_0. The incidence matrices of elements in St_0 have determinant 1, while the incidence matrices of elements in St in the complement of St_0 have determinant -1. This distinction, in conjunction with the geometric construction of the lower Christoffel word, is the motivation for the word theory terminological choice of Christoffel to designate this class of words, and for the Christoffel word of a given slope to stand in for the conjugacy class of that word. In this article then, when the authentic/plagal distinction is at issue, the focus will be on the authentic side.

III. Christoffel Duality and Scale Foldings

In order to better understand distinctions among the modes, we triangulate between the words representing their patterns of step intervals and words to be introduced in this section called *scale foldings*. The latter follow what we earlier called the generation order for a well-formed scale, but in a way that is determined by the mode.

The usual diatonic may be generated by either the perfect fifth or the perfect fourth. Generalizing to well-formed scales generated modulo 1, the generator may be $\theta \in \mathbb{R}$, $0 < \theta < 1$, or $1 - \theta$. Translated into musical terms, we may generate the same (C-major) collection, either by rising perfect fifths (falling perfect fourths), departing from F, or by rising perfect fourths (falling perfect fifths), departing from B: F-C-G-D-A-E-B, or B-E-A-D-G-C-F. Whereas we by convention choose to conceive of and notate our scales as ascending, there is not necessarily a musical reason to prefer fifths to fourths, or, equivalently, rising perfect fifths to descending perfect fifths. Will the mathematics again tell us which to prefer?

In [17], scale foldings for the modes of usual diatonic were defined as follows (the generalization for modes of all well-formed scales follows this model). Given a diatonic mode, understood as an octave species, i.e., without the authentic/plagal distinction, within a fixed C-major collection, a *forward folding* is the sequence of upward perfect fifths and downward perfect fourths, departing from note F within the modal octave, such that all notes of the mode lie within the modal octave, including modal final and excluding the octave above the modal final. For example, for D-Dorian, from the F4 within the D4 to D5 octave one can extend a rising perfect fifth to C5, but then one is forced down a perfect fourth to G4, again down a perfect fourth to the modal final D4, up a perfect fifth to A4, down a perfect fourth to E4, up to B4. From B4 we complete the folding with a downward perfect fourth to the excluded note F♯4 (just as to complete the pattern *abaaaba* of scale step intervals for Dorian we need the excluded upper octave D5: D4-E4-F4-G4-A4-B4-C5-(D5). We choose a different alphabet to encode the forward folding word, with x for rising perfect fifth and y for falling perfect fourth. In the case of Dorian, the forward folding word is thus *xyyxyxy*. Proceeding through the modes in similar fashion, we have a one-to-one mapping between words in the conjugacy class of words representing scale step interval patterns and words representing the respective forward folding patterns (generally a different conjugacy class). Both conjugacy classes are the conjugacy classes of Christoffel words. Referring back to the opening paragraph of Section II, for the usual diatonic these are the conjugacy classes for the dual well-formed diatonic scale classes $(N, g) = (7, 2)$ and $(N, g^{-1}_{\text{mod } N}) = (7, 4)$, the conjugacy classes of lower Christoffel words *aaabaab* of slope 2/5 and *xyxyxyy* of slope 4/3, respectively. On the other hand, given a diatonic mode, the *backward folding* considers the generation by perfect fourth, departing from note B within the modal octave, such that all notes of the mode lie within the modal octave, again including the modal final and excluding the octave above the modal final. Again, for example in D-Dorian, from B4 within the B4 to B5 octave, one finds E4 a perfect fifth below B4, then a rising perfect fourth from E4 to A4, then a descending perfect fifth to the modal final, D4, then ascending perfect fourth to G4, again an ascending perfect fourth to C5, descending perfect fifth to F4, and finally ascending perfect fourth to (excluded) B♭. Encoding the backward folding word again over the alphabet $A = \{x, y\}$, now with x representing descending perfect fifths and with y representing ascending perfect fourths, the backward folding word for Dorian is *xyxyyxy*. This word is different from the forward folding word for Dorian, but it is a member of the same conjugacy class. If we go through the backward folding words for the diatonic modes, we again exhaust the conjugacy class of the lower Christoffel word of slope 4/3.

Table 1 aligns the modes, in circle of fifths order, with their modal scale words, and with their associated forward and backward folding words. Observing the sequence of modal scale

Table 1: *The diatonic modes, their scale words, and associated forward and backward folding words*

final	mode	scale word	forward folding word	backward folding word
F	Lydian	*aaabaab*	*xyxyxyy*	*yxyxyxy*
C	Ionian	*aabaaab*	*yxyxyxy*	*xyxyxyy*
G	Mixolydian	*aabaaba*	*yyxyxyx*	*yxyxyyx*
D	Dorian	*abaaaba*	*xyyxyxy*	*xyxyyxy*
A	Aeolian	*abaabaa*	*yxyyxyx*	*yxyyxyx*
E	Phrygian	*baaabaa*	*xyxyyxy*	*xyyxyxy*
B	Locrian	*baabaaa*	*yxyxyyx*	*yyxyxyx*

words, we see that circle of fifths order coincides with lexicographic order (with $a < b$), beginning with the lower Christoffel word of the conjugacy class. The sequences of the associated folding words are rotations by one position, rotations by one to the left for forward foldings, one to the right for backward foldings. To be more precise, and to motivate the monoid terminology of conjugates/conjugacy, we extend the monoid $A^* = \{x, y\}^*$ to the free group on two letters, F_2. With the adjunction of the inverse letters, $F_2 = \{x, y, x^{-1}, y^{-1}\}^*$, becomes a group, again with concatenation of words as the closed, associative product, the empty word ε as group identity, and usually with the understanding that group elements are represented by reduced words, wherever a letter and its inverse are adjacent and may be canceled. We define *conjugation by* u for $u \in F_2$ by $conj_u : F_2 \to F_2 : w \to u^{-1}wu$. In any group such a mapping defines an inner automorphism of the group, so the group of inner automorphisms of F_2 is $Inn(F_2) = \{conj_u | u \in F_2\}$. The circle of fifths order defines a linear order, since the sequence of diatonic perfect fifths begins with F and ends with B–the lexicographic order on the modal scale words is similarly linear–but conjugation by initial letters defines a cyclic order through the conjugacy class. In the last column of Table 1, aligned with the circle of fifths order of the modes, the words for backward foldings follow the cyclic order of conjugation by initial letters, e.g., beginning with Lydian, $y^{-1}(yxyxyxy)y = xyxyxyy$, $x^{-1}(xyxyxyy)x = yxyxyyx$, $y^{-1}(yxyxyyx)y = xyxyyxy$, $x^{-1}(xyxyyxy)x = yxyyxyx$, $y^{-1}(yxyyxyx)y = xyyxyxy$, $x^{-1}(xyyxyxy)x = yyxyxyx$, and conjugation of $yyxyxyx$ by y returns to the Lydian backward folding word. Similarly, for the forward foldings, aligned with the circle of fifths order of the modes, the words follow a cyclic order, but now of conjugation by the inverses of initial letters (e.g., since $(y^{-1})^{-1} = y$, beginning with the forward folding word for Lydian, we have $conj_{y^{-1}}(xyxyxyy) = yxyxyxy$, forward folding word for Ionian). Put another way, conjugating by initial letters aligns backward foldings with ascending circle of fifths order for the modes, while conjugating by initial letters aligns forward folding with descending circle of fifths order for the modes. Forward and backward foldings coincide for Aeolian.

What does mathematics say as to forward foldings vs. backward foldings? Initially, we must recognize that mathematics knows nothing of foldings, because word theory does not recognize the notes that for the music theorist determine the words, the notes "between" the letters of the words, as it were. The foldings and their directions only arise in the interpretations assigned to the words.

But the mathematics does decide which interpretation makes for the better one-to-one mapping between the respective conjugacy classes. This clarification happens when we lift from the level of the words to the level of the morphisms in St_0. In the process, the cyclic ordering of the words is cut at a certain point to become a linear ordering of the morphisms. This is the level of *Sturmian involution*, first defined in [16]. Under Sturmian involution, a morphism f in St_0 is mapped to the morphism f^* such that f^* is the reversal of f, where every G and $\tilde{G}$ in the composition of

Table 2: *The diatonic modes, morphisms of modal scale words, and morphisms of backward folding words*

final	mode	morphism f	scale word	morphism f^*	backward folding word
C	Ionian	$GGD(a,b) =$	*aaba,aab*	$\tilde{D}GG(x,y) =$	*xy,xyxyy*
D	Dorian	$\tilde{G}GD(a,b) =$	*abaa,aba*	$\tilde{D}G\tilde{G}(x,y) =$	*xy,xyyxy*
E	Phrygian	$\tilde{G}\tilde{G}D(a,b) =$	*baaa,baa*	$\tilde{D}\tilde{G}\tilde{G}(x,y) =$	*xy,yxyxy*
F	Lydian	$GG\tilde{D}(a,b) =$	*aaab,aab*	$DGG(x,y) =$	*yx,yxyxy*
G	Mixolydian	$\tilde{G}G\tilde{D}(a,b) =$	*aaba,aba*	$DG\tilde{G}(x,y) =$	*yx,yxyyx*
A	Aeolian	$\tilde{G}\tilde{G}\tilde{D}(a,b) =$	*abaa,baa*	$D\tilde{G}\tilde{G}(x,y) =$	*yx,yyxyx*
B	Locrian	bad conjugate	*baabaaa*	bad conjugate	*yyxyxyx*

f is left fixed, and every D composing f is replaced in f^* by $\tilde{D}$, and every $\tilde{D}$ composing f is replaced in f^* by D. This is an anti-automorphism of St_0, and it is clear that $f^{**} = f$. For example, $(GGD)^* = \tilde{D}GG$. Moreover, from the definitions it is clear that Sturmian involution exchanges standard morphisms with Christoffel morphisms.

In Table 2 it becomes apparent that the backward folding assignment conforms with Sturmian involution, still using the diatonic modes as the canonical example. If we follow the forward folding assignment, there is a mismatch between the bad conjugate modal scale word, the representative of Locrian, which has a morphic folding word, and the bad conjugate folding word, which corresponds to the morphic scale word for authentic Mixolydian. However, for the major mode, whose historical ancestor is arguably authentic Ionian, there are music theoretical dividends paid by accepting the forward folding interpretation, as detailed in [17] and discussed in Section IV. In the general musical situation, the forward folding interpretation pairs standard scale words with standard folding words, and Christoffel scale words with Christoffel folding words, the associated morphisms being each other's reversal. For example, for the usual pentatonic, the standard word represents the scale C-D-F-G-A, $GD(a,b) = aba, ab$, whose forward folding F-C-G-D-A-(E) is the standard word $DG(x,y) = yx, yxy$. We have the freedom to choose alternative interpretations in the musical context because of the level of structure afforded by the sequences of notes that lie behind the words.

In Table 2 the modes are ordered in scale order as opposed to circle of fifths order. The *conjugation class* of a special Sturmian morphism may be defined (see Section IV), and it carries a natural linear ordering by virtue of the conjugations by single letters, $conj_a$ and $conj_b$ [19]. Starting from the standard morphism of the class as least element, for every morphism f in the class—except for the anti-standard morphism—either $conj_a \circ f$ or $conj_b \circ f$ is in the class and can be identified as the *successor* of f. The anti-standard morphism is the greatest element in the class, in *conjugation order*, $<_{conj}$. In Table 2 the morphisms are ordered from top to bottom in conjugation order, $GGD <_{conj} \tilde{G}GD <_{conj} \tilde{G}\tilde{G}D <_{conj} GG\tilde{D} <_{conj} \tilde{G}G\tilde{D} <_{conj} \tilde{G}\tilde{G}\tilde{D}$.[2] Note that applying Sturmian involution to these morphisms in conjugation order, the images of the resulting morphisms applied to $xy \in \{x < y\}^*$ are words in lexicographic order. This relationship was proven in [16]. If we define a co-lexicographic order (right-to-left, following the order of composition of mappings), where we order $G < \tilde{G}$ and $D < \tilde{D}$, we see that this co-lexicographic order matches the conjugation order of morphisms in the class. This was proven in general for the conjugation class of a special Sturmian morphism in [20]; see also [2].

Having introduced the inner automorphisms $conj_u$ of F_2, it should be admitted that the elements of St_0 may be extended to automorphisms of F_2, by defining the actions of the gen-

[2] In Table 1 the folding words are conjugated by initial letters, closely related to but distinct from conjugation order for morphisms.

erating morphisms on the negative letters. We set $G(a^{-1}) = a^{-1}, G(b^{-1}) = b^{-1}a^{-1}, \tilde{G}(a^{-1}) = a^{-1}, \tilde{G}(b^{-1}) = a^{-1}b^{-1}$, and $D(a^{-1}) = b^{-1}a^{-1}, D(b^{-1}) = b^{-1}, \tilde{D}(a^{-1}) = a^{-1}b^{-1}, \tilde{D}(b^{-1}) = b^{-1}$. Extended in this way to F_2, the elements of St_0 are automorphisms of F_2 (see [18]). We will not need to compute with these automorphisms, but that they sit within $Aut(F_2)$ will be relevant.

IV. Commutativity and Non-Commutativity

The monoid A^* and the free group F_2 are highly non-commutative structures. In Figure 2, drawn from Figure 9 in [17], the modal scales and foldings for Lydian and Ionian are displayed on the two-dimensional lattice, $\mathbb{Z}^2$. The basis $\{(1,0),(0,1)\}$ consists of units in the *width* and *height* dimensions, where the former represents a move along the line of ascending fifths, and the latter represents an ascending generic step on the diatonic scale. Red vectors $(2,1) = a$ and $(-5,1) = b$ represent tones and semitones, respectively, in the ascending modes, while blue vectors $(4,1) = x$ and $(1,-3) = y$ represent upward perfect fifths and downward perfect fourths, respectively, in the forward folding patterns associated with the modes. In the Lydian mode figure the modal final and the initial tone of the folding are identically the note F, and are set at the origin $(0,0)$. Ionian is displayed with a choice of coordinates. On the left, the same fundamental domain as for Lydian is used, with Ionian final C at the point $(0,1)$ and the initial folding tone F at the point $(3,0)$. On the right, the coordinates for the modal final C and initial folding tone F are now $(0,-3)$ and $(-1,0)$, respectively. In any choice of coordinates, though, the vector addition is commutative; any combination of 5 vectors $(2,1)$ and 2 vectors $(-5,1)$ yields the vector $(0,7)$, representing an octave: $5(2,1) + 2(-5,1) = (0,7)$. And yet the *path* through the lattice, *aaaaabb*, has a completely different musical meaning from the Lydian mode: F-G-A-B-C♯-D♯-E-F′. Similarly, we intuitively understand that the diatonic intervals D-F and E-G are both minor thirds, but that the short musical lines D-E-F and E-F-G are different from each other. While the lattice $\mathbb{Z}^2$ is embedded in the two-dimensional real plane, the lattice paths are elements of an infinite-dimensional vector space (see [5]).

The added level of interpretation afforded by the appeal to notes that lie behind the intervallic letters in words bears fruit when we consider the notes that stand in for the divider symbol, in relation to the notes assigned to modal final and initial tone of the folding. In Figure 2, with the change of coordinates for Ionian, where the width coordinate for the final (tonic) is set at zero, and the height coordinate of the initial tone of the forward folding is set at zero, we see that the coordinates for the shared divider note G are (1,1). This property, referred to in [17] as *divider incidence*, is a general property of authentic modes associated with standard words and with the words of the corresponding forward folding (see [21]). In divider incidence, not only does the divider coincide in scale and folding, but the final essential note (leading tone) also coincides for scale and folding, and initial scale tone and folding divider predecessor coincide, as do initial folding tone and scale divider predecessor tone. Divider incidence plays a role in the word-theoretical understanding of the privileged status of Ionian (and of modes of well-formed scales associated with standard words), with such properties as the Sensitive Interval Property and Double-Neighbor Polarity (see Section 4, [17]).

The explanatory power for tonal properties is the dividend, alluded to above, paid by the forward folding interpretation. What if we assume the alternative backward folding interpretation (recalling that by Sturmian involution this is mathematically preferred)? In this instance a form of divider incidence holds for authentic Dorian. As one can read off Figure 2, the shared divider for authentic D-Dorian is A, and the intervals from modal finals to dividers C-G and D-A are both expressed as vector sums $3(2,1) + (-5,1) = (1,4)$, but as words encoding paths, *aaba* and $abaa = rev(aaba)$, respectively, and the intervals from initial folding tones to dividers F-G and

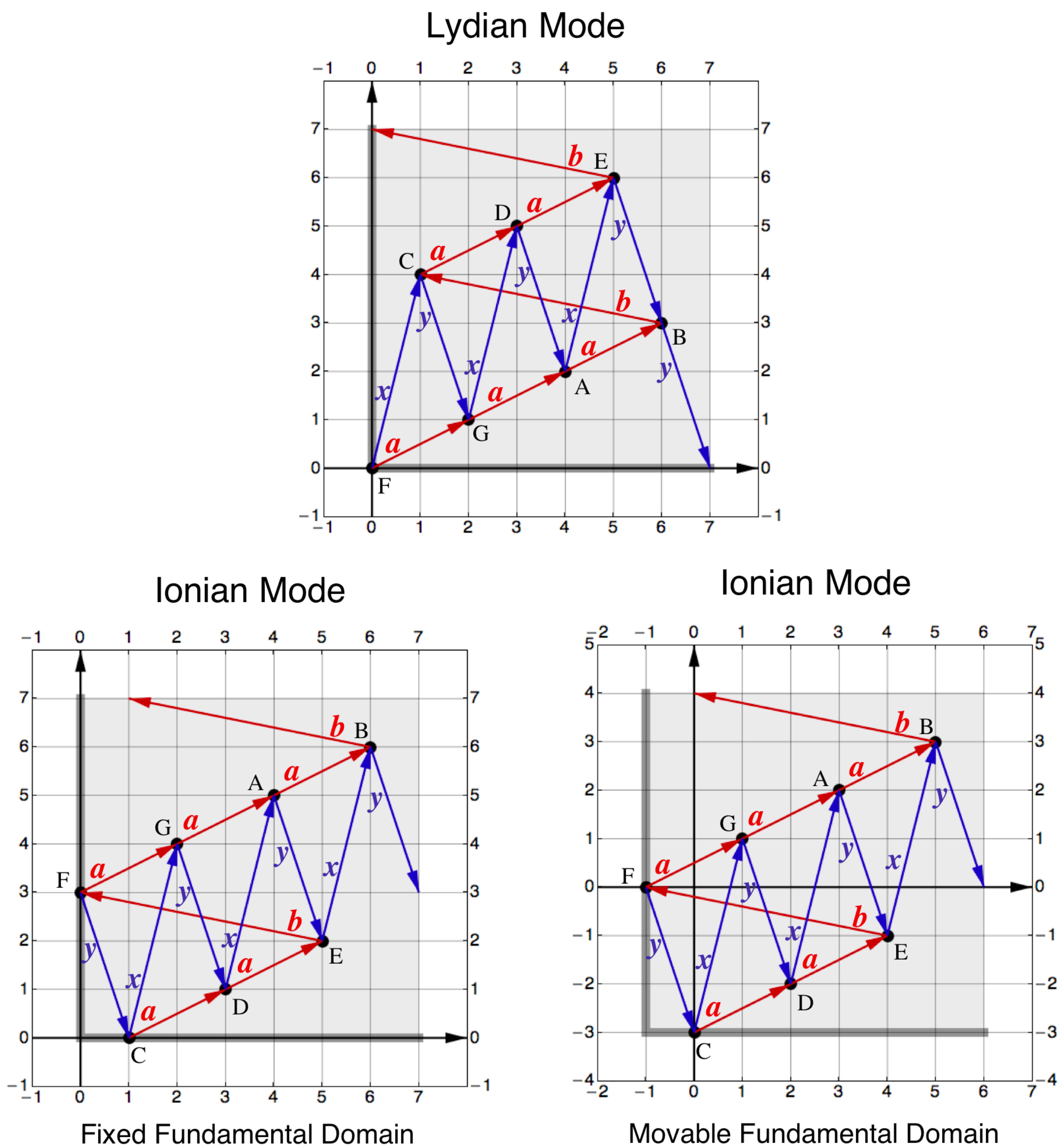

Figure 2: *Representations of Lydian and Ionian via Width-Height Vectors*

B-A are both expressed as vector sums $(1,4) + (1,-3) = (2,1)$, but as words encoding paths, yx and $xy = rev(yx)$, with the meanings of x and y in the latter word as descending perfect fifth and ascending perfect fourth, respectively. The music-historical meaning of the respective assignments must remain completely speculative, but it is suggestive that in the old eight church modes, authentic Dorian was mode 1, while after Glarean's 1547 expansion to the twelve *Dodecachordon* modes and Zarlino's 1571 reordering, authentic Ionian is labeled mode 1.

Returning to the free monoid $A^* = \{a,b\}^*$ of words over a two-letter alphabet, embedded in the free group $F_2 = \{a, b, a^{-1}, b^{-1}\}$, we consider relations with their commutative counterparts, the additive monoid of non-negative integer ordered pairs, $\mathbb{Z}^2_+$ and the additive group $\mathbb{Z}^2$. Consider the monoid homomorphism $V : A^* \to \mathbb{Z}^2_+ : w \to (|w|_a, |w|_b)$. $V(a) = (1,0)$, $V(b) = (0,1)$, which form a basis for $\mathbb{Z}^2_+$. The empty word is the kernel of V, but V is many-to-one: e.g., $V(aabb) = V(abab) = (2,2)$. V is a projection based upon letter count, a monoid epimorphism: if $u, v \in A^*, w = uv$, then $V(w) = V(u,v) = V(u) + V(v)$, and if $z = (z_1, z_2) \in \mathbb{Z}^2_+$, any word w with $|w|_a = z_1$ and $|w|_b = z_2$ is in the inverse image of z under V.

We wish to show that V induces a linear map on $\mathbb{Z}^2_+$, such that we have a commutative square, where f is a morphism (not necessarily Sturmian) of A^*:

$$\begin{array}{ccc} A^* & \underset{f}{\rightarrow} & A^* \\ V\downarrow & & V\downarrow \\ \mathbb{Z}^2_+ & \underset{M_f}{\rightarrow} & \mathbb{Z}^2_+ \end{array}$$

where

$$M_f = \begin{pmatrix} f(a)_a & f(b)_a \\ f(a)_b & f(b)_b \end{pmatrix}.$$

That is, we wish to show that $M_f V(w) = V f(w)$, for all w in A^*. V sends ab to $(1,1)$, while f sends ab to $f(ab) = f(a)f(b)$. Then $V(f(ab)) = V(f(a)f(b)) = V(f(a)) + V(f(b)) = (|f(a)|_a + |f(b)|_a, |f(a)|_b + |f(b)|_b) = M_f\binom{1}{1} = M_f V(ab)$.

Proposition 1 *Let $w = w_1 \dots w_k$ where $w_j \in \{a,b\}, 1 \le j \le k$, and suppose that $|w|_a = m$ and $|w|_b = n$, and let f be a morphism of A^*, then $V(f(w)) = M_f V(w)$.*

Proof:

$f(w) = f(w_1) \dots f(w_k)$, and $V(w) = (m,n)$. Then $V(f(w)) = V(f(w_1) \dots f(w_k))$

$= (m|f(a)|_a + n|f(b)|_a, m|f(a)|_b + n|f(b)|_b) = M_f\binom{m}{n} = M_f V(w)$.

Exemplifying the commutative diagram, beginning with the root word ab and applying the standard morphism GGD we have $GGD(a,b) = aaba, aab$, and we observe that

$$\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

that is, $M_{GGD}(V(ab)) = V(GGD(ab))$.

In the following, f is assumed to be a member of St_0, generated by $\{G, \tilde{G}, D, \tilde{D}\}$. For G and $\tilde{G}$, we have $G(a,b) = a, ab$ and $\tilde{G}(a,b) = a, ba$, and for D and $\tilde{D}$, we have $D(a,b) = ba, b$ and $\tilde{D}(a,b) = ab, b$, so

$$M_G = M_{\tilde{G}} = R = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

and

$$M_D = M_{\tilde{D}} = L = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

R and L freely generate the monoid $SL_2(\mathbb{N})$ (by which is meant the intersection of 2x2 matrices with non-negative entries with $SL_2(\mathbb{Z})$).

Recall that the inner automorphisms of F_2, $Inn(F_2) = \{conj_u | u \in F_2\}$ form a subgroup of the automorphisms of F_2, $Aut(F_2)$. From group theory, the inner automorphisms are a normal subgroup of the group of automorphisms, that is, they form the kernel of a group epimorphism. We appeal to the celebrated result of Nielsen (see chapter 5 of [18]; for a proof see [22]) that the quotient of $Aut(F_2)$ modulo $Inn(F_2)$ is isomorphic to the automorphism group $Aut(\mathbb{Z}^2) = GL_2(\mathbb{Z})$. Thus, $Aut(F_2)/Inn(F_2) = GL_2(\mathbb{Z})$, and the kernel of an epimorphism onto the general linear group is the group of conjugations. This in turn implies that the conjugation class of a special Sturmian morphism f is characterized by the result that all its members share the same incidence matrix M_f.

It follows that all the representatives of the conjugation class of $f \in St_0$ are products of the same sequence of letters G and D, to within the distribution of diacritic ˜ marks attached to these letters. Suppose that the conjugation class of f is characterized by the sequence of letters (morphisms) G and D, $X_1 X_2 \ldots X_n$, $X_i \in \{G, D\}, 1 \leq i \leq n$. Then the incidence matrix $M_f = M_{X_1} M_{X_2} \ldots M_{X_n}$. This matrix product is a product of generating matrices R and L, where $M_{X_i} = R$ or L as $X_i = G$ or D. For example, the conjugation class of the St_0 elements that yield the authentic diatonic modes consists of $\{GGD, \tilde{G}GD, \tilde{G}\tilde{G}D, GG\tilde{D}, \tilde{G}G\tilde{D}, \tilde{G}\tilde{G}\tilde{D}\}$. The associated incidence matrix is then $M_{GGD} = M_G M_G M_D = RRL$, i.e.,

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}.$$

V. Concluding Remark

Musicians are used to adding intervals together commutatively to produce intervals, when only the size of the resulting interval is at issue. But more often we are interested in the process of arriving at the result. Even in the traditional one-dimensional pitch height conception of musical intervals this is the case. For a simple example, the perfect fifth followed by perfect fourth resulting in an octave is a very different process from perfect fourth followed by perfect fifth. In semitones, we are simply saying $7 + 5 = 12 = 5 + 7$; the endpoints of the musical line remain unchanged, but the path is different, C-G-C′ vs. C-F-C′. Consider a much more complicated example, the opening ascent in the subject of J. S. Bach's Fugue in F♯ minor from Book 1 of the WTC. The line is essentially 3 times a motive of tone-semitone, each time resetting one semitone lower: F♯-G♯-A, G♯-A♯-B, A♯ (…)-B♯-C♯, or, in semitones, $(2 + 1) - 1 + (2 + 1) - 1 + (2 + 1) = 7$. The goal of the line is the perfect fifth, C♯, 7 semitones above the tonic, F♯, but achieved in a very slow and highly chromatic ascent, and in a systematic, motivically oriented construction. And this is just with respect to the one-dimensional perspective. If the intersection of scale theory with word theory tells us anything it is the value of a two-dimensional framework. As Figure 2 suggested, both the commutative vector sums on the two-dimensional lattice and the paths traced in the course of these sums, are musically compelling. The points on the lattice are the musical notes that the mathematical words have "forgotten," but that breathe musical life to the study. The mathematics outlined above relating non-commutative algebraic objects with their commutative images suggests further interpretations in the study of scales, modes, and transformations among them.

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