Scales, Counterpoint Triples and their Groups

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Abstract: *General scale systems are defined to be linearly ordered finite sets of musical objects. Apart from the common pitch scales we may also speak of duration and interval scales, major and minor scale schemes, ancient greek trope scale schemes. The fundamental groups of a scale (clock group and group of rows) are discussed. The principal counterpoint triple of a scale Σ consists of the operators R_Σ (Σ-retrograde), T_Σ (Σ-transposition) and I_Σ (Σ-inversion). The group they generate will be referred to as a counterpoint group of Σ. A wide class of counterpoint triples is presented extending the composition material of n-tone music. Variations of twelve tone pieces may be derived by applying these triples. Counterpoint spaces (CP-spaces) are reachable left actions of counterpoint groups. Such an action is actually simply transitive. Major and minor chords are defined with respect to a pair (p, q) of natural numbers playing the role of major and minor thirds respectively. It is shown that (p, q)-consonant chords in a CP-space constitute a CP-space as well.*

***Keywords**: Scale, Clock and Row Groups of a Scale, Counterpoint Groups, Counterpoint Spaces.*

I. Introduction

Scale is a generic notion in music theory ([4], [5], [6], [8]). Mathematically speaking, a scale Σ is a linearly ordered finite set of musical objects called degrees of the scale

$$\Sigma : \sigma_0 < \sigma_1 < \cdots < \sigma_{n-1}$$

This general consideration allows us to unify various musical scale type situations: pitch scales (chromatic, diatonic, pentatonic, whole-tone, octatonic etc.), scales of durations and intervals, major and minor scale schemes, ancient greek trope scale schemes (section 2). The set of degrees $\Sigma_n = \{\sigma_0, \sigma_1, \ldots, \sigma_{n-1}\}$ with the clock addition

$$\sigma_\kappa \oplus \sigma_\lambda = \begin{cases} \sigma_{\kappa+\lambda} & \text{if } \kappa+\lambda < n, \\ \sigma_{\kappa+\lambda-n} & \text{if } \kappa+\lambda \geq n \end{cases}$$

form the first fundamental group $G_1(\Sigma)$ of Σ. Transposition T_Σ (one step shift upwards) and inversion I_Σ (reflection with respect to a fixed center) are exclusively defined in terms of $G_1(\Sigma)$:

$$T_\Sigma(\sigma_\kappa) = \sigma_\kappa \oplus \sigma_1, \quad I_\Sigma(\sigma_\kappa) = -\sigma_\kappa = \sigma_{n-\kappa}, \quad 0 \leq \kappa \leq n-1$$

A Σ-row is a permutation of the set of degrees of a scale Σ ([7], [9]). The set $S(\Sigma)$ of all Σ-rows is closed under composition and constitutes the second fundamental group of Σ, $G_2(\Sigma) = (S(\Sigma), \circ)$. $(R_\Sigma, T_\Sigma, I_\Sigma)$ is the principal counterpoint triple of Σ, where R_Σ is the mirror image operator on

Σ-rows. Scales with the same height have isomorphic the respective fundamental groups. This leads to the notion of the scale type $\mathbb{Z}_n : 0 < 1 < \cdots < n-1$ (section 3).

Groups generated by retrograde, transposition and inversion operators are the subject of section 4. The counterpoint group $(r/t/i)$ is the free group generated by three letters r, t, i subjected to the axioms.

$$r^2 = 1 = i^2, \;\; rt = tr, \;\; ri = ir, \;\; it = t^{-1}i.$$

$(r/t/i)$ has three remarkable subgroups:

- (r/i), copy of the Klein four group $\mathbb{Z}_2 \times \mathbb{Z}_2$ ([9], [7]),
- (r/t), copy of the commutative group $\mathbb{Z}_2 \times \mathbb{Z}_n$, provided t has order n,
- (t/i), copy of the dihedral group D_{2n}, provided t has order n ([2], [3]).

Section 5 is devoted to exhibit new counterpoint triples which enrich the musical material permitting to compose extensive twelve tone structures. By construction, the chromatic scale

$$C = \{c, c\sharp, d, d\sharp, e, e, f, f\sharp, g, g, g\sharp, a, a\sharp, b\}$$

is the disjoint union of the diatonic scale

$$D = \{c, d, e, f, g, a, b\}$$

and the pentatonic scale

$$P = \{c\sharp, d\sharp, f\sharp, g\sharp, a\sharp\},$$

so that apart from the counterpoint triple (R_C, T_C, I_C) mainly used in twelve tone music, we obtain two partial counterpoint triples

$$(R_1 = R_D \vee id_P, T_1 = T_D \vee id_P, I_1 = I_D \vee id_P)$$

$$(R_2 = id_D \vee R_P, T_2 = id_D \vee T_P, I_2 = id_D \vee I_P)$$

where $T_D \vee id_P$, $I_D \vee id_P$ are the operators on C_{12} given by

$$(T_D \vee id_P)(x) = \begin{cases} T_D(x) & \text{if } x \in D_7 \\ x & \text{if } x \in P_5 \end{cases}$$

$$(I_D \vee id_P)(x) = \begin{cases} I_D(x) & \text{if } x \in D_7 \\ x & \text{if } x \in P_5 \end{cases}$$

Moreover, $R_D \vee id_P$ reverses the longest substring of D_7 inside a string of C_{12}

$$(R_D \vee id_P)(w_0 s_1 w_1 \cdots w_{k-1} s_k w_k) = w_0 s_k w_1 \cdots w_{k-1} s_1 w_k.$$

Composing termwise the previous triples we get the triple

$$(R_D \vee R_P, T_D \vee T_P, I_D \vee I_P)$$

whose group has 140 elements instead of 48 elements of the group $(R_C/T_C/I_C)$ used in twelve tone composition. On the other hand the group generated by

$$(R_1, R_2/T_1, T_2/I_1, I_2) = \\ \{R_1^{\kappa_1} \circ R_2^{\kappa_2} \circ T_1^{\lambda_1} \circ T_2^{\lambda_2} \circ I_1^{\mu_1} \circ I_2^{\mu_2} \mid 0 \le \kappa_1, \kappa_2, \mu_1, \mu_2 < 2, \; 0 \le \lambda_1 < 7, \; 0 \le \lambda_2 < 5\}$$

and so the cardinality of this group is $2^2 \cdot 7 \cdot 5 \cdot 2^2 = 560$. Extension to arbitrary scales is provided.

In section 6 we introduce the notion of counterpoint space in order to describe various musical structures in an abstract setting. A *rti*-space is a triple $\mathcal{A} = ((r/t/i), A, a_0)$ consisting of a left group action and an element a_0 from which all elements of A are reachable, that is

$$A = \left\{ t^k \cdot a_0, rt^k \cdot a_0, rt^k i \cdot a_0, t^k i \cdot a_0 \mid k \in \mathbb{Z} \right\}.$$

Actually, the above action is simply transitive. Major and minor chords are defined with respect to a pair (p, q) of "intervals". We show that (p, q)-consonant chords in a counterpoint space constitute a counterpoint space, as well.

II. Scale Systems

i. Definition and Examples

A *scale of height* n is a linearly ordered set of musical objects

$$\Sigma : \sigma_0 < \sigma_1 < \cdots < \sigma_{n-1}.$$

The objects $\sigma_0, \sigma_1, \ldots, \sigma_{n-1}$ are the *degrees* of Σ.

Common pitch scales:

- the chromatic scale

$$C : c < c\sharp < d < d\sharp < e < f < f\sharp < g < g\sharp < a < a\sharp < b,$$

- the diatonic scale

$$D : c < d < e < f < g < a < b \ \text{(white keybords)},$$

- the pentatonic scale

$$P : c\sharp < d\sharp < f\sharp < g\sharp < a\sharp \ \text{(black keybords)},$$

- the whole-tone scale

$$H : c < d < e < f\sharp < g\sharp < a\sharp,$$

- the octatonic scale

$$O : c < d < d\sharp < f < f\sharp < g\sharp < a < b.$$

Durations and intervals may be organized into scales

- $DUR(n) : \frac{1}{2^{n-1}} < \frac{1}{2^{n-2}} < \cdots < \frac{1}{2} < 1$
- $INT : 1p < 2m < 2M < 3m < 3M < 4p < 4^+ < 5p < 6m < 6M < 7m < 7M < 8p$ (p=perfect, m=minor, M=Major, 4^+=augmented)

Any increasing sequence of indexes $0 \leq i_0 < i_1 < \cdots < i_\kappa < n$ induces the subscale of Σ

$$\Sigma[i_0, \ldots, i_\kappa] : \sigma_{i_0} < \sigma_{i_1} < \cdots < \sigma_{i_\kappa}.$$

For instance D, P, H, O are subscales of C:

$$D = C[0,2,4,5,7,9,11], \ P = C[1,3,6,8,10], \ H = C[0,2,4,7,9,11], \ O = C[0,2,3,5,6,8,9,11]$$

ii. Scale Schemes

Let s, t be two symbols connected with the ordering $s < t$ meaning "s smaller than t". Any string of $\{s, t\}^*$

$$w = w_1 w_2 \cdots w_\kappa, \quad w_i \in \{s, t\}$$

is called *scale scheme*; it generates by "prefix ranking" the scale

$$1 < w_1 < w_1 w_2 < \cdots < w_1 w_2 \cdots w_\kappa.$$

For instance, the major and minor scale schemes are

$$M = ttsttts \text{ and } m = tsttstt$$

respectively.

An *interpretation* of schemes into a scale

$$\Sigma : \sigma_0 < \sigma_1 < \cdots < \sigma_{n-1}$$

is a right action

$$\Sigma_n \times \{s, t\}^* \to \Sigma_n$$

fully determined by the assignments

$$(\sigma, s) \mapsto \sigma s, \quad (\sigma, t) \mapsto \sigma t \quad (\sigma \in \Sigma_n).$$

Then the *scale of root* $\sigma \in \Sigma_n$ with respect to the scheme w above, is

$$\sigma < \sigma w_1 < \sigma w_1 w_2 < \cdots < \sigma w_1 w_2 \cdots w_\kappa.$$

Implementing s and t in C as "semitone" and "tone" respectively, we get the ordinary major and minor scales with root $\sigma \in C$.

$$M(\sigma) : \sigma < \sigma t < \sigma tt < \sigma tts < \sigma ttst < \sigma ttstt < \sigma ttsttt < \sigma ttsttts,$$

$$m(\sigma) : \sigma < \sigma t < \sigma ts < \sigma tst < \sigma tstt < \sigma tstts < \sigma tsttst < \sigma tsttstt.$$

The ancient greek trope schemes (GTS) are lexicographically ordered from top to bottom with respect to the relation $s < t$ meaning that s is smaller than t:

s	t	t	s	t	t	t	:	Locrian scheme	(LO)
s	t	t	t	s	t	t	:	Phrygian scheme	(F)
t	s	t	t	s	t	t	:	Aeolian scheme	(A)
t	s	t	t	t	s	t	:	Dorian scheme	(D)
t	t	s	t	t	s	t	:	Mixolydian scheme	(M)
t	t	s	t	t	t	s	:	Ionian scheme	(I)
t	t	t	s	t	t	s	:	Lydian scheme	(LY)

That is

$$GTS : LO < F < A < D < M < I < LY$$

([10]).

III. The Fundamental Groups of a Scale

i. The Clock – Group

Consider the scale

$$\Sigma : \sigma_0 < \sigma_1 < \cdots < \sigma_{n-1}.$$

The clock group of Σ is $G_1(\Sigma) = (\Sigma_n, \oplus)$, with Σ_n denoting the set of degrees, $\Sigma_n = \{\sigma_0, \ldots, \sigma_{n-1}\}$ and the binary operation

$$\oplus : \Sigma_n \times \Sigma_n \to \Sigma_n, \ (\sigma_\kappa, \sigma_\lambda) \mapsto \sigma_\kappa \oplus \sigma_\lambda$$

is the clock addition

$$\sigma_\kappa \oplus \sigma_\lambda = \begin{cases} \sigma_{\kappa+\lambda} & \text{if } \kappa + \lambda < n, \\ \sigma_{\kappa+\lambda-n} & \text{if } \kappa + \lambda \geq n. \end{cases}$$

The neutral element is σ_0 and the opposite of σ_κ is $\sigma_{n-\kappa}$, $-\sigma_\kappa = \sigma_{n-\kappa}$, $\kappa = 0, \ldots, n-1$.

Transposition and Inversion with respect to Σ are exclusively defined in terms of $G_1(\Sigma)$. Precisely

$$T_\Sigma, I_\Sigma : \Sigma_n \to \Sigma_n, \ \ T_\Sigma(\sigma_k) = \sigma_\kappa \oplus \sigma_1, \ \ I_\Sigma(\sigma_\kappa) = -\sigma_\kappa, \ \ 0 \leq \kappa \leq n-1.$$

We observe that $T_C(f) = f\sharp$ and $T_D(f) = g$ and $T_P(c\sharp) = d\sharp$ and $T_C(c\sharp) = d$, etc. Likewise, $I_C(d) = a$ and $I_D(d) = g$, $I_P(a\sharp) = d\sharp$ and $I_C(a\sharp) = d$, etc

Proposition 1. *Consider two scales of the same height*

$$\Sigma : \sigma_0 < \sigma_1 < \cdots < \sigma_{n-1}, \ \ \Gamma : \gamma_0 < \gamma_1 < \cdots < \gamma_{n-1}.$$

Then the function

$$\phi : \Sigma_n \to \Gamma_n, \ \ \phi(\sigma_\kappa) = \gamma_\kappa \ \ (\kappa = 0, 1, \ldots, n-1)$$

is an isomorphism of $G_1(\Sigma)$ onto $G_1(\Gamma)$ preserving transposition and inversion

$$\phi \circ T_\Sigma = T_\Gamma \circ \phi, \ \ \phi \circ I_\Sigma = I_\Gamma \circ \phi$$

where "$\circ$" designates function composition performed from right to left.

Proof. We are going to show that ϕ preserves the clock addition, i.e. that

$$\phi(\sigma_\kappa \oplus \sigma_\lambda) = \phi(\sigma_\kappa) \oplus \phi(\sigma_\lambda) \ \text{ for all } \kappa, \lambda.$$

Indeed, if $\kappa + \lambda < n$, then

$$\phi(\sigma_\kappa \oplus \sigma_\lambda) = \phi(\sigma_{\kappa+\lambda}) = \gamma_{\kappa+\lambda} = \gamma_\kappa \oplus \gamma_\lambda = \phi(\sigma_\kappa) \oplus \phi(\sigma_\lambda).$$

In the case $\kappa + \lambda \geq n$, then

$$\phi(\sigma_\kappa \oplus \sigma_\lambda) = \phi(\sigma_{\kappa+\lambda-n}) = \gamma_{\kappa+\lambda-n} = \gamma_\kappa \oplus \gamma_\lambda = \phi(\sigma_\kappa) \oplus \phi(\sigma_\lambda).$$

The rest of the proof is straightforward. □

According to the previous result, scales with the same height behave alike from the transposition/inversion point of view.

ii. The Group of Σ-rows

Rows in an arbitrary scale will be discussed below. A pivotal axis of twelve-tone music is the restriction in the repetition of each of the twelve pitch classes. In a twelve-tone sequence, a pitch class cannot reappear before the remaining eleven pitch classes are heard, thus creating a permutation of twelve different pitch classes ([7], [9]). An *n-row* or *n-aggregate* is a rearrangement of the numbers $0, 1, \ldots, n-1$, that is a bijective function p of $\mathbb{Z}_n$ onto itself that can be represented by a matrix of the form

$$p = \begin{pmatrix} 0 & 1 & 2 & \cdots & n-1 \\ p(0) & p(1) & p(2) & \cdots & p(n-1) \end{pmatrix}$$

or shortly

$$p = (p(0), p(1), p(2), \cdots, p(n-1)).$$

Transposition and inversion can be pointwise extended on rows:

$$\begin{aligned} T\,(p(0), p(1), \ldots, p(n-1)) &= (T(p_0), T(p_1), \ldots, T(p_{n-1})) \\ &= (p(0) \oplus 1, p(1) \oplus 1, \ldots, p(n-1) \oplus 1), \end{aligned}$$

$$\begin{aligned} I\,(p(0), p(1), \ldots, p(n-1)) &= (I(p_0), I(p_1), \ldots, I(p_{n-1})) \\ &= (-p(0), -p(1), \ldots, -p(n-1)). \end{aligned}$$

Retrograde is the mirror image operator

$$R\,(p(0), p(1), \ldots, p(n-1)) = (p(n-1), \ldots, p(1), p(0)).$$

(R, T, I) will be referred to as a *counterpoint triple*. The set S_n of all n-rows is closed under composition

$$(p(0), p(1), \ldots, p(n-1)) \circ (q(0), q(1), \ldots, q(n-1)) = (p(q(0)), p(q(1)), \ldots, p(q(n-1))$$

and constitutes a group $(S_n, \circ)$.

Given a scale of height n

$$\Sigma : \sigma_0 < \sigma_1 < \cdots < \sigma_{n-1}$$

a Σ-row is a rearrangement of the elements $\sigma_0, \sigma_1, \ldots, \sigma_{n-1}$ represented as

$$\pi = \begin{pmatrix} \sigma_0 & \sigma_1 & \cdots & \sigma_{n-1} \\ \sigma_{p(0)} & \sigma_{p(1)} & \cdots & \sigma_{p(n-1)} \end{pmatrix}$$

or shortly

$$\pi = (\sigma_{p(0)}, \sigma_{p(1)}, \cdots, \sigma_{p(n-1)}),$$

where $(p(0), p(1), \ldots, p(n-1))$ is in S_n.

The set $S(\Sigma)$ of all Σ-rows is closed under composition

$$(\sigma_{p(0)}, \sigma_{p(1)}, \ldots, \sigma_{p(n-1)}) \circ (\sigma_{q(0)}, \sigma_{q(1)}, \ldots, \sigma_{q(n-1)}) = (\sigma_{p(q(0))}, \sigma_{p(q(1))}, \ldots, \sigma_{p(q(n-1))})$$

and constitutes the second fundamental group of Σ, $G_2(\Sigma) = (S(\Sigma), \circ)$.

Proposition 2. *If* Σ, Γ *are scales as in the statement of Proposition 1, then the assignment*

$$(\sigma_{p(0)}, \sigma_{p(1)}, \ldots, \sigma_{p(n-1)}) \mapsto (\gamma_{p(0)}, \gamma_{p(1)}, \ldots, \gamma_{p(n-1)})$$

is an isomorphism of $G_2(\Sigma)$ *onto* $G_2(\Gamma)$.

These data lead to introduce the notion of scale type, a means to classify scales. We call *type of height n* the scale

$$\mathbb{Z}_n : 0 < 1 < 2 < \cdots < n-1.$$

Its associated group $(\mathbb{Z}_n, \oplus)$ is the well known group of modulo n integers. For instance the types of the scales C, D, P, H, O are

$$\begin{aligned}\mathbb{Z}_{12} &: 0 < 1 < 2 < \cdots < 11,\\ \mathbb{Z}_{7} &: 0 < 1 < 2 < \cdots < 6,\\ \mathbb{Z}_{5} &: 0 < 1 < 2 < \cdots < 4,\\ \mathbb{Z}_{6} &: 0 < 1 < 2 < \cdots < 5,\\ \mathbb{Z}_{8} &: 0 < 1 < 2 < \cdots < 7,\end{aligned}$$

respectively. If two musical scales have the same height, then the isomorphisms of their corresponding fundamental groups describe equivalent musical mathematical structures besides the nature of the objects they act upon.

The triple $(R_\Sigma, T_\Sigma, I_\Sigma)$ defined below is the principal counterpoint triple of Σ:

$$\begin{aligned}R_\Sigma(\sigma_{p(0)}, \sigma_{p(1)}, \ldots, \sigma_{p(n-1)}) &= (\sigma_{p(n-1)}, \ldots, \sigma_{p(1)}, \sigma_{p(0)}),\\ T_\Sigma(\sigma_{p(0)}, \sigma_{p(1)}, \ldots, \sigma_{p(n-1)}) &= (\sigma_{p(0)} \oplus 1, \sigma_{p(1)} \oplus 1, \ldots, \sigma_{p(n-1)} \oplus 1),\\ I_\Sigma(\sigma_{p(0)}, \sigma_{p(1)}, \ldots, \sigma_{p(n-1)}) &= (\sigma_{-p(0)}, \sigma_{-p(1)}, \ldots, \sigma_{-p(n-1)}),\end{aligned}$$

where $\oplus$ is the $\bmod n$ addition and $-p(k)$ is the opposite of $p(k)$ with respect to this addition.

Proposition 3. *In a scale Σ of height n the following equalities hold*

$$R_\Sigma^2 = id = I_\Sigma^2, \quad T_\Sigma^n = id, \quad R_\Sigma \circ I_\Sigma = I_\Sigma \circ R_\Sigma,$$

$$R_\Sigma \circ T_\Sigma = T_\Sigma \circ R_\Sigma, \quad I_\Sigma \circ T_\Sigma = T_\Sigma^{n-1} \circ I_\Sigma.$$

Proof. We only establish the last equality. First we show that for every $\sigma_\kappa \in \Sigma_n$ we have

$$(I_\Sigma \circ T_\Sigma)(\sigma_\kappa) = \left(T_\Sigma^{n-1} \circ I_\Sigma\right)(\sigma_\kappa).$$

Indeed

$$\begin{aligned}(I_\Sigma \circ T_\Sigma)(\sigma_\kappa) &= I_\Sigma\left(T_\Sigma(\sigma_\kappa)\right) = I_\Sigma(\sigma_\kappa \oplus \sigma_1) = -(\sigma_\kappa \oplus \sigma_1) = (-\sigma_\kappa) \oplus (-\sigma_1) =\\ &= (-\sigma_\kappa) \oplus \sigma_{n-1} = T_\Sigma^{n-1}(-\sigma_\kappa) = T_\Sigma^{n-1}\left(I_\Sigma(\sigma_\kappa)\right) = \left(T_\Sigma^{n-1} \circ I_\Sigma\right)(\sigma_\kappa).\end{aligned}$$

Furthermore, for every $(\sigma_{p(0)}, \sigma_{p(1)}, \ldots, \sigma_{p(n-1)}) \in S(\Sigma)$ we have

$$\begin{aligned}(I_\Sigma \circ T_\Sigma)(\sigma_{p(0)}, \sigma_{p(1)}, \ldots, \sigma_{p(n-1)}) &= \left((I_\Sigma \circ T_\Sigma)(\sigma_{p(0)}), \ldots, (I_\Sigma \circ T_\Sigma)(\sigma_{p(n-1)})\right) =\\ &= \left(\left(T_\Sigma^{n-1} \circ I_\Sigma\right)(\sigma_{p(0)}), \ldots, \left(T_\Sigma^{n-1} \circ I_\Sigma\right)(\sigma_{p(n-1)})\right) = \left(T_\Sigma^{n-1} \circ I_\Sigma\right)(\sigma_{p(0)}, \sigma_{p(1)}, \ldots, \sigma_{p(n-1)})\end{aligned}$$

hence the desired result. □

Remark. Given that the diatonic scale may be constructed with fifths or fourths, the pentatonic with stacked fifths, etc. the scales listed in II.i can all be considered as symmetrical generated constructions. As Andreatta points out, group is the dominating algebraic structure utilized to describe symmetry in music ([1]).

IV. Counterpoint Groups

They are groups generated by retrograde, transposition and inversion operators playing a significant role in serial composition. More precisely, the $(r/t/i)$–group is the free group generated by three letters r, t, i subjected to the axioms

$$r^2 = 1 = i^2, \quad ri = ir, \quad rt = tr, \quad it = t^{-1}i. \tag{1}$$

Its elements are of the form

$$t^\kappa, \quad rt^\kappa, \quad rt^\kappa i, \quad t^\kappa i \quad (\kappa \in \mathbb{Z}).$$

Three subgroups of the above group are of interest: (r/i), (r/t) and (t/i). The first one $(r/i) = \{1, r, iri\}$ is isomorphic to the Klein four group $\mathbb{Z}_2 \times \mathbb{Z}_2$ ([7], [9]). The other two groups are $(r/t) = \{t^\kappa, rt^\kappa \mid \kappa \in \mathbb{Z}\}$ and $(t/i) = \{t^\kappa, t^\kappa i \mid \kappa \in \mathbb{Z}\}$.

Notice that the groups $(r/t/i)$, (r/t) and (t/i) are infinite unless t has finite order, say n, in which case (r/t) and (t/i) coincide with the commutative group $\mathbb{Z}_2 \times \mathbb{Z}_n$ and the dihedral group D_{2n} respectively ([2], [3]). In addition $\mathrm{card}(r/t/i) = 4n$, $\mathrm{card}(r/t) = 2n = \mathrm{card}(t/i)$. In the sequel we deal with left group actions of the form

$$(r/t/i) \times A \to A, \quad (u, a) \mapsto u.a$$

Due to the free character of the group $(r/t/i)$ the above action is completely determined by the values $r.a$, $t.a$, $i.a$ $(a \in A)$ compatible with equalities (1), which means that

$$r^2.a = a = i^2.a, \quad ri.a = ir.a, \quad rt.a = tr.a, \quad it.a = t^{-1}i.a, \tag{2}$$

for all $(a \in A)$.

Examples. The operation of row composition defines a left action of $(r/t/i)$ on S_n in the following manner:

$$r.p = R \circ p, \quad t.p = T \circ p, \quad i.p = I \circ p,$$

for all $p = (p(0), p(1), \ldots, p(n-1)) \in S_n$. For $n = 12$, choosing the Schoenberg op.36 aggregate $p = (0, 1, 6, 2, 7, 9, 3, 4, 10, 11, 5, 8)$ we get

$$\begin{aligned} r.p &= (8, 5, 11, 10, 4, 3, 9, 7, 2, 6, 1, 0) \\ i.p &= (0, 11, 6, 10, 5, 3, 9, 8, 2, 1, 7, 4) \\ t^3 i.p &= (3, 2, 9, 1, 8, 6, 0, 11, 5, 4, 10, 7). \end{aligned}$$

More generally, for any scale Σ the relations

$$r.\tau = R_\Sigma \circ \tau, \quad t.\tau = T_\Sigma \circ \tau, \quad i.\tau = I_\Sigma \circ \tau \quad (\tau \in S(\Sigma))$$

actually define an action of $(r/t/i)$ on $S(\Sigma)$.

Further actions:

- $(r/t/i) \times \mathbb{Z}_{12}^* \to \mathbb{Z}_{12}^*$ with
$$r.s = s_\kappa \cdots s_2 s_1, \; t.s = (s_1 \oplus 1)(s_2 \oplus 1) \cdots (s_\kappa \oplus 1), \; i.s = (12 - s_1)(12 - s_2) \cdots (12 - s_\kappa),$$
for all $s = s_1 s_2 \cdots s_\kappa$ in $\mathbb{Z}_{12}^*$

Likewise

- $(r/t/i) \times \mathbb{Z}^* \to \mathbb{Z}^*$, with
$$r.s = s_\kappa \cdots s_2 s_1, \; t.s = (s_1 + 1)(s_2 + 1) \cdots (s_\kappa + 1), \; i.s = (-s_1)(-s_2) \cdots (-s_\kappa),$$
for all $s = s_1 s_2 \cdots s_\kappa$ in $\mathbb{Z}^*$.

In all these cases the conditions (2) are verified.

V. Partial Counterpoint Operators

The aim of this section is to present new counterpoint triples, which enrich the musical material permitting to compose extensive twelve tone structures. By construction, the chromatic scale is the disjoint union of the diatonic scale and the pentatonic scale, $C = D \cup P$.

Therefore, apart from (R_C, T_C, I_C), we obtain two partial counterpoint triples

$$(R_D \vee id_P, T_D \vee id_P, I_D \vee id_P) \text{ and } (id_D \vee R_P, id_D \vee T_P, id_D \vee I_P), \tag{3}$$

where $T_D \vee id_P$ (resp. $id_D \vee I_P$) is the operator on C_{12} coinciding with T_D (resp. T_P) on the set D_7 (resp. P_5) and leaving unchanged the elements of $C_{12} - D_7$ (resp. $C_{12} - P_5$). Likewise, $(I_D \vee id_P)(x) = I_D(x)$ if $x \in D_7$, $= x$, else. Finally, $R_D \vee id_P$ reverses only the longest string of D_7^* occurring inside any string of C_{12}^*

$$(R_D \vee id_P)(w_0 s_1 w_1 \cdots w_{\kappa-1} s_\kappa w_\kappa) = w_0 s_\kappa w_1 \cdots w_{\kappa-1} s_1 w_\kappa$$

for all $s_i \in D_7$, $w_i \in (C_{12} - D_7)^*$. Similar definitions for $id_D \vee I_P$ and $id_D \vee R_P$ can be stated. In the context of mod 12 integers the above data are encoded as follows:

$$\begin{array}{rl} & \underline{0}\ 1\ \underline{2}\ 3\ \underline{4}\ \underline{5}\ 6\ \underline{7}\ 8\ \underline{9}\ 10\ \underline{11} \\ R_D \vee id_P = & (11,1,9,3,7,5,6,4,8,2,10,0) \\ T_D \vee id_P = & (11,1,0,3,2,4,6,5,8,7,10,9) \\ I_D \vee id_P = & (0,1,11,3,9,7,6,5,8,4,10,2) \end{array}$$

$$\begin{array}{rl} & 0\ \underline{1}\ 2\ \underline{3}\ 4\ 5\ \underline{6}\ 7\ \underline{8}\ 9\ \underline{10}\ 11 \\ id_D \vee R_P = & (0,10,2,8,4,5,6,7,3,9,1,11) \\ id_D \vee T_P = & (0,3,2,6,4,5,8,7,10,9,1,11) \\ id_D \vee I_P = & (0,1,2,10,4,5,8,7,6,9,3,11) \end{array}$$

Furthermore, composing termwise the triples (3), we get the triple

$$(R_D \vee R_P, T_D \vee T_P, I_D \vee I_P)$$

where

$$(T_D \vee T_P)(x) = \begin{cases} T_D(x), & \text{if } x \in D_7 \\ T_P(x), & \text{if } x \in P_5 \end{cases}$$

and so on.

In terms of pitch classes

$$\begin{aligned} R_D \vee R_P &= (11,10,9,8,7,5,6,4,3,2,1,0) \\ T_D \vee T_P &= (11,3,0,6,2,4,8,5,10,7,1,9) \\ I_D \vee I_P &= (0,1,11,10,9,7,8,5,6,4,3,2) \end{aligned}$$

Obviously $T_D \vee T_P \neq T_C$, $I_D \vee I_P \neq I_C$, $R_D \vee R_P \neq R_C$. The order of $T_D \vee T_P$ into the group $S(C)$ is $7 \cdot 5 = 35$ and so the group $(R_D \vee R_P / T_D \vee T_P / I_D \vee I_P)$ has $4 \cdot 35 = 140$ elements instead of 48 elements of the group $(R_C / T_C / I_C)$ used in twelve tone music. Consequently, we are able to speak of a multiple enrichment of the organisation of the pitch material, beyond the already known ways of managing it in twelve-tone music and without the encroachment of its main principles.

Our next task will be to determine the group generated by the six partial operators

$$\begin{aligned} R_1 = R_D \vee id_P, \quad R_2 = id_D \vee R_P, \\ T_1 = T_D \vee id_P, \quad T_2 = id_D \vee T_P, \\ I_1 = I_D \vee id_P, \quad I_2 = id_D \vee I_P. \end{aligned} \tag{4}$$

By taking into account that generators (4) commute to each other except that

$$I_1 \circ T_1 = T_1^{-1} \circ I_1 = T_1^6 \circ I_1 \text{ and } I_2 \circ T_2 = T_2^{-1} \circ I_2 = T_2^4 \circ I_2$$

we get

$$\begin{aligned} &(R_1, R_2/T_1, T_2/I_1, I_2) = \\ &\quad = \left\{ R_1^{\kappa_1} \circ R_2^{\kappa_2} \circ T_1^{\lambda_1} \circ T_2^{\lambda_2} \circ I_1^{\mu_1} \circ I_2^{\mu_2} \mid 0 \le \kappa_1, \kappa_2, \mu_1, \mu_2 < 2,\ 0 \le \lambda_1 < 7,\ 0 \le \lambda_2 < 5 \right\} \end{aligned}$$

and thus

$$\mathrm{card}(R_1, R_2/T_1, T_2/I_1, I_2) = 2^2 \cdot 7 \cdot 5 \cdot 2^2 = 560.$$

Let us discuss more complex situations. Suppose that $(A_1, \ldots, A_\kappa)$ is a partition of a scale Σ

$$\Sigma_n = A_1 \cup \cdots \cup A_\kappa,\ A_i \cap A_j = \emptyset \ \text{ for } i \neq j$$

and denote by $\bar{A}_i$ the set complement of A_i in Σ_n, $\bar{A}_i = \Sigma_n - A_i$, $1 \le i \le \kappa$. Since subsets of Σ are scales with the induced ordering, we may define the counterpoint triples

$$R_i = R_{A_i} \vee id_{\bar{A}_i},\ T_i = T_{A_i} \vee id_{\bar{A}_i},\ I_i = I_{A_i} \vee id_{\bar{A}_i},\ 1 \le i \le \kappa.$$

Theorem 4. *The cardinality of the group generated by the above operators is*

$$\mathrm{card}((R_i)_i/(T_i)_i/(I_i)_i) = 2^\kappa \cdot \mathrm{ord}(T_1) \cdots \mathrm{ord}(T_\kappa) \cdot 2^\kappa$$

where $\mathrm{ord}(T_i)$ *is the order of* T_i *in the group* $S(\Sigma)$.

In another direction we may replace R_i, T_i, I_i *by a triple of permutations* $(\rho_i, \tau_i, \iota_i)$ *on* A_i, $1 \le i \le \kappa$.

VI. Counterpoint Spaces

The algebraic structure of counterpoint space (CP-space) is proposed, in order to found basic musical notions in a formal framework.

An *rti-space* is a triple $\mathcal{A} = ((r/t/i), A, a_0)$ formed by a left group action

$$(r/t/i) \times A \to A$$

and an initial element $a_0 \in A$ (playing the role of the pitch c).

These data must comply with the following axiom:

S) the elements of A are accessible from a_0, that is

$$A = \left\{ t^k \cdot a_0, rt^k \cdot a_0, rt^k i \cdot a_0, t^k i \cdot a_0 \mid k \in \mathbb{Z} \right\}.$$

The *ti*-spaces are formulated as before except we replace $(r/t/i)$ by the group (t/i). Axiom S) takes the form

$$A = \left\{ t^k \cdot a_0, t^{ki} \cdot a_0 \mid k \in \mathbb{Z} \right\}.$$

Examples. • $\mathcal{A} = ((r/t/i), A_n, (0,1,\ldots,n-1))$, where

$$A_n = \{T^\kappa, R \circ T^\kappa, R \circ T^\kappa \circ I, T^\kappa \circ I \mid \kappa = 0,1,\ldots,n-1\}$$

Special case $\mathcal{A}_{12} = ((r/t/i), A_{12}, (0,1,\ldots,11))$.

- Given an arbitrary scale Σ of height n, we have the following CP-space

$$\mathcal{A}_\Sigma = ((r/t/i), A_\Sigma, (\sigma_0, \sigma_1, \ldots, \sigma_{n-1}))\,,$$

where

$$A_\Sigma = \{T^\kappa_\Sigma, R_\Sigma \circ T^\kappa_\Sigma, R_\Sigma \circ T^\kappa_\Sigma \circ I_\Sigma, T^\kappa_\Sigma \circ I_\Sigma \mid \kappa = 0,1,\ldots,n-1\}$$

Standard CP-spaces:

- $\mathcal{Z}_n = ((t/i), \mathbb{Z}_n, 0)$ instances $n = 12, 7, 5$
- $\mathcal{Z} = ((t/i), \mathbb{Z}, 0)$

Remark. It should be noticed that the last two examples of actions in section 4 can not be organized into CP-spaces since they are deprived initial elements.

Simply transitive actions are often encountered in mathematical music theory ([3]). Recall that a left group action

$$G \times A \to A,\ (g,a) \mapsto ga,$$

is *simply transitive* whenever for any pair $(a_1, a_2) \in A^2$ there exists a unique $g \in G$ so that $a_2 = ga_1$.

Proposition 5. *If $\mathcal{A}$ fulfils the additional axiom*

S') the function $g \mapsto ga_0$ is injective,

then the action $(r/t/i) \times A \to A$ is simply transitive.

Proof. We follow the guideline proof arguments of the corresponding result in [3]

(Existence). According to the axiom S) any pair $(a_1, a_2) \in A^2$ is written $a_1 = g_1 a_0$, $a_2 = g_2 a_0$ and so $a_0 = g_1^{-1} a_1$ and $a_2 = g_2 g_1^{-1} a_1$.

(Uniqueness). Assume, now, that $a_2 = u_1 a_1 = u_2 a_1$ then $u_1 g_1 a_0 = u_2 g_1 a_0$ and so by axiom S') $u_1 g_1 = u_2 g_1$ and by right cancellation $u_1 = u_2$. □

Corollary 6. *The counterpoint spaces $\mathcal{A}_n$, $\mathcal{A}_\Sigma$, $\mathcal{Z}_n$, $\mathcal{Z}$ fulfill the axiom S') and thus the corresponding actions are simply transitive.*

We are going to indicate how basic musical notation can be defined in the setup of counterpoint spaces. In the traditional context consonant chords are built by overposing thirds. A chord is a triple of simultaneously played pitches. A major (resp. minor) chord consists of a root pitch, a second pitch four (resp. three) semitones above the root and a third pitch seven semitones above the root. Major (resp. minor) chords are successive transpositions of the C-major chord $(0,4,7)$ (resp. f-minor chord $(5,8,0)$). Moreover $(5,8,0)$ is the inversion of $(0,4,7)$:

$$\begin{array}{ccc}
C = (0,4,7) & \xrightarrow{\ I\ } & (5,8,0) = f \\
\downarrow T & & \downarrow T \\
C\sharp = (1,5,8) & & (6,9,1) = f\sharp \\
\downarrow T & & \downarrow T \\
D = (2,6,9) & & (7,10,2) = g \\
\downarrow & & \downarrow \\
\vdots & & \vdots \\
\downarrow T & & \downarrow T \\
B = (11,3,7) & & (4,7,11) = e
\end{array}$$

The foundation of chord theory can be realized in a ti-space $\mathcal{A} = ((t/i), A, a_0)$. First we need a building pair (p, q) of natural numbers abstracting the pair $(4, 3)$ for thirds. Now, the (p, q) major chords in $\mathcal{A}$ are all triples of the form $(t^\kappa \cdot a_0, t^{\kappa+p} \cdot a_0, t^{\kappa+p+q} \cdot a_0)$, κ running over the set $\mathbb{Z}$ of integers. The (p, q)-minor chords in $\mathcal{A}$ are all triples of the form $(t^\kappa i \cdot a_0, t^{\kappa+q} i \cdot a_0, t^{\kappa+p+q} i \cdot a_0)$, $\kappa \in \mathbb{Z}$.

All together (p, q)-major chords and (p, q)-minor chords form the set $(p, q) - CC(\mathcal{A})$ of (p, q) *consonant chords* in $\mathcal{A}$

$$(p, q) - CC(\mathcal{A}) = (p, q) - MC(\mathcal{A}) \cup (p, q) - mC(\mathcal{A}),$$

where $(p, q) - MC(\mathcal{A})$, $(p, q) - mC(\mathcal{A})$ stand for the sets of major and minor chords in $\mathcal{A}$ respectively.

Remark. Since an ordinary chord consists of simultaneously sounding pitches it is not necessary to insist on a particular ordering of the pitches inside a chord. We adopt this mathematical abuse also in our general setting, i.e. any chord (a_1, a_2, a_3) will be identified with the chord (a_3, a_2, a_1)

The following result confirms that the group (t/i) acts on the left on $(p, q) - CC(\mathcal{A})$.

Proposition 7. *i) Left multiplication by i changes the arity of the chords, that is converts major to minor chords and vice versa.*
ii) Left multiplication by t preserves the arity of chords.

Proof. Indeed, if $(t^\kappa \cdot a_0, t^{\kappa+p} \cdot a_0, t^{\kappa+p+q} \cdot a_0)$ is in $(p, q) - MC(\mathcal{A})$, then

$$\begin{aligned} i(t^\kappa \cdot a_0, t^{\kappa+p} \cdot a_0, t^{\kappa+p+q} \cdot a_0) &= (it^\kappa \cdot a_0, it^{\kappa+p} \cdot a_0, it^{\kappa+p+q} \cdot a_0) \\ &\overset{(+)}{=} (t^{-\kappa} i \cdot a_0, t^{-\kappa-p} i \cdot a_0, t^{-\kappa-p-q} i \cdot a_0) \\ &\overset{\text{Remark}}{=} (t^{-\kappa-p-q} i \cdot a_0, t^{-\kappa-p} i \cdot a_0, t^{-\kappa} i \cdot a_0) \in (p, q) - mC(\mathcal{A}) \end{aligned}$$

where equality (+) comes from the axiom $it = t^{-1}i$. The other assertions can be proved by similar arguments. □

Proposition 8. *If $\mathcal{A} = ((t/i), A, a_0)$ is a ti-space, then*

$$(p, q) - chord(\mathcal{A}) = \big((t/i), (p, q) - CC(\mathcal{A}), (a_0, t^p \cdot a_0, t^{p+q} \cdot a_0)\big)$$

is also a ti-space. Moreover, if $\mathcal{A}$ satisfies the axiom S'), then $(p, q) - chord(\mathcal{A})$ also satisfies S').

Proof. First we show that all major chords are accessible from the initial major chord $(a_0, t^p \cdot a_0, t^{p+q} \cdot a_0)$:

$$(t^\kappa \cdot a_0, t^{\kappa+p} \cdot a_0, t^{\kappa+p+q} \cdot a_0) = t^\kappa (a_0, t^p \cdot a_0, t^{p+q} \cdot a_0).$$

On the other hand, minor chords are accessible from the generic minor chord $(t^{-p-q} i \cdot a_0, t^{-p} i \cdot a_0, i \cdot a_0)$:

$$(t^{-\kappa-p-q} i \cdot a_0, t^{-\kappa-p} i \cdot a_0, t^{-\kappa} i \cdot a_0) = t^{-\kappa} (t^{-p-q} i \cdot a_0, t^{-p} i \cdot a_0, i \cdot a_0).$$

Notice that when κ runs over $\mathbb{Z}$, the minor chord $(t^{-\kappa-p-q} i \cdot a_0, t^{-\kappa-p} i \cdot a_0, t^{-\kappa} i \cdot a_0)$ runs over the whole set $(p, q) - mC(\mathcal{A})$.

Finally,

$$(t^{-p-q} i \cdot a_0, t^{-p} i \cdot a_0, i \cdot a_0) = i(a_0, t^p \cdot a_0, t^{p+q} \cdot a_0).$$

Hence axiom S) is valid in $(p,q)-\mathrm{chord}(\mathcal{A})$. To establish S') we have to show the injectivity of the function $u \mapsto u(a_0, t^p \cdot a_0, t^{p+q} \cdot a_0)$. We have

$$u_1(a_0, t^p \cdot a_0, t^{p+q} \cdot a_0) = u_2(a_0, t^p \cdot a_0, t^{p+q} \cdot a_0) \text{ implies}$$
$$(u_1 a_0, u_1 t^p \cdot a_0, u_1 t^{p+q} \cdot a_0) = (u_2 \cdot a_0, u_2 t^p \cdot a_0, u_2 t^{p+q} \cdot a_0) \text{ implies}$$
$$u_1 \cdot a_0 = u_2 \cdot a_0$$

which, by axiom S') for $\mathcal{A}$, gives $u_1 = u_2$ and the proof is achieved. □

Corollary 9. *The action*

$$(t/i) \times (p,q) - CC(\mathcal{A}) \to (p,q) - CC(\mathcal{A})$$

is simply transitive, provided $\mathcal{A}$ satisfies S').

Proof. It is a consequence of proposition 3. □

VII. Conclusion-Future Work

A generic notion in music theory is that of a scale. It refers to a finite linear ordered set of musical objects called degrees of the scale. This definition covers various musical scale situations. Basic tools in this setup are the fundamental groups of a scale: clock group and group of rows. Scales are classified according to their cardinality: two equivalent scales have isomorphic their corresponding fundamental groups.

Major and minor scale schemes as well as ancient greek musical trope schemes are discussed. Groups generated by retrograde/transposition/inversion operators are used to enrich twelve tone composition techniques. Essentially, a novel view of partiality of the twelve-tone aggregate is proposed, which is not based on the division of the chromatic pitch set into trichords, tetrachords or hexachords, but on the partition $C = D \cup P$. Counterpoint spaces are domains suitable to develop an abstract music theory. The main result is that consonant chords in such a space form also a counterpoint space. A future task will be the study of neo-Riemannian theory in the present general context, which will provide a novel view of the relationship between voice-leading and counterpoint. Infinite scales could have also interest to be investigated.

Acknowledgement

The author is indebted to the anonymous referees for their fruitful comments.

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