Common Rhythm as Discrete Derivative of Its Common-Time Meter

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Abstract: *Musicians recognize two important functions over the sixteen points in time, or beat classes, distributed evenly over a common-time measure: metric weight and onset frequency. Existing scholarship acknowledges that these functions are similar but not identical, and researchers tend toward one or the other as a model for metric entrainment. However, if the discrete metric-weight function is converted into a continuous curve, then the two functions strongly correlate: the ordering of each beat class by its backwards discrete derivative on this curve perfectly matches the ordering of each beat class by its onset frequency in a classical corpus.*

Keywords*: Rhythm. Meter. Calculus. Conducting. Fourier.*

I. Two Functions Over Moments in Common Time

The bar graphs of Figures 1 and 2, representing two functions, offer two perspectives of a musical measure divided into sixteen equal unit spans of time. The aim of this essay is to elucidate, in both mathematically elegant and musically intuitive ways, not only the structures of each function but also their mutual correlation. Before discussing the graphs individually, a few words are needed about how this information is presented generally. The moments beginning each unit span will be referred to as "beat classes" (bcs), because points in time in previous or subsequent measures are deemed as equivalent to those in this measure. Each function maps each beat class to what I will generally call a "level."[1] The measure that these two graphs describes would be typically indicated in musical notation as lasting the duration of a whole note, with a time signature of $\frac{4}{4}$, common time, or cut time, although other time signatures such as $\frac{2}{4}$ readily accommodate a sixteen-fold division. Although it is customary in musical terminology to assign the first element of a series of beats in a measure (or notes in a scale) to the number 1, the horizontal values for these graphs instead use zero-based numbering. This follows both many mathematical conventions in general and music theory's notation of beat class in particular, whereby "beat" in "beat class" describes not only moments of tactus onset—the more standard use of the word "beat"—but also the beginning of any unit duration that evenly divides a longer span as a modular (i.e. quotient) space, such as a measure.[2] I will mix both meanings of "beat" in what follows. In both graphs, my inversion—higher as lesser—of the more typical orientation of a Cartesian vertical axis—higher as greater—is deliberate and will be explained later.

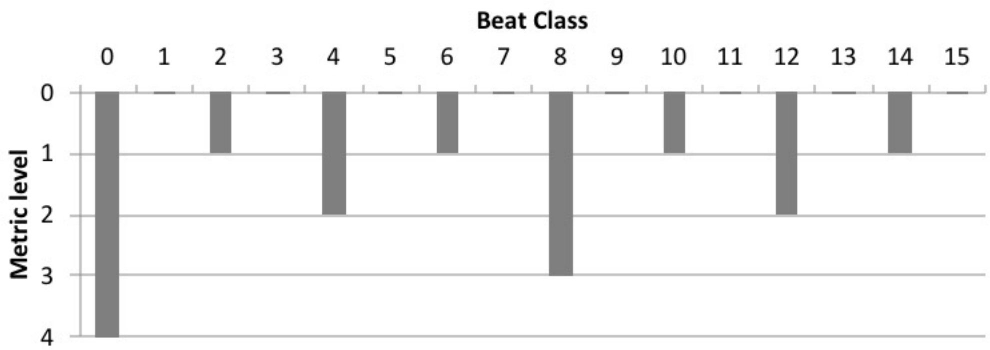

Figure 1: *Metric level of the 16 beat classes (f_1).*

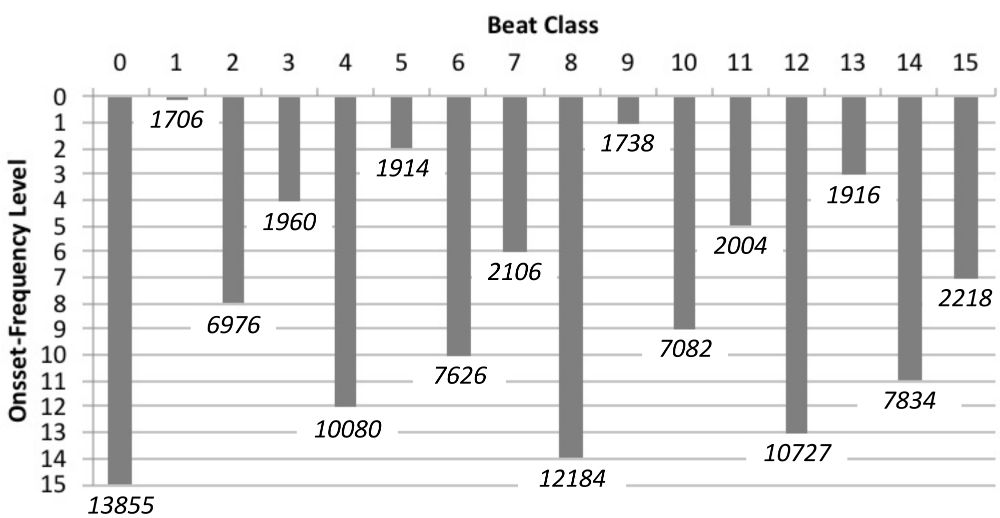

Figure 2: *Number of onsets (in italics) in common-time string-quartet movements by Joseph Haydn assessable by computer, and their ranking into 16 frequency levels (f_2).*

Figure 1 displays a standard interpretation of the relative weights or strengths of metric accent in such a measure.[3] In what follows, the more neutral term "level" encapsulates more subjective expressions like "weight" or "strength" of metric accent. The downbeat (bc 0) is on the highest level, because it carries the most weight of all of the beat classes in this or any other measure. If the lowest level is numbered 0, then this downbeat should be numbered as level 4, as there are five different degrees of metrical accent for this number of beat classes. The third beat (bc 8)—which occurs halfway through the measure—carries the second strongest metrical accent (level 3), and the second and fourth beats (bcs 4 and 12) tie for the third strongest metrical accent (level 2). The four remaining even-numbered beat classes (bcs 2, 6, 10, 14)—the "di" moment in the Takadimi [7] system of metric notation—tie for the fourth strongest metrical accent (level 1), and the eight odd-numbered beat classes—the "ka" and "mi" in Takadimi—tie for the least strong metrical accent (level 0). This mapping of beat-class number (x) to metric-accent level number (f_1) can be generalized for all "pure duple" meters (meters whose adjacent pulses relate by a factor of two) as follows: for a measure of 2^n units ($n \in \mathbb{Z}^+$), $f_1(x,n) = log_2(gcd(2^n, x))$, where gcd stands for "greatest common divisor."[4] For example, in a measure of sixteen units (n = 4), bc 4 has an accent level of $log_2(gcd(2^4, 4)) = log_2(gcd(16, 4)) = log_2(4) = 2$. Another way to define $f_1(x,n)$ is as the number of contiguous rightmost zeros in a binary (radix-2) representation of x. For example, the binary representation of 4 is 100, in which there are two contiguous rightmost zeros; therefore, bc 4 is on level 2.

For a common-time measure in Western classical music, Figure 2 displays a common ranking of onset frequencies among the sixteen equidistant beat classes. In particular, this bar graph shows in italics how many onsets occur in the sixteen beat-class positions for all of the string quartet movements by Joseph Haydn in $\frac{4}{4}$ assessable by computer [4]. These sixteen counts are then ranked into levels. With such a large data set, such that the frequencies of onsets for two different beat classes are highly unlikely to be the same, Figure 2 unsurprisingly has sixteen different levels, numbered 0 to 15, unlike the five of Figure 1. Thus, when described as a function—I will call it $f_2(x,n)$, parallel to f_1's definition—this mapping of beat class to level is bijective, whereas f_1 is surjective. However, despite the higher specificity, the ranking among these sixteen beat classes tends to be the same or quite similar for other Western common-practice corpora of sufficient size. One such instance appears in an article by David Huron and Ann Ommen [8] about syncopation in American popular music: their tally of onsets among common-time monophonic songs in the Essen Folksong Database well matches the rankings in Figure 2.[5]

The distribution of Figure 2 can be related to an understanding of Western common-practice styles. First, an onset chosen at random in a work related to a corpus from which f_2 values are computed is more likely to appear at a beat class with a higher f_2 value, regardless of whether the music is polyphonic (like Haydn's quartet movements) or monophonic (like the folksongs of the Essen Database). Second, as shown in Figure 3, each of the sixteen levels in Figure 2 corresponds to a rhythm within a $\frac{4}{4}$ measure, composed of onsets on the beat classes on that level or higher. For example, the level 13 rhythm is half-quarter-quarter, because these three onsets correspond to beat

[1] See [1] for a relatively early use of this term, although the concept is an older one.

[2] Here, "tactus" refers to a primary pulse near the range of 85 to 120 beats per minute. Justin London [14, pp.30-33] provides a summary of the concept.

[3] Lerdahl and Jackendoff [13, p.19] provide a well-known example.

[4] Cohn [2, p. 194] coins the term "pure duple."

[5] The published bar graph in [8, p. 215] does not contain specific counts. However, in June 2015 Huron kindly shared his data with me, which I will call $f_3(x,4)$ = [19221, 24, 2116, 412, 10226, 25, 5661, 430, 14280, 24, 2380, 348, 13458, 39, 5964, 634]. Of the 120 pairs of 16 beat classes that be ordered, only two of these 120 between $f_2(x,4)$ (my Haydn count) and $f_3(x,4)$ (Huron and Ommen's Essen count) are not the same: $f_2(3,4) < f_2(11,4)$ but $f_3(3,4) > f_3(11,4)$, and $f_2(1,4) < f_2(9,4)$ but $f_3(1,4) = f_3(9,4)$.

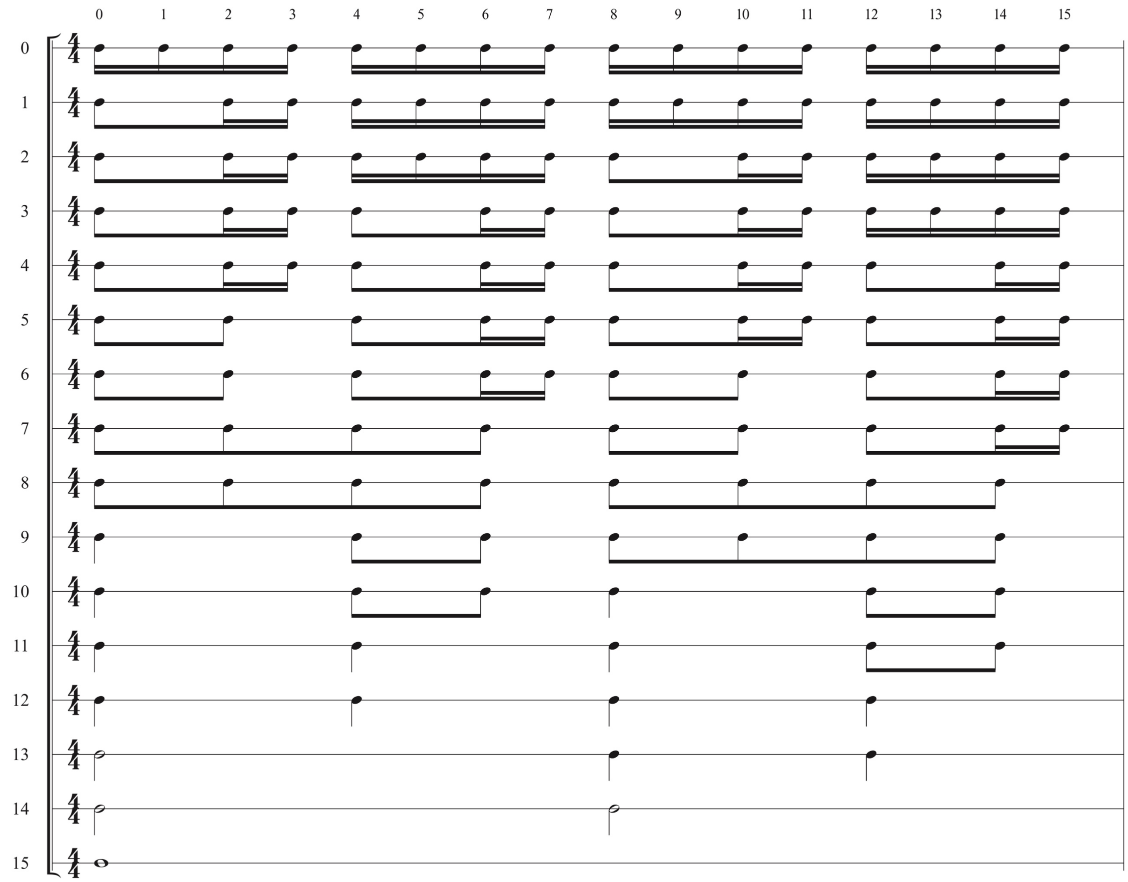

Figure 3: *Sixteen common rhythms in $\frac{4}{4}$, corresponding to the 16 levels and 16 beat classes from Figure 2.*

classes mapped to levels 13, 14, and 15 in Figure 2. Each of these rhythms is relatively idiomatic of, and rather common within, Western classical music. For example, in Johann Joseph Fux's 1725 treatise *Gradus ad Parnassum*, when the student writes a quarter-quarter-half rhythm within a common-time measure, the teacher gently recommends the half-quarter-quarter rhythm as a better solution.[6] These norms may be generalized by the time-honored preference for beginning longer inter-onset intervals on metrically stronger moments, which pushes shorter durations toward the end of metrical spans like measures, half-measures, and beats. To be sure, other rhythms besides those of Figure 3 are also quite probable in Western common-practice music. However, I speculate that, of two rhythms spanning the same kind of measure with the same number of onsets, the one that is more common in Western common-practice music is more likely than not to be the one whose sum of Figure-2 levels is higher.[7]

In the aforementioned article by Huron and Ommen, the authors note how two of their graphs, which correspond to my Figures 1 and 2, "are similar though not identical" [8, p. 214] to each other, acknowledging the imbalance between the frequency of onsets for the second and fourth beats (bcs 4 and 12) that I also cited earlier. They nonetheless continue with the recognition of "a notable correspondence between the hierarchy of event onsets [the counterpart of my Figure

[6] [5, p. 67].
[7] Temperley [17, p. 358] recognizes and addresses the shortcomings of this approach.

2] and the conventional metrical hierarchy [the counterpart of my Figure 1].” However, “[t]he causal relationship here is unknown” to the authors, although they speculate that “[i]t is possible that the metrical hierarchy originates in the distribution of event onsets... or the distribution of event onsets might simply reflect a pre-existing metrical hierarchy that influences the composition of music.” The primary purpose of this article is to propose such a causal relationship between Figures 1 and 2. My first step toward this proposal is to appreciate that *both* Figures 1 and 2 exhibit comparable symmetries and formulaic generalizations.

II. Symmetries and Formulas for Each Function

When one looks at both bar graphs, Figure 1’s symmetry is probably discerned more immediately: for example, all of its rankings invert around both bcs 0 and 8; that is, $f_1(x) = f_1(0–x \bmod 16) = f_1(8–x \bmod 16)$. Put in colloquial terms, assuming that the beat-class assignment continues cyclically into adjacent measures, Figure 1 exhibits a vertical mirror symmetry around both an axis at bc 0 and an axis at bc 8. In other words, if two beat classes sum to 0 or 8 mod 16, then they will have the same level in Figure 1. Moreover, Figure 1 admits formulation: earlier, I proposed two formulas for $f_1(x,n)$, one using a logarithm and the other using binary representation.

It is quite reasonable to perceive and understand Figure 2 as a distortion of Figure 1, especially with the latter’s conspicuous symmetry. It is even more reasonable to do so when the data of Figure 2 is displayed on a vertical axis whose units are onset counts and not ranking levels, as Huron and Ommen do in their article. From this vantage point, my Figure 2 may be seen as a distortion not only of Figure 1 but also of the scale of the onset distribution: for example, bcs 7 and 15 are adjacent in the ranking (levels 6 and 7) and differ by 112 onsets, while bcs 15 and 8 are also adjacent in the ranking (levels 7 and 8) but differ by 4758 onsets.

However, this recalibration of the graph’s range helps to reveal Figure 2’s own internal consistency that is at once independent from that of Figure 1, while also obliquely affiliated with it. For example, $f_2(x) = 15 – f_2(1–x \bmod 16)$. In colloquial terms, assuming that beat-class designations continue into adjacent measures, Figure 2 exhibits a vertical mirror symmetry around both the point equidistant from bcs 0 and 1 and the point equidistant from bcs 8 and 9. In other words, if two beat classes sum to 1 mod 16, then they correspond, not with the *same* ranking as they do in the Figure 1 mirror, but rather *opposite* rankings. For one example, bcs 0 and 1 sum to 1 mod 16, and bc 0 hosts the most frequent onsets (level 15), while bc 1 hosts the fewest (level 0). For another example, bcs 12 and 5 sum to 1 mod 16, and bc 12 hosts the third-most frequent onsets (level 13) while bc 5 hosts the third-fewest frequent onsets (level 2).

Furthermore, f_2 may be written formulaically. At the heart of one such formula is a function from applied mathematics that also employs binary representation, a form of representation I used earlier for a formulation of $f_1(x)$. This function is the bit-reversal permutation ($\mathrm{rev_n}$), which maps a set of integers $0, 1, \ldots, 2^n–1$ ($n \in \mathbb{Z}^+$) to itself by mapping each integer to the reversal of the bits in its binary representation.[8] For example, $\mathrm{rev}_4(12) = 3$, because the four-bit binary representation of 12 is 1100, and the retrograde of 1100 is 0011, which is 3 in decimal form. Generalized for any span divided into 2^n equidistant beat classes, $f_2(x,n) = (2^n–1) – \mathrm{rev_n}(-x \bmod 2^n)$. Table 1 provides the computation of this equation for n = 4 in particular, matching the mapping of beat classes to the rankings of Figure 2.

[8] The bit-reversal permutation has been applied to FFT (Fast Fourier Theorem) algorithms, as demonstrated in [3, p. 918].

Table 1: *Demonstration of $f_2(x,n) = (2^n-1) - rev_n(-x \bmod 2^n)$, for $n = 4$.*

beat class (x)	$f_2(x,4)$	-x mod 16	-x mod 16 in binary	rev_4(-x mod 16) in binary	rev_4(-x mod 16)	$(2^4-1) - rev_4$(-x mod 16)
0	15	0	0000	0000	0	15
1	0	15	1111	1111	15	0
2	8	14	1110	0111	7	8
3	4	13	1101	1011	11	4
4	12	12	1100	0011	4	12
5	2	11	1011	1101	13	2
6	10	10	1010	0101	5	10
7	6	9	1001	1001	9	6
8	14	8	1000	0001	1	14
9	1	7	0111	1110	14	1
10	9	6	0110	0110	6	9
11	5	5	0101	1010	10	5
12	13	4	0100	0010	2	13
13	3	3	0011	1100	12	3
14	11	2	0010	0100	4	11
15	7	1	0001	1000	8	7

III. Relating the Two Functions to Each Other

The resemblances between the manners in which I have described the symmetries and formulations of f_1 and f_2 hint at, but do not themselves furnish, a deep-seated connection between them. The use of binary representation in defining both f_1 and f_2 intimates a utility in a component-wise disassembly and reassembly of both metrical and rhythmic wholes, and the different positions of the f_1 and f_2's axes of symmetry suggests that metrical symmetries hinge on the beat classes themselves, while rhythmic regularities operate in between these beat classes somehow. My investigation into a causal relationship between Figures 1 and 2 continues by revisiting Huron and Ommen's article. At one point, they set two psychological theories of temporal regularity head-to-head:

> Some psychologists have proposed that the metrical hierarchy arises from integrally-related mental oscillators that coordinate auditory attending. However, recent psychological research more strongly suggests that rhythmic perceptions arise from simple exposure to rhythmic stimuli rather than via mental oscillators. This research suggests that patterns of auditory attending arise through the mechanism of statistical learning. Listeners are sensitive to the frequency of occurrence of sound events, and these distributions appear to become internalized as mental "schemas." [8, p. 214].

On the one hand, a series of synchronized mental oscillators operating at different frequencies, all multiples of the downbeat frequency, correlate with metrical weighting schemes such as that in Figure 1. On the other hand, the various schematized rhythms in Figure 3 that follow from the statistical distribution of Figure 2 entrain an acculturated listener to a metrical orientation and flow. The words "rather than" in the quotation above suggest that these two models are mutually incompatible. But could they be mutually reinforcing? Is it possible to replace "rather than" with "in addition to" or perhaps even "derived from"? I believe the answer to both of these questions could be "yes."

The use of a mental oscillator as a model for entrainment to a periodicity expresses not only the regularity of this periodicity and its persistence in the absence of constant support, but also the continuous nature of metric experience, in contrast to the discrete and discontinuous design of Figure 1 and f_1's requirement of integer input. Meter has been characterized as the fluctuation of a listener's attention [6]. The greater the metrical weight, the more a listener is paying attention. This correlation probably stems from evolutionary efficiency. Changes in music, such as shifts to a new harmony, often occur on metrically accented moments. Therefore, it one wishes to discern as much information about the music as possible with the least amount of attentional energy, it makes sense to attend considerably more to downbeats or downbeat-like moments in particular and considerably less to the moments in between. But one cannot change this energy non-continuously. Thus, the model of an oscillator—such as a spinning circle, a swinging pendulum, a vibrating spring, and so forth—captures this continuous change, and has been called upon by scholars to model meter, particularly in the work of Edward Large.[9]

A common visualization of such an oscillating function is as a continuous periodic wave, peaking at the moment of greatest metrical weight and bottoming out at the moments of least metrical weight (although this verticality metaphor has been, and could be, inverted). These waves can be binarily categorized in two ways. First, music scholars have constructed these waves either more causally as visual aids or precisely as mathematical functions. Second, music scholars have constructed these waves so that either its curvature—that is, its second derivative—at any point has the same sign or not. For example, the musicologist and conductor Viktor Zuckerkandl drew inverted cycloid-like waves to depict the periodicity of the downbeat in Chopin's "Military" Polonaise op. 40 no. 1.[10] Although Zuckerkandl's argument is more philosophical than mathematical, his downbeats as the sharp points of an inverted cycloid's maxima suggests the notion that the rate in which one is gaining attentional energy as a downbeat approaches is always increasing. This acceleration and sharpness appears to emulate an idealized metric state, such as the crispness of a conductor's beat pattern, in which the hand moves more quickly as it both approaches and leaves a beat.

This sharpness qualitatively differs from modeling downbeats as the plateaus of a sine wave or an iterated normal distribution, where the rate in which one is gaining attentional energy as a downbeat approaches slows down at some significantly earlier point, which provides a better model for how listeners might "hedge their bets," adjusting to small fluctuations in a periodic stimulus. The use of von Mises distributions [11] allows for a continuum between the two types of curves: a wave with a high κ (its measure of concentration) produces sharp peaks whereas a wave with a low κ creates gently sloped plateaus.

Large and co-author Caroline Palmer [12] also innovated the combination and display of component waves into a composite wave that represents multiple periodicities of a metrical hierarchy. For example, their model of triple meter combines two periodic von Mises distributions of equal amplitude but with one distribution's period three times as long as the other. In an analogous fashion, Figure 4 models the $\frac{4}{4}$-plus-eighths-and-sixteenths meter of Figure 1 by combining—in the manner of Fourier synthesis—five periodic parabolic functions of equal amplitudes but with periods two times as long as the next shorter function. These five periodic parabolic functions signify the five metric levels in a $\frac{4}{4}$ measure subdivided into sixteen equal parts. While other types of waves—such as sine or von Mises—could be used for the components of this composite metric function, the periodic parabolic wave has a constant second derivative, or acceleration of attentional gain. This captures a general intuition that, as a relatively strong beat is approached, the rate in which one is gaining attentional energy is always increasing. This intuition aligns with a conductor-musician synchronization study that concluded that "absolute acceleration along the movement trajectory. . . was the main cue used by participants to synchronize with the conductors' gestures" [15, p. 470]. (Zuckerkandl's cycloid would also provide this continuous acceleration and yield the same results below, but a parabolic function is mathematically easier

[9] [9], [10], and [11] are three early examples.

[10] [18, p. 171].

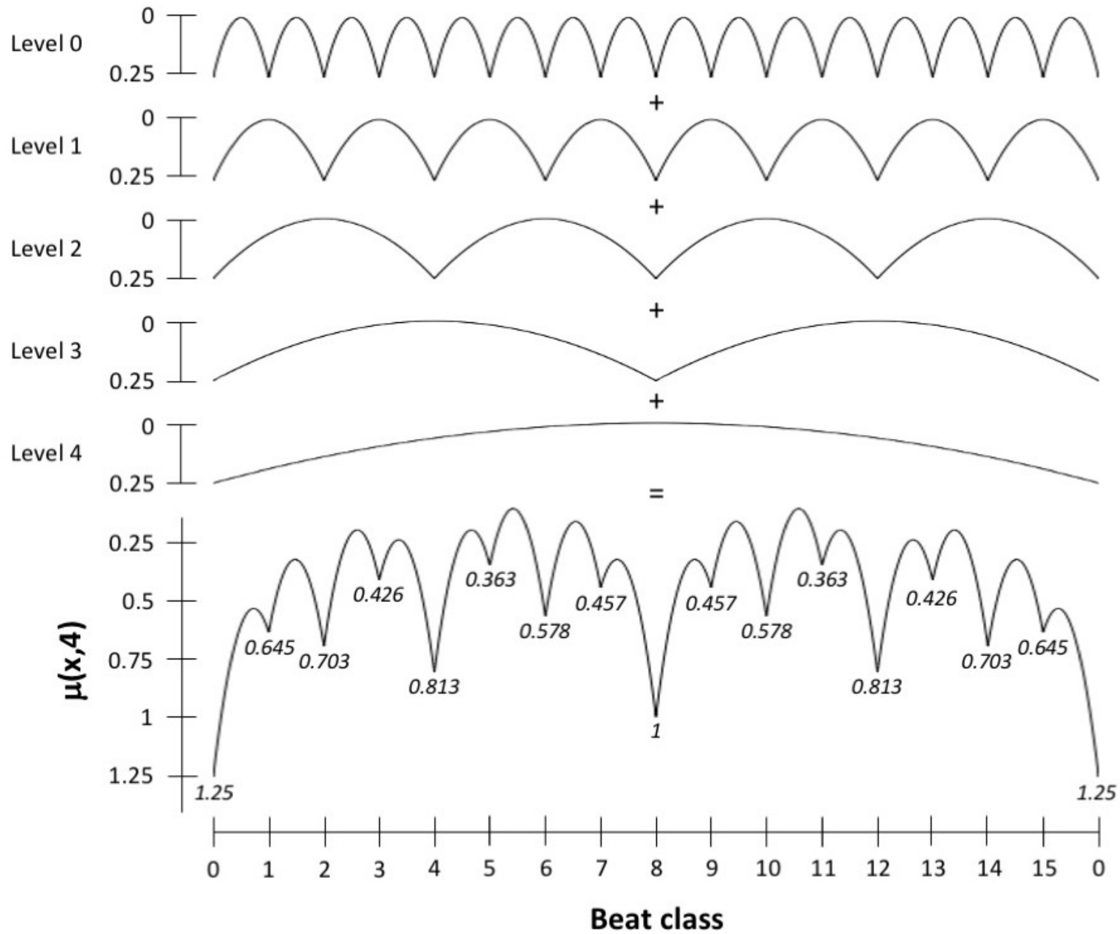

Figure 4: *Demonstration of the synthesis of a composite metric wave composed of five periodic parabolic waves at different frequencies, with values for the 16 beat classes provided in italics, rounded to the nearest thousandth.*

to work with.) This rate then drops non-continuously to a lower value when the strong beat is passed, and then begins to rise once again in its approach to the next. I will generalize this composite function as μ(x,n) for all pure duple meters that are 2^n units long (therefore, Figure 4's composite wave is μ(x,4)) as follows, where $\lfloor\rfloor$ represents the floor function that produces the function's periodicity:

$$\mu(x,n) = \sum_{i=0}^{n} \left(\left\lfloor \frac{-x}{2^i} \right\rfloor + \frac{x}{2^i} + \frac{1}{2} \right)^2$$

This composite wave is attractive to me as a musician and a theorist for multiple reasons, which I will attempt to convey through a listening and kinematic activity that I encourage the reader to try. (It is because of this activity and the ideas that stem from it that the customary orientation of the vertical axis has been consistently reversed in all of my graphs.) Audiate a pure duple meter with five different metric levels such that Level 0's periodicity is somewhere between 200 beats per minute (closer to a $\frac{4}{4}$ measure) and 30 beats per minute (closer to eight measures of $\frac{4}{4}$). Now, in time with the music, move your arm up and down within a range of one or two vertical feet to connect the cusps of $\mu(x,4)$ while flexing your wrist so that your hand (which is perhaps holding a baton) moves along the curves of $\mu(x, 4)$. First, despite its limitation to a single spatial dimension, this still feels to me a lot like conducting, a gestural embodiment of the many levels within the meter. Second, the vertical position of your hand at beat classes 0, 4, 8, and 12 corresponds exactly to the metric weight graphed in Figure 1, although the same cannot be said for the other beat classes.

However, this non-correspondence of the other twelve beat classes sets up a third correspondence. In a $\frac{4}{4}$ conducting pattern, the downbeat is preceded by the largest descending motion of the hand. In fact, in standard conducting patterns, the downbeat's significance is experienced somatically by the conductor and indicated visually to the ensemble not by the relatively low position of the hand, but rather by the relatively large *downward* change of position that precedes this low position. It is a *down*beat, not a *low*beat. The same can be said for conducting $\mu(x, 4)$: from the immediately preceding beat class, the arm descends the most into beat class 0 than into any other beat class. The values for the 16 beat classes provided in Figure 4 help to see this distinction: the ordered difference between $\mu(0, 4)(1.25)$ and the immediately preceding $\mu(15, 4)(\approx 0.645)$ is ≈ 0.605, which is larger than the next-largest ordered difference of 0.543 between $\mu(8, 4)(1)$ and the immediately preceding $\mu(7, 4)(\approx 0.457)$. Described using terms from calculus, beat class 0 has the highest backwards difference, or backwards discrete derivative (∇), of all 16 values in $\mu(x, 4)$: $\nabla\mu(0, 4) > \nabla\mu(x, 4)$ for all $x \in \mathbb{Z}_{16}$, $x \neq 0$, where $\nabla\mu(0, 4) = \mu(x, n) - \mu(x - 1$ mod16, n).[11]

Beat class 0's superlative position regarding $\nabla\mu(x, 4)$ correlates with its highest level in both Figures 1 and 2. While this correlation only extends to the second level in Figure 1, each beat class's ranking by $\nabla\mu(x, 4)$ perfectly corresponds to its ranking by onset count shown in Figure 2. Table 2 shows this exact correspondence. In mathematical terms, for all x,y $\in \mathbb{Z}_{16}$, $\nabla\mu(x, 4) > \nabla\mu(y, 4)$ if and only if f_2(x,4) > f_2(y,4). In practical and embodied terms, the distance one's hand travels downwards to a beat class when conducting $\nabla\mu(x, 4)$ matches the degree of likelihood that an onset (or more onsets, if polyphonic) will occur at that beat class in a common-practice work.

This exact correspondence strikes me as more than coincidental. An even division of time focuses a listener's attention more toward some moments equidistant in time and less toward other moments in between. This change of attentional degree is necessarily continuous to some degree. Due to this constant flux, those moments of greater attention can be distinguished not only by the absolute high state of attentional level but also by the *change* of this level from a preceding moment on a lower level to the current moment on a higher level. For relatively strong beats like downbeats, both the state of the moment and the change of state into the moment highly correlate and become interchangeable as indicators. Multiple periodicities, each with their own continuous attentional functions, constitute a composite attentional function. In such music with multiple periodicities, certain points in time that are equivalent in their *state* of composite metrical attention will nonetheless be preceded by different degrees of *change* of state of composite metrical attention, differentiating them. Owing to an overgeneralization inherent in the aforementioned interchangeability of state and change, it follows that a moment preceded by a greater change of state—such as the fourth beat in a $\frac{4}{4}$ measure—could be experienced as more downbeat-like than a moment of equivalent state preceded by a lesser change of state—such as the second beat in a $\frac{4}{4}$ measure.

Earlier, I quoted Huron and Ommen's proposal that "patterns of auditory attending arise through the mechanism of statistical learning" of common rhythmic patterns, such as those in Figure 3 for common time. Nothing I have presented here calls this into question. However, I hope that what I have presented here is a reasonable hypothesis for why at least one meter and its constituent periodicities give rise to certain rhythmic patterns—and not others—affiliated with this meter in the first place.

[11] Reale [16] offers a recent application of calculus's backwards discrete derivative to metric dissonance.

Table 2: *Demonstration of correspondence between $f_2(x,4)$ and $\nabla\mu(x,4)$.*

beat class (x)	$f_2(x,4)$	μ(x,4)	∇μ(x,4) = μ(x,4) – μ(x–1 mod 16,4)	∇μ(x,4) ranking
0	15	1.25	0.60546875	15
1	0	0.64453125	-0.60546875	0
2	8	0.703125	0.05859375	8
3	4	0.42578125	-0.27734375	4
4	12	0.8125	0.38671875	12
5	2	0.36328125	-0.44921875	2
6	10	0.578125	0.21484375	10
7	6	0.45703125	-0.12109375	6
8	14	1	0.54296875	14
9	1	0.45703125	-0.54296875	1
10	9	0.578125	0.12109375	9
11	5	0.36328125	-0.21484375	5
12	13	0.8125	0.44921875	13
13	3	0.42578125	-0.38671875	3
14	11	0.703125	0.27734375	11
15	7	0.64453125	-0.05859375	7

IV. Extensions

I leave it as an exercise to the reader to adapt and extend f_1, f_2, and my proposed manner of relating them to other common-practice meters besides common time, such as what Cohn [2] calls "pure triple" (such as $\frac{9}{8}$) or "mixed meters" (such $\frac{3}{4}$, $\frac{6}{8}$, and $\frac{12}{8}$), each with a potential variety of subdivisions. My preliminary forays into doing so are producing encouraging results. However, to the reader interested in this exercise, I offer a recommendation that I have already built into the present study: use continuous functions like periodic parabolic or cycloid waves. In my initial research into correlating Figures 1 and 2, I built μ from constituent sine waves, and the correspondence between f_2 and $\nabla\mu$ was just as exact. However, neither sine waves nor von Mises curves—regardless of the value of κ—yield nearly as close of a correspondence for mixed meters, at least when the amplitudes for each constituent function are the same, as periodic parabolic or cycloid waves, two continuous functions that are arguably more in line with conducting motions.

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