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Contents

Foreword

We are glad to announce the release of the second number of the fourth volume of *MusMat – Brazilian Journal of Music and Mathematics*. Six original articles integrate this number. **Marco Feitosa** introduces the concept of *Partitional Harmony*, an original field of research that relates the Theory of Integer Partitions to several fields of Post-Tonal Theory. **Robert Morris** provides an in-depth analysis of Feldman's *Last Pieces* for piano solo, bringing to light information that can help the pianists perform the cancelling effect requested by the composer. **Gabriel Pareyon** combines Matthai philosophy with Category Theory, using Yoneda lemma, suggesting that the latter can support a robust philosophy of music within the scope of Category Theory. **Robert Peck** investigates the inversion operation, in terms of cycles, and its application to a dramaturgical context, through the examination of the Aristotelian concept of *peripeteia*, as observable in Birtwistle's opera *Punch and Judy*. **Paulo de Tarso Salles** explores Forte's Genera Theory (as well as other proposals that deal with similarities between pitch-class sets) and demonstrates the application of this theory in some works by Villa-Lobos. **Pauxy Gentil-Nunes** discusses the *Partitioning Complexes* and their application in musical practice by examining three situations: textural planning in the context of compositional processes, observation of the relationship between textural configurations and coupling of the body (*Performative Partitioning*), and *Spatial Partitioning*.

Liduino Pitombeira

December 2020

Partitional Harmony: The Partitioning of Pitch Spaces

*Marco Feitosa
Federal University of Rio de Janeiro (UFRJ)
marco.feitosa@ufrj.br
Orcid: 0000-0001-8754-2352
DOI: 10.46926/musmat.2020v4n2.01-27

Abstract: *In this preliminary work, we seek to present a brief historical review of the use of partitions in music, to provide a concise introduction to the theory of partitions, and lastly, through an extensive bibliographic revision and a thoughtful theoretical reflection, to lay the foundations of what we call partitional harmony – a comprehensive harmonic conception which relates the theory of partitions to several fields of post-tonal music theory. At the end, some basic operations (pitch, transposition, inversion, and multiplication) are defined and an illustrative musical application is provided, followed by our research prospects.*

Keywords*: Post-Tonal Music Theory. Pitch-Class Set Theory. Pitch Spaces. Theory of Partitions. Partitional Harmony.*

Sed omnia in mensura et numero et pondere disposuisti.

Liber Sapientiae, 11:20

I. Introduction

Partitions[1] play important roles in several areas of mathematics such as combinatorics, Lie theory, representation theory, mathematical physics, theory of special functions, and so on. However, it is in the field of number theory [8, 80, 107] that they are investigated in greater depth and where their fundamental studies and main theoretical advances are concentrated. The

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[1]Throughout this work, partitions refer mostly, but not exclusively, to *integer partitions*. For definitions, see Section II.

simple notion of dividing an object into sub-objects makes partitions find use in the most varied fields of knowledge, from, for example, non-parametric statistics to particle physics [9, pp. xv-xvi].

The use of partitions in music dates back to the incursion of mathematical combinatorics into music theory in the 17th century [94, 127], and comprise a wide range of applications with great potential for both musical composition and analysis. Among some of the works that associate partitions and music in a more explicit way, we highlight the early studies of twelve-tone partitioning, in particular the *source set* approaches of Babbitt [13, 14, 15], Perle [129, 130, 131] and Martino [108], in addition to other studies in *combinatoriality* by Forte [56], Howe [85] and Gamer [65]. In the same direction, we also mention the works of Wintle [175], Bazelow and Brickle [20], Morris and Starr [117, 124, 161, 162, 163], Haimo and Johnson [77], and the further partitional or *mosaic* approaches of Mead [110, 111], Kurth [97, 98, 99, 100], Morris and Alegant [2, 3, 4, 5, 6, 122, 123]. Morris still uses partitions to shape some of his *compositional designs*, especially the Design VI [118, pp. 265-270]. Although this is an extensive list, even chronologically speaking, it is worth noting that most of these works employ *partitions of the aggregate*, or rather *set partitions*,[2] almost exclusively. Other relatively recent works present varied musical applications for partitions with a more mathematical approach.[3]

In Brazil, a seminal work relating partitions to musical texture is the *partitional analysis*, attributed to Gentil-Nunes [66], which consists of an original approach to musical composition and analysis based on a convergence between the theory of partitions and the textural analysis proposed by Berry [26]. Through an exhaustive taxonomy for the texture parameter, it offers a topological and metric mapping of textural configurations, and presents formal structures that can be applied to various fields of texture. It also includes some original concepts, such as the *agglomeration* and *dispersion* indices, the *partitional operators*, as well as the software *Parsemat*® and its graphical tools–the *indexogram* and *partitiogram*.

In the wake of partitional analysis, many other works have been carried out,[4] especially within the MusMat research group,[5] associating partitions with varied musical parameters or aspects such as rhythm, timbre, melody, contours, gestures, events, etc. Moreover, not only new tools have been developed, but the theory itself has been considerably expanded, mainly by the recent works of Gentil-Nunes [67, 68] and Sousa [157, 158, 159].

Given the strong influence of Berry's textural approach, inherent to the very conception of partitional analysis [69], it is natural that most of these works are concerned with musical texture and its elements. Therefore, although Berry addresses harmony in the first chapter of his book (tonality) and even in the second (texture)–precisely when discussing the concepts of *density* and *dissonance* [26, pp. 209-213], essentially harmonic (vertical) aspects, or more specifically, aspects related to the musical organization and structuring of pitches have not yet been deeply explored in the light of partitional analysis.

In this regard, as Gentil-Nunes [67, p. 107] points out, "each partitioning has its own idiosyncrasy and has to be evaluated from scratch in this respect, since each handled material has its own nature." Bearing that in mind, we propose a way of conceiving harmony and harmonic relations through the partitioning of pitch spaces. As we shall see, this comprehensive harmonic conception, which we call *partitional harmony*, relates the theory of partitions to several fields of post-tonal

[2] *Set partitions* concern the ways in which a set of n elements can be split up into a set of disjoint subsets [9, p. 214]. A partition of a set is "a collection of disjoint sets whose union is the given set" [86]. For example, the set $\{1, 2, 3\}$ has 5 set partitions: $\{1, 2, 3\}$; $\{1, 2\}, \{3\}$; $\{1, 3\}, \{2\}$; $\{2, 3\}, \{1\}$; $\{1\}, \{2\}, \{3\}$. An alternative representation may be: 123, 12|3, 13|2, 1|23, 1|2|3. See also [106].

[3] See [43, 51, 61, 62, 63, 87, 91, 105, 109, 169].

[4] Publications related to partitional analysis are available on Gentil-Nunes' personal website: `https://pauxy.net/partitional-analysis-publications`.

[5] The list of works published by the group is available at: `https://musmat.org/publicacoes-papers`.

music theory, especially *pitch-class set theory*, but also *transformational theory* and *diatonic set theory*, showing great potential for integration with partitional analysis itself, in addition to other musical and even mathematical theories. Last but not least, such a partitional approach to harmony allows us to establish *harmonic directionality* and *functionality* in post-tonal musical contexts, paving the way for the development of a *post-tonal system*, if we can speak of such a thing, and finally–in a broader sense–a *post-tonal harmony*.

II. The Theory of Partitions

According to Andrews [8, p. 149], "the theory of partitions is an area of additive number theory, a subject concerning the representation of integers as sums of other integers." By his definition, a *partition* of a positive integer n is a representation of n as a sum of positive integers, called *summands* or *parts* of the partition, the order of which is irrelevant. More formally [9, p. 1], a partition of a positive integer n is a finite non-increasing sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, such that $\sum_{i=1}^{k} \lambda_i = n$. Each λ_i corresponds to a part of the partition and if λ is a partition of n, then we write either $\lambda \vdash n$ or $|\lambda| = n$. We call λ a *k-partition* if λ has k parts, and the number of parts of λ is also called the *length* of λ, being denoted by $\ell(\lambda)$ [160, p. 58].

The number of distinct partitions of a positive integer n is given by $p(n)$, called the *partition function*. So, for example, $p(4) = 5$, as there are altogether 5 different ways to write the number 4 as a sum of positive integers: $4 = 3+1 = 2+2 = 2+1+1 = 1+1+1+1$. It is clear that the partitions of n correspond to the set of solutions $(j_1, j_2, \ldots, j_n)$ of the Diophantine equation $1j_1 + 2j_2 + \ldots + nj_n = n$ [173]. Therefore, in our example, the partitions (4), $(3,1)$, $(2,2)$, $(2,1,1)$, $(1,1,1,1)$ correspond, respectively, to the solutions $(j_1, j_2, j_3, j_4) = (0,0,0,1)$, $(1,0,1,0)$, $(0,2,0,0)$, $(2,1,0,0)$, $(4,0,0,0)$ of the equation $1j_1 + 2j_2 + 3j_3 + 4j_4 = 4$. By definition, $p(n) = 0$, when n is negative, and by convention, $p(0) = 1$, referring to the empty partition (with no parts). Table 1 shows the values of $p(n)$ for each corresponding n.

Table 1: *Values of* $p(n)$, *for* $n = \{0, 1, \ldots, 12\}$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$p(n)$	1	1	2	3	5	7	11	15	22	30	42	56	77

The partition function, despite its conceptual simplicity, is not at all obvious or trivial. As n grows, the number of partitions increases quite rapidly, making the calculation more and more difficult and complex. For example, $p(50) = 204226$, $p(100) = 190569292$ and $p(200) = 3972999029388$. Not surprisingly, the study of this function has been an object of interest to notable mathematicians since the mid-18th century, playing a central role in the development of number theory. Among some of the most relevant works, we can highlight the generating function for $p(n)$, developed by Euler [53], the asymptotic formula by Hardy and Ramanujan [79], perfected two decades later by Rademacher [137, 138], and–more recently–the algebraic formula by Bruinier and Ono [34], which expresses $p(n)$ as a finite sum of algebraic numbers, in addition to the discovery of the "fractal behavior" of partitions by Folsom, Kent and Ono [55].

While $p(n)$ comprises the set of all *unrestricted partitions* of n, other functions are limited to subsets of *restricted partitions*, i.e., partitions whose parts satisfy some condition. A function like $p(S, n)$, for example, denotes the number of partitions of n that belong to a subset S of partitions whose parts can be even, odd, distinct, prime, etc. [9, p. 2]. Other functions such as $p_k(n)$ and $p(j, k, n)$ denote, respectively, the number of partitions of n with exactly k parts and the number of partitions of n into at most k parts, with largest part at most j [160, p. 58]. Still others may be constrained to even smaller subsets or establish even more specific conditions. Functions involving restricted partitions can be simply represented by the generic formula $p(n \mid [\text{condition}])$ [11, p. 6] – for example, $p(n \mid \text{odd parts})$, $p(n \mid \text{distinct parts in } \{3, 6, 12, 24, \dots\})$, $p(n - r \mid \text{all parts} \leq r)$, and so forth. Table 2 shows the values of $p_k(12)$, or $p(12 \mid k \text{ parts})$, for each corresponding k.

Table 2: *Values of* $p_k(12)$*, for* $k = \{1, 2, \dots, 12\}$.

k	1	2	3	4	5	6	7	8	9	10	11	12
$p_k(12)$	1	6	12	15	13	11	7	5	3	2	1	1

In general, partitions are represented as arithmetic series, with terms (parts) arranged in non-increasing order. However, they can also be represented with more convenient configurations [174], depending on the context and their applications. The most common form is the *standard* or *natural representation* which simply gives the sequence of numbers in the representation–e.g., $(2, 1, 1)$ for the number $4 = 2 + 1 + 1$. The *multiplicity representation* instead gives the number of times each number occurs together with that number–e.g., $(2, 1)$, $(1, 2)$ for $4 = 2 \cdot 1 + 1 \cdot 2$. Besides that, the parts may also be grouped into sequences, lists, tuples or vectors, may or may not be enclosed in brackets, be separated by commas, periods or spaces, or even have no separators. Their order can be increasing or non-increasing, decreasing or non-decreasing, lexicographic or anti-lexicographic, etc. Furthermore, partitions can be written even more concisely, through the *frequency representation* ("power notation"), in which the number of occurrences of each part is denoted by a corresponding exponent.[6] Table 3 presents some possibilities of representing the partition $7 + 5 + 5 + 3 + 2 + 2 + 2 + 1 + 1 + 1 + 1$ of 30.

Table 3: *Possible representations of* $7 + 5 + 5 + 3 + 2 + 2 + 2 + 1 + 1 + 1 + 1$.

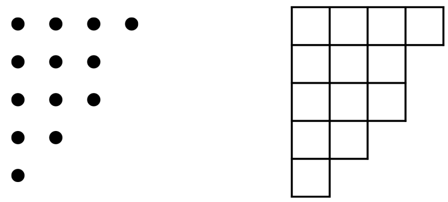
$(7, 5, 5, 3, 2, 2, 2, 1, 1, 1, 1)$	$(1, 1, 1, 1, 2, 2, 2, 3, 5, 5, 7)$
$(7\ 5\ 5\ 3\ 2\ 2\ 2\ 1\ 1\ 1\ 1)$	$(1\ 1\ 1\ 1\ 2\ 2\ 2\ 3\ 5\ 5\ 7)$
(75532221111)	(11112223557)
$(7^1 5^2 3^1 2^3 1^4)$	$(1^4 2^3 3^1 5^2 7^1)$
$(75^2 32^3 1^4)$	$(1^4 2^3 35^2 7)$

Apart from such numerical configurations, the partitions can also be represented graphically, through the *Ferrers diagram* or the *Young diagram*, in which the parts correspond, respectively, to dots or squares distributed horizontally and vertically in the plane, according to their size and multiplicity. As an example, the graphical representations of the partition $4 + 3 + 3 + 2 + 1$ of 13, in the respective diagrams, are shown in Figure 1.

Graphical representations are often useful for illustrating or performing transformations in partitions and also for demonstrating certain properties called *partition identities* [1]. These identities consist of correspondence or congruence relations between restricted partitions. One of the simplest identity can be obtained through *conjugation*, a transformation of the partition operated in the Ferrers diagram, in such a way that the rows of the first diagram become the columns of the second (and vice versa), resulting in a partition with the greatest part equal to the number of parts of the original partition, i.e., $p(n \mid k \text{ parts}) = p(n \mid \text{greatest part is } k)$ [11, p. 17].

We can notice in Figure 2 that the transformation by conjugation corresponds to the rotation of the original diagram around its diagonal axis (highlighted by the darker dots). In this way, the parti-

[6] When a part occurs only once its exponent may be omitted.



(a) *Ferrers diagram* **(b)** *Young diagram*

Figure 1: *Graphical representations of* $(43^2 21)$.

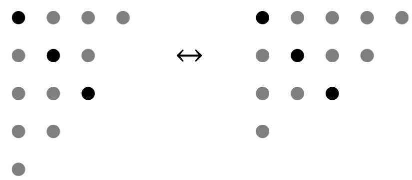


Figure 2: *Conjugation between partitions* $(43^2 21)$ *and* (5431).

tions $(43^2 21)$ and (5431) map to each other and are therefore called *conjugate partitions*.[7] It is worth pointing out that the sum of the parts of the conjugated partitions is always the same, which means that they are invariably partitions of the same positive integer. Finally, from other conjugation properties, we can come to new partition identities [11, pp. 17–18]–for example, $p(n \mid \leq k \text{ parts}) = p(n \mid \text{all parts} \leq k)$, $p(n \mid \text{distinct parts}) = p(n \mid \text{parts of every size from 1 to the largest part})$, $p(n \mid \text{self-conjugate}) = p(n \mid \text{distinct odd parts})$, and so on.

Many other identities can be obtained in a similar way or through more sophisticated mathematical methods. Moreover, other graphic representations, such as *Young's lattice*, *Young tableaux*, *Hasse diagram*, *Durfee square*, etc., are equally useful and have numerous applications. However, we limit ourselves here to just mentioning them, since the description of their attributes would go beyond the scope of this preliminary work.

As we have seen, the order of the parts of a partition is irrelevant, which characterizes an *unordered partition*. On the other hand, when the order of the parts is considered, then we have what we call an *ordered partition* or, simply, a *composition*[8] [9, p. 54]. Thereby, a composition

Table 4: *Values of* $c(n)$, *for* $n = \{0, 1, \ldots, 12\}$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$c(n)$	1	1	2	4	8	16	32	64	128	256	512	1024	2048

Table 5: *Values of* $c_k(12)$, *for* $k = \{1, 2, \ldots, 12\}$.

k	1	2	3	4	5	6	7	8	9	10	11	12
$c_k(12)$	1	11	55	165	330	462	462	330	165	55	11	1

[7] A partition that is its own conjugate is called *self-conjugate*–e.g., $(43^2 1)$.

[8] Despite the term *composition* already has a specific meaning in music, we use it here since it is well established in mathematics and for its practical advantages. Nevertheless, the disambiguation between *musical composition* and *integer composition* should be easily inferred from the context in which each term is used.

of n can be thought of as an expression of n as an ordered sum of integers [160, p. 17]. More precisely, a composition of n is a finite (ordered) sequence of positive integers $\alpha = \langle \alpha_1, \alpha_2, \ldots, \alpha_k \rangle$,[9] such that $\sum_{i=1}^{k} \alpha_i = n$. For example, for $n = 4$, besides the 5 partitions: (4), (31), (2^2), (21^2), (1^4); there are 8 distinct compositions: $\langle 4 \rangle$, $\langle 31 \rangle$, $\langle 13 \rangle$, $\langle 22 \rangle$, $\langle 211 \rangle$, $\langle 121 \rangle$, $\langle 112 \rangle$, $\langle 1111 \rangle$. The number of all compositions of a positive integer n is given by the *composition function* $c(n) = 2^{n-1}$, and the number of compositions of n with k parts[10] is given by $c_k(n) = \binom{n-1}{k-1} = \frac{(n-1)!}{(k-1)!\,(n-k)!}$. Likewise, $c(n) = 0$, when n is negative, and $c(0) = 1$, the empty composition (with no parts). If a composition α has k parts, then we call α a *k-composition*. We can also express functions involving *restricted compositions* using the generic formula $c(n \mid [\text{condition}])$. Tables 4 and 5 show, respectively, the values of $c(n)$ and $c_k(12)$, or $c(12 \mid k \text{ parts})$, for each corresponding n and k.

As Andrews points out [9, p. 55], compositions may also be represented graphically. To the composition $\langle a_1, a_2, \ldots, a_k \rangle$ of n we can associate k segments of the interval $[0, n]$, so that the first segment is of length a_1, the second of length a_2, and so on. As an example, the composition $\langle 32312 \rangle$ of 11 can be represented as showed in Figure 3.

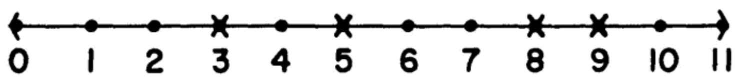


Figure 3: *Graphical representation of the composition $\langle 32312 \rangle$. Reproduced from Andrews [9, p. 55]. Copyright 1984 by Cambridge University Press. Reproduced with permission of Cambridge University Press through PLSclear.*

It is important to observe that many aspects of partitions naturally extend to compositions, which also present interesting properties and identities.[11] In addition, due to their flexibility and versatility, both find applications in different segments of discrete and combinatorial mathematics, such as set theory, group theory, graph theory, modern algebra, etc.[12] By way of illustration, we mention here the methods used in the combinatorics of pattern avoidance and pattern enumeration in set partitions [106], and the close connection between subsets of a set and compositions of a non-negative integer in enumerative combinatorics [160].

Before we close this brief introduction to the theory of partitions, it is useful to introduce the concept of *multipartition*, which has only recently been studied for its own intrinsic interest [10]. By Fayers definition [54, p. 115], a multipartition of n with k components is a k-tuple $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(k)})$ of partitions, such that $|\lambda^{(1)}| + |\lambda^{(2)}| \cdots + |\lambda^{(k)}| = n$. If k is understood, we shall just call this a multipartition of n. As with partitions, 0 (zero) has a unique multipartition– the empty one, and if $\boldsymbol{\lambda}$ is a multipartition of n, then we write $\boldsymbol{\lambda} \vdash n$ or $|\boldsymbol{\lambda}| = n$. For example, $((431), (42^3), (321))$ is a multipartition of 24. If the order of the parts is relevant, then we can extend this definition to the compositions and arrive at the analogous concept of *multicomposition*. So, $\langle \langle 143 \rangle, \langle 2^2 42 \rangle, \langle 132 \rangle \rangle$ is a multicomposition of 24.

Having seen so far some of the elementary properties of integer partitions, let's check below how they can be useful for the formulation of a harmonic conception through the partitioning of pitch spaces.

[9] In this work, we differentiate partitions from compositions by using round ( ) and angle ⟨ ⟩ brackets, respectively.

[10] A composition whose parts may be equal to 0 (zero) is called a *weak composition*. The number of all weak compositions of n is obviously infinite, but the number of weak compositions of n with k parts is given by $c_k(n) = \binom{n+k-1}{k-1} = \frac{(n+k-1)!}{n!\,(k-1)!}$.

[11] See, for example, [23, 75, 76, 95, 113, 125, 126].

[12] For an introduction to many of these topics, see [74].

III. The Partitioning of Pitch Spaces

In acoustics, *pitch*[13] characterizes one of the perceptual attributes of sound[14] and is defined as "that aspect of auditory sensation in terms of which sounds may be ordered on a scale extending from *low* to *high*, such as a musical scale" [24, p. 397]. It is a subjective quantity despite being related to the sound *frequency*, which in turn is a definite physical quantity that can be measured on physical instruments without any reference to the ear. Thus, if frequency is a physical measure of vibrations or cycles per second–given in hertz (Hz), pitch is the corresponding perceptual experience of frequency [105, p. 13]. In other words, pitch is "our subjective evaluation of sound frequency" [17, p. 112], or "the intersubjective correlate of frequency" [118, p. 23].

Generally speaking, for any given frequency there will be perceived a certain pitch, i.e., low frequencies correspond to low pitches, and conversely. In the case of pure tones,[15] pitch is directly related to the frequency of the sound wave, and in the case of more complex sounds, it is determined by what the ear judges to be the most fundamental wave frequency of the sound [82]. However, this correspondence is not exact, since pitch is also determined by other factors such as frequency range[16] and region, sound complexity, duration, intensity, and even by hearing disorders [17, p. 112]. The perception may be different depending on the situations, so that a specific frequency will not always have the same pitch. But for practical and musical purposes, given that those factors have little effect on the pitch of musical tones,[17] we may disregard them and use the two terms frequency and pitch as essentially synonymous when referring to musical sounds of definite pitch.[18]

We may also think of these pitched sounds, or rather this *pitch domain* [141], as a generic *musical space*[19] of pitches, i.e., a generic *pitch space*. According to Harley [81, pp. 90–107], the recognition of the spatial quality of pitch relationships is common to many theories of musical space. These relationships are spatial in an abstract sense and may be formalized and generalized through mathematical notions of space (e.g., continuum, metric space, vector space, and so on). Thus, the vertical dimension of music can be described as a system of relationships that comprises two aspects of pitch–one which changes monotonically with frequency (as an approximate logarithmic function of frequency [12, 16, 133]) and one rotating in a cycle of octave repetitions (*octave equivalence*). In modern psychoacoustic terminology, this distinction is expressed respectively

[13] According to the current American National Standard on Acoustical Terminology (ANSI/ASA S1.1-2013) [7], pitch is "that attribute of auditory sensation by which sounds are ordered on the scale used for melody in music." The pitch of a sound "may be described by the frequency of that simple tone having a specified sound pressure level that is judged by listeners to produce the same pitch." In addition, we can consider as practical dimensions for pitch: "(1) the perceptual dimension of pitch height, which defines the position of the tone along a scale from low to high, normally monotonic with the physical dimension of frequency; (2) the circular dimension of pitch class, which defines the position of the tone within the octave."

[14] The others are loudness, timbre, duration, and spatial perception.

[15] A *tone* is a sound wave capable of exciting an auditory sensation having pitch. A *pure tone*, or *simple tone*, is a sound wave, the instantaneous sound pressure of which is a sinusoidal function of time (a sine wave) [7].

[16] The frequency range of human hearing is between approximately 20 to 20000 Hz, varying from person to person according to age and gender. Pitch discrimination is very poor and drops off near the lower limit (below about 30 Hz) and also at high frequencies (above about 5000 Hz) [105, p. 159], being practically nonexistent above about 10000 Hz [17, p. 111].

[17] A *musical tone* is a kind of (periodic) sound produced by most of our musical instruments, which lasts long enough and is steady enough for the ear to ascribe to it three characteristics–loudness, quality (timbre), and pitch [17, p. 94].

[18] The frequency range of pitched musical sounds, i.e., the useful range of fundamental frequencies of tones produced by musical instruments is considerably narrower than that of audible frequencies, being approximately 27 to 4200 Hz–slightly wider than the piano's range. The high-frequency region above that accommodates the harmonics of the high tones and other sounds of indefinite pitch (e.g., sounds produced by unpitched percussion instruments, noises such as bow scrapings, clinking of keys, and so forth) [17, p. 111].

[19] We refer here to the term as it is broadly used by Lewin [104]. For a thorough review about music and space, see [81].

by the concepts of *pitch height* (linear dimension) and *pitch chroma* (cyclic dimension).[20] Both dimensions can be combined in a geometric model of a cylindrical spiral, generating a helical representation of pitch known as the *pitch helix*.[21] Figure 4 shows Shepard's pitch helix, in which the rectilinear scale of pitch is deformed into a simple regular helix having one complete turn per octave (height), and collapsed into a circle where there is complete perceptual identity of all tones in the octave relation (chroma) [154, p. 352]. This duality between the rectilinearity and circularity of pitch underlies the disambiguation of *pitches* and *pitch classes*,[22] as well as the conceptualization of linear and cyclic (modular) pitch spaces, namely and respectively, *pitch spaces* and *pitch-class spaces*.

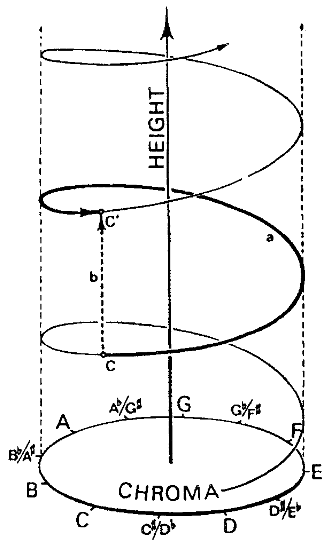


Figure 4: *The pitch helix. Reproduced from Shepard [153, p. 105]. Copyright 1965 by the Board of Trustees of the Leland Stanford Jr. University. Reproduced with permission of Stanford University Press.*

At this point, we ought to dwell on the following question: how do we select out of the whole range of pitches those to be used musically, or rather, how do we construct useful musical scales, or even musical systems,[23] out of the continuum of available pitches? Since, by the fairly universal concept of *octave generalization*,[24] pitches separated by an octave are in some sense musically similar or equivalent (have the same chroma), in numerous cultures, musical scales are often defined by specifying the intervals[25] within an octave,[26] whether from a natural (physical) basis or not. Once these intervals are determined and replicated successively across each octave, the pitch continuum is then discretized into a finite number of pitches. Therefore, despite various–let's

[20] These concepts were originally defined by Bachem [16] as *tone height* and *tone chroma*. For related uses and further developments, see [32, 48, 49, 50, 89, 96, 101, 136, 141, 144, 151, 153, 154, 168, 171, 172].

[21] Other multidimensional representations are also possible, see [154].

[22] A *pitch class* is "a collection of pitches related by octave and enharmonic equivalence" [165, p. 5].

[23] For a deep survey on the "domain of musical systems," see [29, 30].

[24] For an in-depth discussion about musical intervals, scales, and tuning systems in different cultures from a perceptual perspective, see [36].

[25] An *interval* is "the spacing in pitch [perceptually] or frequency [physically], as indicated by context, between two tones." The frequency spacing is "expressed by the ratio of the frequencies or by a specified logarithm of this ratio" [7]. See also [133].

[26] Although less common, there are also musical systems based on intervals of equivalence (*modular intervals*) other than the octave–e.g., the *naturale*, *durum*, and *molle* hexachords of medieval solmization ("Guidonian") system, or any arbitrary system that divides other intervals, or that deliberately ignores the acoustic similarity of the octave.

say–non-structural pitch glides (*glissandi*, *portamenti*, trills, and other ornamentation), most musical systems are predicated on discrete intervals, so that practical music is limited to a relatively small set of discrete pitch relationships.

It may seem odd or counter-intuitive, but in such musical systems pitches are actually determined by intervals, and not the other way around. Furthermore, the sonic identity or profile of a musical system is determined not so much by its proper pitches (unless one has absolute pitch) as by the intervals between them. In this sense, intervals precede pitches. This paradoxical aphorism–particularly true in Western music[27]–not only is the cornerstone of our present harmonic conception, but also underpins many theoretical works on post-tonal music which deal, to some extent, with pitch relationships through an intervallic approach.[28]

In fact, particularly with regard to harmony, pitches are not as musically relevant as intervals. Most of us rarely treat individual pitches as auditory units or tie particular behaviors to them, instead, our pitch perception is "relationally determined", and that is because pitch is a "morphophoric" (form-bearing) medium, in which the same musical patterns or configurations may have different locations and still preserve their perceived structural identities [12, pp. 147–148]. Then we are generally more responsive to the relations and ratios between pitches than to those pitches themselves [154, p. 344]. In other words, "the quality of a musical pitch depends critically on its relations to other musical pitches" [96, p. 112], and such relations, or "configural properties," are precisely the intervals. This is quite clear, for example, when a song is transposed, or when an orchestra is tuned to a pitch other than the standard $A = 440$ Hz, and yet the original musical relationships–and the music itself–remain essentially the same. This is also true for a music student who learns the interval patterns of each type of scale/chord (major, minor, etc.) before building them on any tonic/root. Hence, intervals are determinant not only for the way we perceive and recognize music, but also for how we understand and conceive its most fundamental and structural relationships.[29]

As stated above, intervals play a crucial role in the construction of most musical systems. In a similar fashion, since "the basic notion of interval, i.e., distance, is crucial for the musical structuring of pitch, and the mathematical structuring of space (metric space)" [81, p. 97], they are also essential for defining pitch spaces. Roughly speaking, a pitch space is basically what the term implies–"a space of pitches" [118, p. 346]. The concept of space here, though rather vague, should be understood mostly in the mathematical sense of the term, namely as an *abstract space*.[30] Thus, a pitch space corresponds to "a theoretical model setting forth 'distance' relationships between pitches" [116]. Depending on their complexity, different pitch spaces correspond to varied models, often mathematical,[31] such as geometric shapes (line, circle, helix, torus, etc.), in addition to graphs, groups, grids, and lattices (*Tonnetz*), or even more abstract formulations like equations, ordered pairs, multidimensional vectors, and so forth. In general, pitches with equivalent musical

[27] Most Western musical systems, like Pythagorean tuning, Just intonation, Meantone temperament, Well and Equal temperaments, among others, are interval-based (i.e., frequency-ratio-based) systems. For further explanation on this and comparisons between different systems, see [17, 19, 21, 36, 105, 133, 152, 170].

[28] See [18, 38, 39, 40, 41, 42, 45, 57, 64, 78, 118, 119, 142, 155, 156]. Some of these works are discussed at length and in historical perspective in [150], see also [25].

[29] Notwithstanding this logical deduction, Krumhansl's experiment indicates that "all intervals of equal size are not perceived as equal when the tones are heard in tonal contexts. Instead, the degree to which listeners judge tones to be related depends on their functions in the key" [96, p. 120].

[30] An *abstract space* is "a formal mathematical system consisting of undefined objects and axioms of a geometric nature" (e.g., Euclidean spaces, metric spaces, topological spaces, vector spaces, etc.). On the other hand, a mathematical *system* is (1) "a set of quantities having some common property" (e.g., the system of even integers, the system of lines passing through the origin, etc.), or (2) "a set of principles concerned with a central objective" (e.g., a coordinate system, a system notation, etc.) [86].

[31] A *mathematical model* is "a mathematical construct designed to study a particular real-world system or phenomenon" [70, p. 60], or basically, a description or representation of a system in mathematical terms (concepts and language).

relationships are equally spaced within those models, and closely related pitches are placed next to one another (and conversely). Also, octave-related pitches may or may not be distinguished, which characterizes a pitch or a pitch-class space, respectively.

If, on the one hand, pitch spaces usually correspond to mathematical models, on the other, considering contemporary approaches to perception, they are not entirely mathematical. Recent cognitive studies show that there is considerable evidence on the phenomenical aspects of pitch spaces in musical perception.[32] Still, since they are somehow models of musical systems (pitch systems in our particular case),[33] they can be somewhat classified by the same criteria common to *mathematical modelling* and *systems theory* literature.[34] Therefore, pitch spaces can be concrete, representational, or abstract at first,[35] but also natural vs. artificial (designed), linear vs. non-linear, static vs. dynamic, implicit vs. explicit, continuous vs. discrete, deterministic vs. probabilistic (stochastic), mechanistic vs. empirical (statistical), open vs. closed, and so on. We will not go further here on all of those properties and their definitions, since they are well known in the specific literature and, to a certain degree, self-explanatory. For the moment, it is sufficient to notice that some of them make more sense and are more relevant to pitch spaces than others. Figure 5 shows a preliminary classification of pitch spaces in mathematical terms, in the form of a schematic diagram, partly based on those properties which we judge to be the most significant for our present purposes, and partly in other geometric and algebraic concepts.[36]

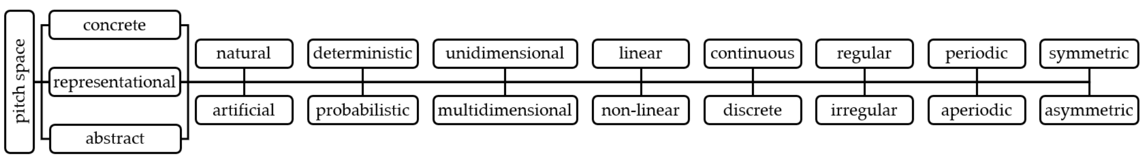


Figure 5: *Classification of pitch spaces in mathematical terms.*

Different approaches and taxonomies from various authors use some of those among other original criteria to define and classify not only pitch spaces, but also musical spaces in general, including spaces of chords, scales, harmonic progressions, tonal regions and keys, rhythms, timbres, and so on.[37] Morris [118], for example, defines five types of pitch spaces: the *contour space* (*c-space*), which corresponds to the pitch continuum, where the sizes of intervals between the pitches are undefined or ignored; the linear *pitch space* (*p-space*) and *unequal space* (*u-space*), which have, respectively, equal and unequal intervals between their successive adjacent pitches; and, finally, the cyclic *pitch-class space* (*pc-space*) and *modular space* (*m-space*), which are the collapsed pitch-class versions of the previous two, respectively. Figure 6 presents Gentil-Nunes' graphical representations of those pitch spaces.

Naturally, as we can see, the pitch continuum is the most fundamental pitch space. In brief,

[32] For an interesting discussion about the paradoxes of pitch spaces involving mathematical models of musical space and embodied models of musical perception, see [33].

[33] *System* here should be understood in the light of the *General System Theory* (GST), i.e., as "a complex of interacting elements" [27, p. 55], or rather, as "a set of elements or parts that is coherently organized and interconnected in a pattern or structure that produces a characteristic set of behaviors" [112, p. 188]. Klir [93, pp. 4–5] defines a system as "a set or arrangement of things so related or connected as to form a unity or organic whole", formalizing this common-sense definition through the equation $S = (T, R)$, where S, T, R, denote, respectively, a *system*, a *set of things* distinguished within S, and a *relation* (or, possibly, a set of relations) defined on T.

[34] For an introductory reference on both of these topics, see [22, 27, 47, 52, 70, 90, 93, 112, 114, 149].

[35] Although common in the referred literature, these terms and the concepts behind them were borrowed from the Concrete-Representational-Abstract (CRA) educational approach [84].

[36] Further details on this classification will be provided on another occasion.

[37] We highlight here [31, 46, 88, 103, 104, 109, 118, 166, 167, 176]. For others, see again [81, 96].

by partitioning it we derive the discrete pitch spaces, and by collapsing these latter we obtain the discrete pitch-class spaces.[38] Continuous pitch spaces are useful in many contexts, however, for obvious reasons, we can only infer definite harmonic relationships from the discrete ones. The harmonic properties of a discrete pitch space depend essentially on its intervallic structure, which, again, may be evenly or unevenly constructed, i.e., with regular or irregular intervals (or interval patterns). Different structures imply different results for transformational operations on musical objects. For example, the transposition or inversion of a musical motif performed within a diatonic space will be different from that within a chromatic space, precisely because the "step distances" [102, 103]–the distances in diatonic or chromatic adjacent steps[39]–are not the same in both spaces. In this regard, each pitch space consists of a unique musical universe with its own sonic characteristics and harmonic peculiarities. Pitch spaces with irregular or asymmetric interval environments tend to determine tonal centers and confer a unique status on each pitch, since every one has a unique relation to the others in its own characteristic way. Such a framework gives rise, for example, to the typical dynamisms of tonal music (motion and rest, tension and resolution, etc.). Conversely, regular and symmetric pitch spaces tend to confer the same status on all pitches, weakening the sense of musical centricity and harmonic motion.

Often, for practical or theoretical musical reasons, we may perceive or think of pitch spaces as homogeneous spaces abstracted from their interval configuration, i.e., without distinguishing the actual distances between their adjacent pitches. For example, the successive steps of a diatonic scale (the scale degrees) may be perceived as in some sense equal, and indeed they are notated that way ($\hat{1}$, $\hat{2}$, $\hat{3}$, etc., for tonic, supertonic, mediant, and so on), despite the asymmetric interval construction of the scale. In this sense, the notion of step distances within discrete pitch spaces presupposes, or at least suggests, that the successive pitches in those spaces are to some extent

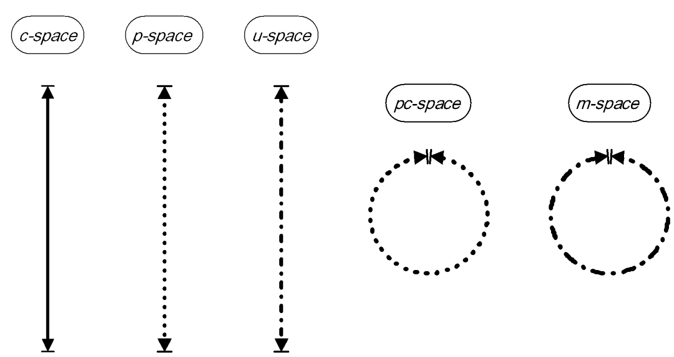


Figure 6: *Graphical representations of Morris' pitch spaces. Reproduced from Gentil-Nunes [66, p. 28]. Copyright 2009 by Gentil-Nunes. Reproduced with permission of the author.*

[38] A pitch space may also be generated from another pitch space. In addition, "certain subsets of elements taken from an equal-interval pitch space can produce a pitch space of uneven intervals" [118]. E.g., a diatonic scale drawn from a chromatic scale, the black keys of a piano keyboard, etc.

[39] In *diatonic set theory* [44], this distinction is usually made through the concepts of *generic interval* and *specific interval*, respectively. Other equivalent terms are *diatonic length* (*dlen*) and *chromatic length* (*clen*) [43], *diatonic distance* and *chromatic distance* [146, 147, 148], *d distance* and *c distance* [88], and so forth.

categorically perceived[40] as functionally regular,[41] with the eventual interval distances between them being disregarded. According to Shepard [154, p. 373], once the pitches are "categorically mapped into the discrete nodes of an internal representation that is functionally regular, it is the structural properties inherent in that regular representation that are important." Therefore, regardless of whether a given discrete pitch space is evenly or unevenly constructed, we may conceive of it (or part of it, if multidimensional) as what we call here a *categorical space*, i.e., a *simple directed graph*[42] which represents its most fundamental structure, homogenizing any irregular interval spacing, so that the vertices corresponding to the successive pitches or pitch classes are equally spaced. We can then define two types of categorical spaces, the *categorical pitch space* (linear) and the *categorical pitch-class space* (cyclic), as follows.

Def. 3.1. Categorical pitch space (cp-space). Let $V = \{v_1, v_2, \ldots, v_n\}$ be a set of n pitches from a pitch space, arranged from low to high and labeled successively with consecutive integers, a *categorical pitch space (cp-space)* is a directed graph $G = (V, E)$, of order $|V| = n$ and size $|E| = n - 1$, with $E = \{(v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n)\}$.

Def. 3.2. Categorical pitch-class space (cpc-space). Let $V = \{v_1, v_2, \ldots, v_n\}$ be a set of n pitch classes from a pitch-class space, labeled successively from 0 to $n - 1$, a *categorical pitch-class space (cpc-space)* is a directed graph $G = (V, E)$, of order $|V| = n$ and size $|E| = n$, with $E = \{(v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n), (v_n, v_1)\}$.

Inasmuch as categorical spaces are directed graphs, we may associate the edges between vertices with *directed intervals* between pitches or pitch classes.[43] And for this we may use the concept of *walk*,[44] so that the directed interval between two pitches or pitch classes is equal to the *length*[45] of the corresponding walk. Accordingly, we define a *categorical interval* as follows.

Def. 3.3. Categorical interval (ci). Let $G = (V, E)$ be a categorical pitch or pitch-class space, a *categorical interval (ci)* between x and y, with $x, y \in V$, is given by $ci_{x\text{-}y} = l$, where l is the length of a x-y walk in G.

Remark. In a categorical pitch-class space, if $l < |E|$, then the categorical interval is called *simple*, whereas if $l \geq |E|$, then it is called *compound*.[46] In this sense, we can rewrite a categorical interval as $ci_{x\text{-}y} = l = i + |E| \cdot d$, where $i = (y - x) \bmod |E|$, $d \in \mathbb{N}$ is the *dimensional factor*, $d = 0$ if the

[40] Originally, *categorical perception* "refers to the concept that some stimuli can only be responded to on an absolute basis, i.e., discrimination is limited by identification" [35, p. 457]. In other words, it is "the phenomenon by which the categories possessed by an observer influences the observers' perception" [71, p. 69], or simply, "a phenomenon that occurs when signals that vary over a continuous physical scale are perceived as belonging to a small number of discrete groups" [92, p. 878]. However, in this work, by *category* "is meant a number of objects that are considered equivalent" [145, p. 5], or simply, "a set of entities that are grouped together" [72, p. 276]. Therefore, as will be defined, a *categorical space* is a space of those objects or entities (pitches or pitch-classes in our case), and a *categorical interval* is the distance between them within that respective space.

[41] The term *functional* here should not be confused with tonal or mathematical function, but understood in its broad sense, i.e., relating to how useful something is in a certain context, or relating to the way in which it works or operates in that context.

[42] A *simple graph* is a graph without loops and with at most one edge between any two vertices. A *directed graph*, or *digraph*, is a graph in which directions are assigned to the edges, being called *directed edges*. For an introduction to *graph theory*, see again [74].

[43] The reasoning behind this is analogous to that of Lewin's interval definition, i.e., "a directed measurement, distance, or motion" [104, p. xxix].

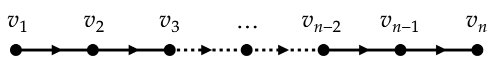
[44] A *walk* is an alternating sequence of vertices and edges of a graph.

[45] The *length* of a walk is the number of edges in the walk. A walk containing no edges is called *trivial*.

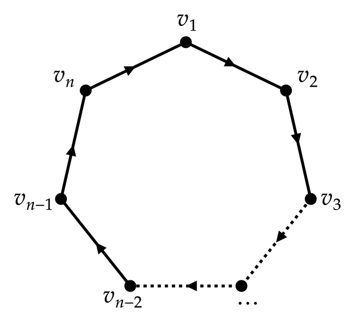
[46] From these conditional statements the octave is considered here not a simple but a compound interval, i.e., the octave is a *compound unison* (or *prime*). By analogy, the notion of simple and compound categorical intervals may be extended to categorical pitch spaces, insofar they correspond to unfolded categorical pitch-class spaces.

categorical interval is simple, and $d \geq 1$ if it is compound. Categorical intervals are called *adjacent* if they correspond to adjacent walks in G.

Before proceeding, let's summarize and clarify a bit more the previous definitions. If a pitch or pitch-class space is itself a model (often mathematical), then a categorical space is a functionally regular representation of that model–a simple directed graph. Since the graph is directed, its edges may be associated with directed intervals. Hence, the length of a walk between two vertices corresponds to a categorical interval between two pitches or pitch classes. Here some observations are important: 1) When a pitch or pitch-class space is already equally spaced (regular) in its original representation, the corresponding categorical space (the graph) will somehow coincide with that representation. 2) Insofar as a pitch space may contain undefined pitches and may be theoretically infinite, a categorical pitch space must correspond to a part or segment of that unlimited space. 3) Walks in a categorical pitch space are always *open*, since it is linear (a *path graph*), whereas in a categorical pitch-class space they may be *open* or *closed*, since it is cyclic (a *cycle graph*).[47] 4) In a categorical pitch-class space, a walk can be of any length, which implies that a categorical interval can also be of any size. 5) The dimensional factor of a categorical interval corresponds to the number of octaves it comprises. Figure 7 provides a visual glimpse of the previous definitions and observations.



(a) *Categorical pitch space (cp-space).*



(b) *Categorical pitch-class space (cpc-space).*

Figure 7: *Categorical spaces of pitches and pitch classes.*

Now, if we look closely at the graph of Figure 7a, we will see that it is very similar to that of Figure 3, which suggests that we may associate the categorical intervals (walk lengths) with the parts of a composition (interval segments). And indeed, we can. If a composition is basically a sequence of integers (parts) partitioning a larger integer, in a similar way, we may think of a sequence of categorical intervals partitioning a categorical pitch or pitch-class space. From this analogy, we can define what we call an *interval composition*.

Def. 3.4. Interval composition (ic). Let $G = (V, E)$ be a categorical pitch or pitch-class space, an *interval composition (ic)* of G is a sequence of adjacent categorical intervals $\alpha = \langle \alpha_1, \alpha_2, \ldots, \alpha_k \rangle$, such that $\sum_{i=1}^{k} \alpha_i = |E| \cdot d$, where $d \in \mathbb{N}$ is the *dimensional factor*, $d = 1$ if G is a categorical pitch space, and $d \geq 1$ if G is a categorical pitch-class space.

Remark. An interval composition is called a *linear interval composition (lic)* or a *cyclic interval composition (cic)* if G is respectively a categorical pitch or pitch-class space. The elements of an

[47] An *open walk* is a walk that begins and ends at different vertices, if no edge is repeated it is called a *trail*, and if no vertex occurs more than once it is called a *path*. Contrarily, a *closed walk* begins and ends at the same vertex, this time, if no edge is repeated it is called a *circuit*, and if no vertex occurs more than once it is called a *cycle*.

interval composition are called *parts*. The *length* of an interval composition, denoted by $\ell(\alpha)$, is the number of its parts. A subsequence of consecutive parts of an interval composition is called an *interval subcomposition (isc)*.

And since compositions are ordered partitions, the reverse is also true, partitions are unordered compositions. Then, we can also define what we call an *interval partition*.

Def. 3.5. Interval partition (ip). Let $G = (V, E)$ be a categorical pitch or pitch-class space, an *interval partition (ip)* of G is a non-decreasing sequence of categorical intervals $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, such that $\sum_{i=1}^{k} \lambda_i = |E| \cdot d$, where $d \in \mathbb{N}$ is the *dimensional factor*, $d = 1$ if G is a categorical pitch space, and $d \geq 1$ if G is a categorical pitch-class space.

Remark. An interval partition is called a *linear interval partition (lip)* or a *cyclic interval partition (cip)* if G is respectively a categorical pitch or pitch-class space. The elements of an interval partition are called *parts*. The *length* of an interval partition, denoted by $\ell(\lambda)$, is the number of its parts. A subsequence of parts of an interval partition is called an *interval subpartition (isp)*.

Once again, some additional observations should be considered: 1) In a categorical pitch-class space, the dimensional factor corresponds to the number of overlapping turns around the graph. 2) Interval compositions correspond to *ordered* interval configurations between pitches or pitch classes, while interval partitions correspond to *unordered* interval configurations, being classes of interval compositions and, therefore, more abstract representations. 3) Interval partitions are arranged in a non-decreasing order for the sake of consistency with the conventional (left-compacted) models of *normal form* and *prime form* from the pitch-class set theory.

Through these concepts, which we call *interval structures*[48] (in opposition to *pitch structures*), we are now able to partition the categorical spaces of pitches or pitch classes, in order to determine harmonic (musical and mathematical) relationships between those structures. But before presenting an illustrative musical application, let's define some basic operations.

IV. Basic Operations

For the definitions and examples[49] of the operations presented below, we mostly follow Morris [118] notation for pitches and pitch classes, as well as for pitch sets, pitch-class sets, and set classes, in addition to the same operators labels. None of these operations is really new, on the contrary, they are all well known in the main literature [58, 118, 128, 140, 165]. However, the use of them through interval structures, specifically interval compositions, is something relatively original.

Def. 4.1. Pitch (P_x). Let $\alpha = \langle \alpha_1, \alpha_2, \ldots, \alpha_k \rangle$ be an interval composition of a categorical pitch or pitch-class space $G = (V, E)$, the *pitch* operation, defined by $P_x\alpha$, where $x \in V$, sets x as the *root vertex*[50] of G and, consequently, the initial vertex (pitch or pitch-class) of α.

[48] Despite the nuances of our concepts and the original paths we take here to reach them, they resemble many others in literature. Interval compositions are somewhat similar to Bacon's *harmonies* [18], Hanson's *sonorities* (*scales*) [78], Forte's *interval successions* [58], Chrisman's *successive interval arrays* [39, 40, 41, 42], Regener's *chords* [142], Carter's *successive intervals* [38], Morris' *spacing*, INT_1, $CINT_1$, and $PCINT$ [118, 119, 120], Sorderberg's *CORD* and *interval strings* [155, 156], Straus' *spacing intervals* [165], and so on. On the other hand, interval partitions resemble Forte's *basic interval patterns (bips)* [57, 58], Jedrzejewski's *partitions* [87], Keith's *interval sets* [91], Coelho de Souza's *PCORD* [45], among others.

[49] For simplicity, all examples set out here use the twelve-tone equal temperament system (12-TET) as the basis for the categorical pitch or pitch-class spaces, which means that $|E| = 12$ in the latter case.

[50] A *root vertex* is a vertex labeled in a special way so as to distinguish it from the other vertices. A graph with one or more root vertices is called a *rooted graph*.

Remark. An interval composition α under the pitch operation P_x is called a *pitched interval composition* $P_x\alpha$, and is equivalent to a pitch or pitch-class set $P = \{p_1, p_2, \ldots, p_k\}$, $P \subseteq V$, given respectively by $P = \{x + \sum_{i=1}^{j} \alpha_{i-1}\}_{j=1}^{k+1}$ or $\{(x + \sum_{i=1}^{j} \alpha_{i-1}) \bmod |E|\}_{j=1}^{k}$. Conversely, the pitched interval composition equivalent to a successively arranged[51] pitch or pitch-class set $P = \{p_1, p_2, \ldots, p_k\}$, $P \subseteq V$, is given respectively by $P_x\alpha = P_{p_1}\langle p_{i+1} - p_i\rangle_{i=1}^{k-1}$ or $P_{p_1}\langle (p_{(i+1) \bmod k} - p_{i \bmod k}) \bmod |E|\rangle_{i=1}^{k}$, where $p_0 = p_k$ in the latter case. The x in P_x is called the *pitch index*.

Ex. $P_0\langle 3,4,5\rangle = \begin{cases} \{0,3,7,12\}, & \text{for a cp-space,} \\ \{0,3,7\}, & \text{for a cpc-space.} \end{cases}$

$$P_9\langle 1,1,2,1,3,1,3\rangle = \begin{cases} \{9,10,11,13,14,17,18,21\}, & \text{for a cp-space,} \\ \{9,10,11,1,2,5,6\}, & \text{for a cpc-space.} \end{cases}$$

$$\{2,4,5,7,9\} = \begin{cases} P_2\langle 2,1,2,2\rangle, & \text{for a cp-space,} \\ P_2\langle 2,1,2,2,5\rangle, & \text{for a cpc-space.} \end{cases}$$

Def. 4.2. Transposition (T_x). Let $\alpha = \langle \alpha_1, \alpha_2, \ldots, \alpha_k\rangle$ be a pitched interval composition of a categorical pitch or pitch-class space $G = (V, E)$, the *transposition* operation is respectively defined by $T_x P_y\alpha = P_{x+y}\langle \alpha_1, \alpha_2, \ldots, \alpha_k\rangle$ or $P_{(x+y) \bmod |E|}\langle \alpha_1, \alpha_2, \ldots, \alpha_k\rangle$, where $x \in \mathbb{Z}$.

Remark. The transposition operation T_x results in a *translation* or a *rotation* of the pitched interval composition $P_y\alpha$ by x, if G is respectively a categorical pitch or pitch-class space.[52] The x in T_x is called the *transposition index*. T_0 is an identity operation.

Ex. $T_2P_3\langle 3,3,6\rangle = \begin{cases} P_{2+3}\langle 3,3,6\rangle = P_5\langle 3,3,6\rangle, & \text{for a cp-space,} \\ P_{(2+3) \bmod 12}\langle 3,3,6\rangle = P_5\langle 3,3,6\rangle, & \text{for a cpc-space.} \end{cases}$

$$T_8P_7\langle 1,2,1,3,5\rangle = \begin{cases} P_{8+7}\langle 1,2,1,3,5\rangle = P_{15}\langle 1,2,1,3,5\rangle, & \text{for a cp-space,} \\ P_{(8+7) \bmod 12}\langle 1,2,1,3,5\rangle = P_3\langle 1,2,1,3,5\rangle, & \text{for a cpc-space.} \end{cases}$$

$$T_{-5}P_2\langle 2,1,2,3,4\rangle = \begin{cases} P_{-5+2}\langle 2,1,2,3,4\rangle = P_{-3}\langle 2,1,2,3,4\rangle, & \text{for a cp-space,} \\ P_{(-5+2) \bmod 12}\langle 2,1,2,3,4\rangle = P_9\langle 2,1,2,3,4\rangle, & \text{for a cpc-space.} \end{cases}$$

Def. 4.3. Inversion (I, I_x). Let $\alpha = \langle \alpha_1, \alpha_2, \ldots, \alpha_k\rangle$ be an interval composition of a categorical pitch or pitch-class space $G = (V, E)$, the *inversion* operation is defined by $I\alpha = \langle \alpha_k, \alpha_{k-1}, \ldots, \alpha_1\rangle$.

Remark. If α is pitched, then the transposition operation T_x must be associated with the inversion operation I, resulting in the composite operation T_xI, abbreviated as I_x, such that $I_xP_y\alpha = P_{x+y}\langle \alpha_k, \alpha_{k-1}, \ldots, \alpha_1\rangle$ or $P_{(x+|E|-y) \bmod |E|}\langle \alpha_k, \alpha_{k-1}, \ldots, \alpha_1\rangle$, if G is respectively a categorical pitch or pitch-class space, where $x \in \mathbb{Z}$. The inversion operation I_x results in a *reflection* of the pitched interval composition $P_y\alpha$ around an *axis of symmetry* $s = (x + y + |E|)/2$ or $(x \bmod |E|)/2$, if G is respectively a categorical pitch or pitch-class space. The x in I_x is called the *inversion index*. I_0 is equivalent to I.

Ex. $I\langle 4,3,5\rangle = \langle 5,3,4\rangle$

$$I_9P_5\langle 1,2,2,3,4\rangle = \begin{cases} P_{9+5}\langle 4,3,2,2,1\rangle = P_{14}\langle 4,3,2,2,1\rangle, & \text{for a cp-space,} \\ P_{(9+12-5) \bmod 12}\langle 4,3,2,2,1\rangle = P_4\langle 4,3,2,2,1\rangle, & \text{for a cpc-space.} \end{cases}$$

[51] Unlike for a pitch set, "successively arranged" does not necessarily mean "arranged in ascending order" for a pitch-class set, but rather "arranged in cyclic order."

[52] In the former case, G is also translated by x, such that $V = \{v_i + x\}_{i=1}^{n}$. In the latter case, it is the root vertex of G that is rotated by x.

$$I_{-6}P_{10}\langle 1,2,1,1,2,5\rangle = \begin{cases} P_{-6+10}\langle 5,2,1,1,2,1\rangle = P_4\langle 5,2,1,1,2,1\rangle, & \text{for a cp-space,} \\ P_{(-6+12-10) \bmod 12}\langle 5,2,1,1,2,1\rangle = P_8\langle 5,2,1,1,2,1\rangle, & \text{for a cpc-space.} \end{cases}$$

Def. 4.4. Multiplication (M_x). Let $\alpha = \langle \alpha_1, \alpha_2, \ldots, \alpha_k\rangle$ be an interval composition of a categorical pitch or pitch-class space $G = (V, E)$, the *multiplication* operation is respectively defined by $M_x\alpha = \langle x\cdot\alpha_1, x\cdot\alpha_2, \ldots, x\cdot\alpha_k\rangle$ or $\langle |x\cdot\alpha_k|, |x\cdot\alpha_{k-1}|, \ldots, |x\cdot\alpha_1|\rangle$, for $x \geq 0$ or $x < 0$, where $x \in \mathbb{Q}$.

Remark. If α is pitched and G is a categorical pitch space, then $M_xP_y\alpha = P_{x\cdot y}\langle x\cdot\alpha_1, x\cdot\alpha_2, \ldots, x\cdot\alpha_k\rangle$ or $P_{x\cdot y}\langle |x\cdot\alpha_k|, |x\cdot\alpha_{k-1}|, \ldots, |x\cdot\alpha_1|\rangle$, respectively for $x \geq 0$ or $x < 0$, where $x \in \mathbb{Q}$. If α is pitched and G is a categorical pitch-class space, then $M_xP_y\alpha = P_{(x\cdot y) \bmod |E|}\langle x\cdot\alpha_1, x\cdot\alpha_2, \ldots, x\cdot\alpha_k\rangle$ or $P_{(x\cdot y) \bmod |E|}\langle |x\cdot\alpha_k|, |x\cdot\alpha_{k-1}|, \ldots, |x\cdot\alpha_1|\rangle$, respectively for $x \geq 0$ or $x < 0$, where $x \in \mathbb{Q}$. The multiplication operation M_x results in a *homothety*, i.e., a homogeneous *dilation* or *contraction* of the pitched interval composition $P_y\alpha$ by x, respectively if $|x| > 1$ or $0 < |x| < 1$. The x in M_x is called the *multiplication index*. M_{-1} is equivalent to the inversion operation I. M_0 results in an empty interval composition. M_1 is an identity operation.

Ex. $M_5\langle 2,3,2,1,4\rangle = \langle 10,15,10,5,20\rangle$

$$M_7P_3\langle 2,1,3,6\rangle = \begin{cases} P_{7\cdot 3}\langle 7\cdot 2, 7\cdot 1, 7\cdot 3, 7\cdot 6\rangle = P_{21}\langle 14,7,21,42\rangle, & \text{for a cp-space,} \\ P_{(7\cdot 3) \bmod 12}\langle 7\cdot 2, 7\cdot 1, 7\cdot 3, 7\cdot 6\rangle = P_9\langle 14,7,21,42\rangle, & \text{for a cpc-space.} \end{cases}$$

$$M_{-\frac{1}{3}}P_6\langle 3,9,24\rangle = \begin{cases} P_{-\frac{1}{3}\cdot 6}\langle |-\tfrac{1}{3}\cdot 24|, |-\tfrac{1}{3}\cdot 9|, |-\tfrac{1}{3}\cdot 3|\rangle = P_{-2}\langle 8,3,1\rangle, & \text{for a cp-space,} \\ P_{(-\frac{1}{3}\cdot 6) \bmod 12}\langle |-\tfrac{1}{3}\cdot 24|, |-\tfrac{1}{3}\cdot 9|, |-\tfrac{1}{3}\cdot 3|\rangle = P_{10}\langle 8,3,1\rangle, & \text{for a cpc-space.} \end{cases}$$

Considering this is an introductory work, we will not go beyond the definitions of those so-called *canonical* operations–transposition, inversion, and multiplication–which are sufficient for now. Nevertheless, other operations (rotation, merge, split, transference, interference, conjugation, complementation, etc.) are in progress and will be properly formalized in due course.

V. Musical Application

Let's take as an example the musical miniature for piano shown in Figure 8, which was composed using the cyclic interval composition $\langle 3,1,4,4\rangle$ and the basic operations (P_x, T_x, I_x, M_x). At first, we have $P_2\langle 3,1,4,4\rangle = \{2,5,6,10\}$ and its inversion $I_0P_2\langle 3,1,4,4\rangle = P_{10}\langle 4,4,1,3\rangle = \{10,2,6,7\}$, disposed respectively on outer and inner parts and transposed successively by T_{-1} throughout the entire piece, except for mm. 5, where the outer parts are in turn multiplied by $M_2P_{11}\langle 3,1,4,4\rangle = P_{10}\langle 6,2,8,8\rangle = \{10,4,6,2\}$, before completing the chromatic sequence a perfect fifth ($P5$) away from the starting point (i.e., $ci_{2\text{-}9} = ci_{10\text{-}5} = 7$).

At first glance, it may seem that there is no significant difference between this approach and that used in pitch-class set theory, not least because the operations are basically the same. However, with a little more attention, we realize that the interval representation has great advantages for describing structural relationships, making them more explicit and comprehensible. In this sense, the interval structures and basic operations presented here are not only very useful, but also–possibly–the first steps on a long journey towards the development of a post-tonal system, or rather, a post-tonal harmony.

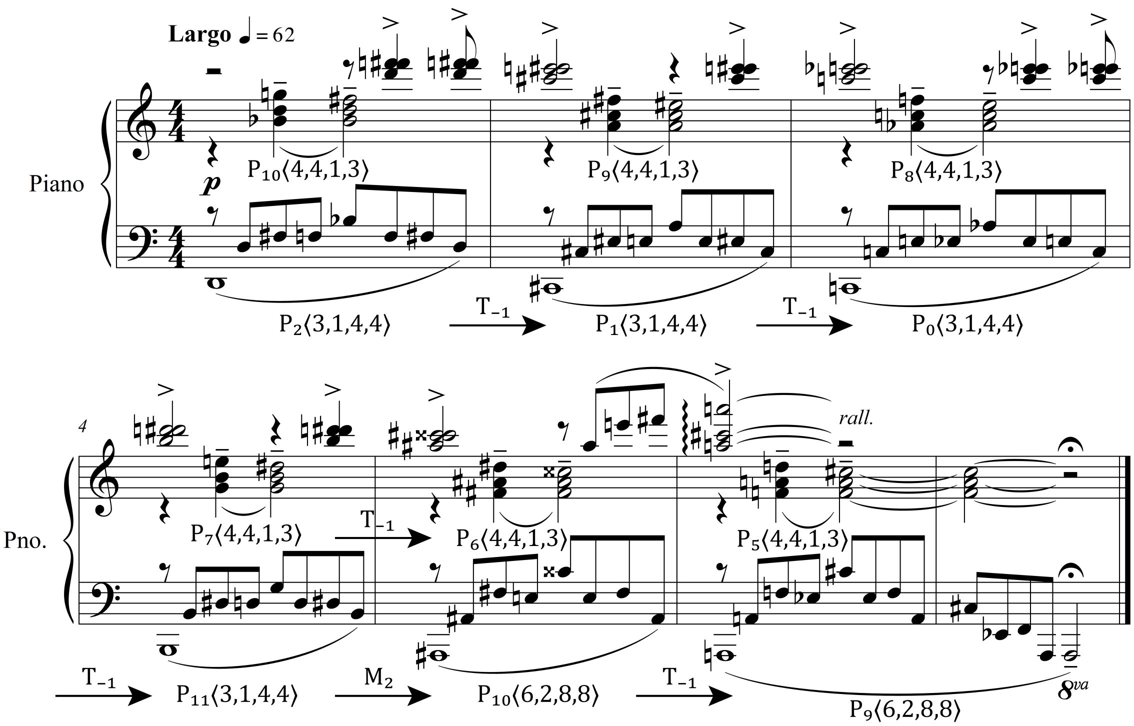


Figure 8: *Musical miniature using the cyclic interval composition* $\langle 3, 1, 4, 4\rangle$ *and the basic operations* (P_x, T_x, I_x, M_x).

VI. Concluding Remarks

In this preliminary work, we have sought to present a brief historical review of the use of partitions in music, to provide a concise introduction to the theory of partitions, and lastly, through an extensive bibliographic revision and a thoughtful theoretical reflection, to lay the foundations of what we have called *partitional harmony*. As we could see, at least fundamentally, this conception runs through many subjects of post-tonal music theory and different fields of mathematics, presenting great potential for musical applications, whether analytical or compositional.

As we progress in our research, our theoretical scope will be gradually expanded, including other operations of transformation and ordering of the interval structures, enumeration of those structures,[53] algorithms to normal and prime forms, new approaches to interval content (*interval orbits*, *interval multicompositions* and *multipartitions*), inclusion, similarity, and equivalence relations, *harmonic identities*, *voice-leading spaces* [28, 45, 121, 164], in addition to the development of an application software (a calculator) and an exhaustive taxonomy for interval structures. In the long run, the codification or systematization of the harmonic aspects of the post-tonal language through the use of partitions, or other *partitional systems*,[54] may be a point of convergence between *partitional analysis* [66] and the modeling of *compositional systems* [134, 135].

Finally, we hope that many of the concepts and the valuable bibliography presented here will be useful to those interested in post-tonal music theory and related topics.

[53] For an introduction to enumeration in music, see [37, 60, 61, 62, 63, 73, 83, 87, 91, 143].

[54] Since the notion of categorical space may be extended to musical entities other than pitches or pitch-classes, such as chords, scales, harmonic progressions, etc., we can likewise partition those spaces and establish relationships between their elements through the use of partitions. Hence the concept of *partitional systems*.

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Aspects of Performance Practice in Morton Feldman's *Last Pieces*

*Robert Morris
Eastman School of Music - University of Rochester
rmorris@esm.rochester.edu
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Abstract: *Morton Feldman's* Last Pieces *for piano solo of 1959 poses an interesting interpretive problem for the performer. As in many Feldman compositions of the 1950s and 60s, the first movement of the work is notated as a series of "sound events" to be played by the performer choosing the durations for each event. The only tempo indications are "Slow. Soft. Durations are free." This situation is complicated by Feldman's remark about a similar work from 1960, "[I chose] intervals that seemed to erase or cancel out each sound as soon as we hear the next." I interpret this intension to keep the piece fresh and appealing from sound to sound. So, how the pianist supposed to play* Last Pieces *in order to supplement the composers desire for a sound to "cancel out" preceding sounds? To answer this question, I propose a way of assessing the salience of each sound event in the first movement of Last Pieces, using various means of associating each of its 43 sound events according chord spacing, register, center pitch and bandwidth, pitch intervals, pitch-classes, set-class, and figured bass. From this data, one has an idea about how to perform the work to minimize similarity relations between adjacent pairs of sound events so that they can have the cancelling effect the composer desired. As a secondary result of this analysis, many cohesive compositional relations come to light even if the work was composed "intuitively".*

Keywords*: Morton Feldman. Musical Set Theory Analysis. Performance of Twentieth-Century Music. Musical Contour Analysis.*

I. Introduction

Writing in 1962 on his compositional evolution, Morton Feldman states:

> After several years of writing graph music, I began to discover its most important flaw (...) if the performers sounded bad it was less because of their lapses in taste than I was still involved with passages and continuity that allowed their presence to be felt. [3]

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Figure 1: *First system of Morton Feldman's* Last Pieces *(1959).*

Feldman's solution was to compose a series of "sound events" to be played at a given tempo with the performer choosing the durations for each event. One such work is *Last Pieces* (1959) for piano in four movements. The sound events are single chords or tones (each notated with unstemmed quarter notes) occasionally accompanied by a grace note or a fermata. The first movement, for instance, is written on three systems, has 43 sound events and is marked "Slow. Soft. Durations are free." Figure 1 shows the first system of the work; Figure 2 shows the 43 sound events, numbered and lined up in tens for easy reference. Grace notes are not distinguished from the other notes.

It would seem that performing such a piece is straightforward enough. But playing the sounds events one by one, listening carefully to each, gives rise to questions of nuance that in performance can radically alter the flow and character of the musical experience. Commenting about a sister piece, Durations, Feldman offers this: "[I chose] intervals that seemed to erase or cancel out each sound as soon as we hear the next" [3]. This desire for sonic particularity was shared by many composers of the early 1950s—Stockhausen, Cage, Messiaen. But in works like *Last Pieces*, the performer has the responsibility of projecting the "suchness" of each sound event without fail over the entire extent of the piece. We might think of the performer's task as similar to the curator of an art show. Given a number of paintings on a theme to be displayed in a certain order on the walls of a hallway, how should they be hung so that each painting is presented to its best advantage, not diminished or overshadowed by another. Questions of lighting, wall position, grouping of adjacent painting will naturally arise and be highly dependent on the nature of each painting. The difference, of course, is that the musician has to do this with sound events in real time.

To this end, it is useful to carefully consider the nature of each sound event from a number of points of view, not to group the events into classes or hierarchies, but to fully explore each event's particularity and the ways in which one can project its character via nuances of piano touch, voicing, pedaling, and duration. I will therefore examine the sound events in the first piece of *Last Pieces* using concepts and techniques from twentieth-century music theory. I will show there are overlapping patterns of association in the work, which provide criteria for projecting various interpretations that fulfill Feldman's desire for local sonic presence.

But before I begin, I want briefly to address Feldman's strong and often negative views toward music theory. For instance, when the then young composer/pianist, Frederik Rzewsky referred to one of Feldman's pieces as, "You know that canon for two pianos?," Feldman mentally retorted, "Canon, me, my canon? Oh yes, that free-durational piece. It was a canon, I suppose. To tell you the truth, if I thought it was a canon, it would have caused me to commit suicide" [2]. But it was

Figure 2: *The 43 sound events of movement I from Feldman's* Last Pieces *(1959).*

not only that Feldman did not recognize his music as defined by inappropriate technical terms, Feldman's whole aesthetic enterprise was designed to transcend the influences and habits of the past, and to avoid at all costs what he called "compositional rhetoric".

Nonetheless, Feldman did not consider sounds to be context free; there is context, that of other sounds, but not theoretic models and systems, on one hand, or traditions and past practices, on the other. So a suitable methodology for examining Feldman's music must concern musical materials before they are conscripted into generic relationships with each other according to some syntax or practice—as William Carlos Williams put it, "no ideas but in things." Oddly enough, the principles of musical set theory are just what is necessary for this task. Feldman himself made this point, if grudgingly, "I do not deny the validity of the pitch set, but in relation to sonic experience today, it seems to me as equivalent to the baby's playpen, and just as full of toys and pacifiers. [1]" Of course, if he knew of any musical set theory in the 1960s, it would have been Allan Forte's work, which only treats sets of pitch-classes and in music that did not interest Feldman. But by 1980, his descriptions of his compositional method is conceptually set-theoretic. In fact, in his last years, Feldman often referred to himself as closet serialist.

But if I need any further justification for my method of analysis, in 1963 Feldman wrote,

> When sound is considered as a horizontal series of events all its properties must be extracted in order to make it pliable to horizontal thinking. How one extracts these properties now has become for many the compositional process. In order to articulate a complexity of such close temporal ordering one might say differentiation has become here the prime emphasis on the composition. [5, p. 12]

The properties I shall extract from the 43 sound events of the first movement of Last Pieces, will be from three interconnected domains: contour, pitch, and pitch-class. We shall assess a sound event's singularity with respect to a given property, by placing it within the range of values presented by all the sound events having that property. In this way, we shall locate sounds events that are singular in pitch, range, interval content, and other properties of this kind. In order to address Feldman's requirement that sounds should erase or cancel out previous ones, we will attend to the degree of disjunction between successive events and the associative connections between both adjacent and non-adjacent events. This approach is based on three hypothetical principles:

- If a sound event is followed by something very different it will be eclipsed by the novelty of the new event;
- If a sound event X is noticed to be like another from before, the presence of the sound events immediately before X will be diminished since they are perceived to be sonically remote from X and the previous event which X resembles;
- From the above, adjacent sounds are relatively alike, they tend to be grouped and thereby lose some of their individuality, but if the group is followed by a different sound, the new sound's suchness is contextually highlighted.[1] Thus, a sound event's singularity is affected not only by its relations to all of the others, but to its local context.

Let me begin by attending to some terminology. I shall henceforth use the term *chord* to stand for sound event or verticality, even though some of the "chords" in the piece are dyads or single tones. The term *width* will stand for bandwidth, the number of semitones from the bottom to top of a chord. *Center* will stand for that pitch or semitone dyad that is equidistant between the highest and lowest notes of a chord.

[1] Stockhausen's idea of "Experiential time."

Looking over the 43 chords in Figure 2, we note they are grouped into three sections by Feldman's use of fermatas. Chord #1 sets the tone, and two sequences of chords follow; #2-#20, #21-#43. In this soft and slow piece, it's possible that Feldman wanted the fermatas to indicate that a sound is to be sustained until extinction. Feldman says, "The attack of a sound is not its character....Decay, however, this departing landscape, this expresses where the sound exists in our hearing—leaving us rather than coming toward us."[4] The other 39 sounds events are undifferentiated in notation and their durations are free; they may range from a second or two to much longer and from performance to performance.

The 43 chords come in different sizes and shapes. Dyads are the most frequent, and the few chords of five and six tones are arpeggiated. A look at Figure 2 shows that there are runs of dyads and chords of three and four notes here and there. Taking the 43 chords at once, they encompass a width of 71 semitones, from the lowest D of the piano, which is played alone in chord #31, to the highest ♭ of event #23. Apart from the two single-note chords, #2 and #31, the narrowest chord is #38 of 19 semitones; the largest is chord #23 of 64 semitones. Thus there are no multiple-note chords of less than an octave and a fifth, obliging them to be played with two hands. The most frequent width is 21 semitones, shared by 7 chords, three of which occur successively as #27-29; another adjacency is #39 and #40.

Figure 3 shows what pitches and pitch-classes constitute the chords. We see a fairly normal distribution of pitches, but some are infrequent or omitted in the middle range, and the peak is not at the center but at two pitches — C♯ below and B above middle-C – because the chords are played by the two hands. All this indicates that the overall distribution of notes is not a global characteristic of the particularity of the piece. The chart also helps us to locate those rare pitches whose scarcity induces a certain charisma, or points out the common pitches between chords that associate them. (By the way, there are many instances of each pitch-class, with C and C♯ dominating and F♯ G♯ and B less frequent.)

We can also group the 43 chords into types by register, center, and what I have called *spacing-types*. As I mentioned before, there are no narrow chords in this movement. The average width is 31 semitones, with just about as many widths above and below this number. Figure 4 shows the multiple-pitch chords with the lowest and highest centers; it also shows that the two one-pitch chords are lower than any center of a multiple-pitch chords. Chord #42, the next to last, has the highest center pitch. The example additionally shows the largest set of chords that share the same center, the dyad middle C and D♭. The set includes the first chord plus #14, the only chord of more than 2 tones that has mirror symmetry around its center. No two adjacent chords have the same center, but the example shows two pairs that have adjacent centers.

There are six spacing types: *even*, *uneven*, *overtone*, *inverse-overtone*, *centered*, and *barbell*. Figure 5 illustrates the spacing types. Almost half of the chords are of even spacing, but most of these are trivial since a chord of one or two notes must have even spacing. Overtone spacing, with the intervals becoming narrower at the top of a chord, are next most frequent; these produce a clear and resonant, bell-like sound. Barbell spacing is also frequent with its two groups of neighboring pitches situated at the top and bottom of a chord with a wider interval between the groups; these chords fit nicely under the two hands. Two spacing types are uniquely represented: chord #14, mentioned above, is centered and is the only multiple-note chord that has mirror-symmetry. Chord #7 has inverse-overtone spacing; its singularity is heightened by its low E♭ grace-note, which is the next to lowest pitch in the piece.

pitch distribution

octave 1	octave 2	octave 3	octave 4
0123456789AB	0123456789AB	0123456789AB	0123456789AB
11 121	21 13412 22	45 32 12232	322433 31345
		x	x
	x	xx	x xx
	xx	xx x x	x xxx x xxx
x	x xx x xx	xx xx xxxx	xxxxxx x xxx
xx xxx	xx xxxxx xx	xx xx xxxxx	xxxxxx xxxxx

octave	octave 6	octave 7
0123456789AB	0123456789AB	0123456789AB
3 323313231	31312 22111	12
x x xx x x	x x	
x xxxx xxx	x x x xx	x
x xxxxxxxxx	xxxxx xxxxx	xx

pitch-class distribution

0	1	2	3	4	5	6	7	8	9	A	B
13	13	10	11	11	9	8	11	8	11	12	8
x	x										
x	x									x	
x	x		x	x			x		x	x	
x	x	x	x	x			x		x	x	
x	x	x	x	x	x		x		x	x	
x	x	x	x	x	x	x	x	x	x	x	x
x	x	x	x	x	x	x	x	x	x	x	x
x	x	x	x	x	x	x	x	x	x	x	x
x	x	x	x	x	x	x	x	x	x	x	x
x	x	x	x	x	x	x	x	x	x	x	x
x	x	x	x	x	x	x	x	x	x	x	x
x	x	x	x	x	x	x	x	x	x	x	x
x	x	x	x	x	x	x	x	x	x	x	x

Figure 3: *Pitch and Pitch-class distribution in Feldman's* Last Pieces *(1959).*

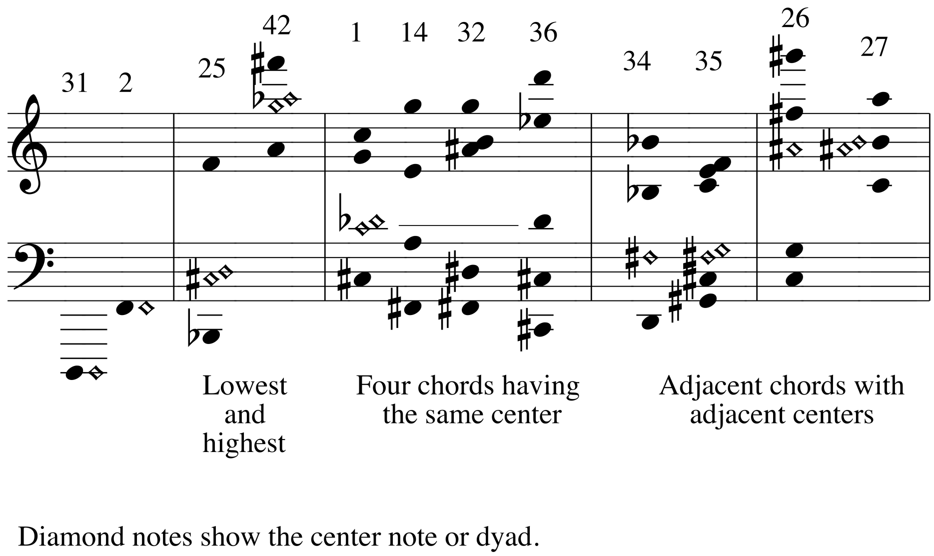

Figure 4: *Centers of Chords in Feldman's* Last Pieces

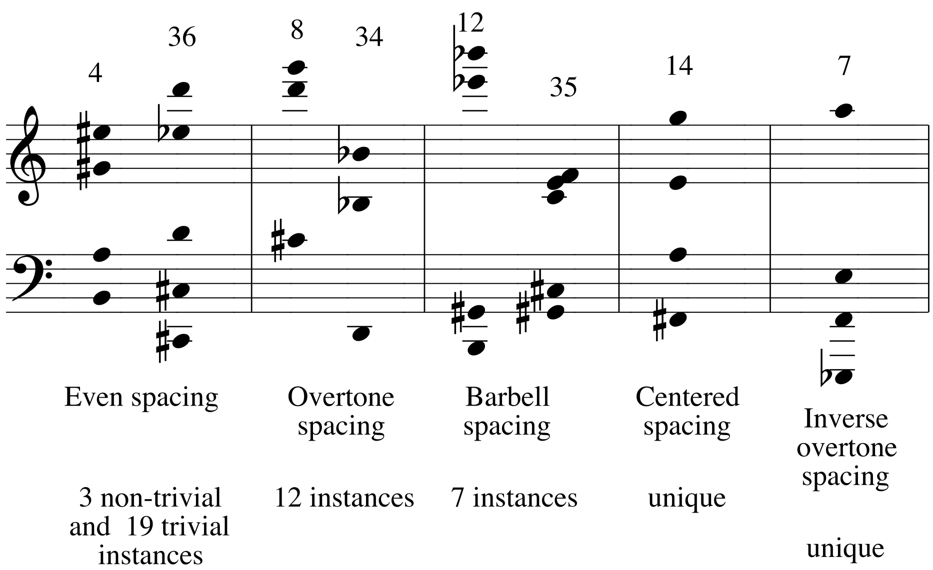

Figure 5: *Chord Spacing Types in Feldman's* Last Pieces *(1959).*

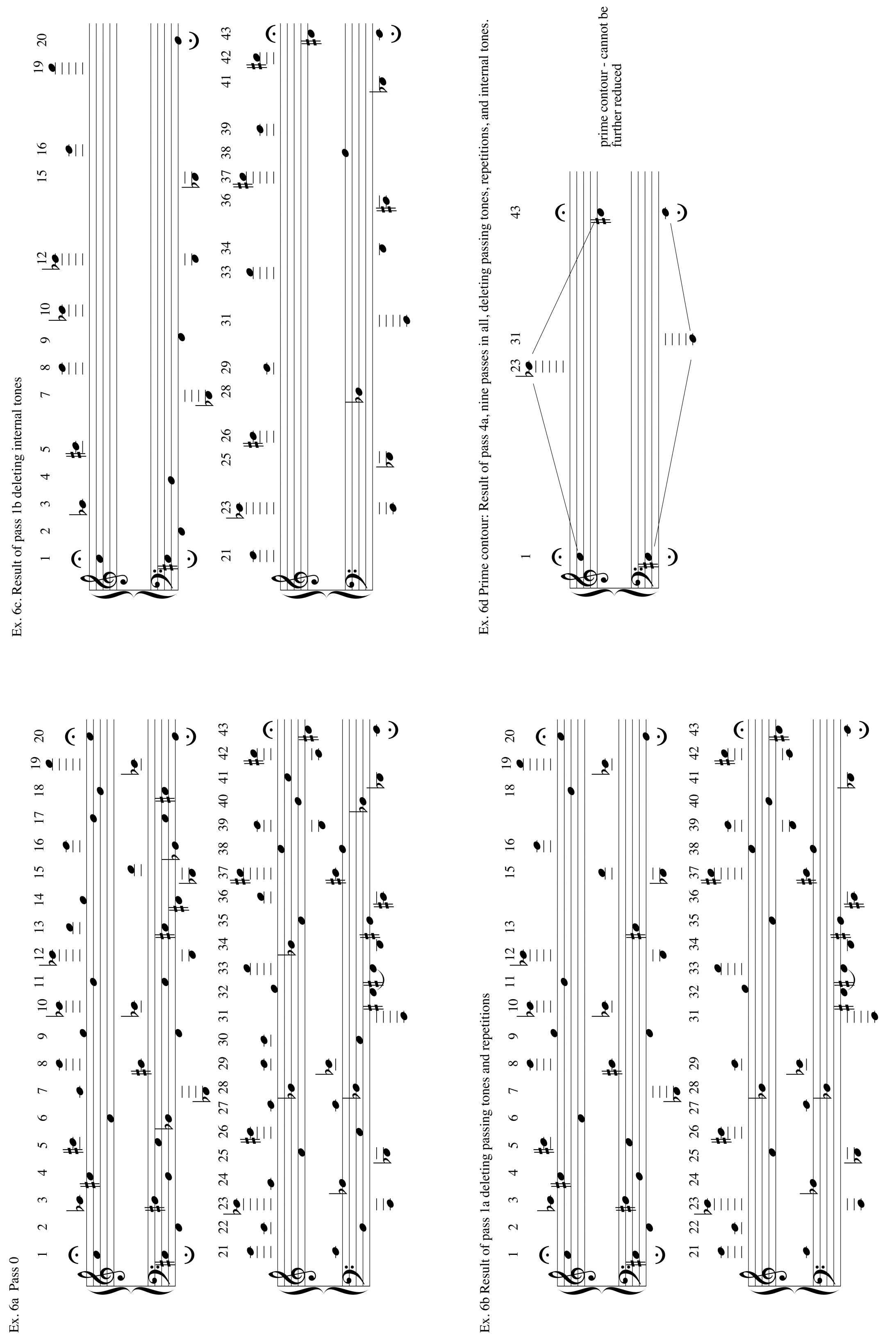

Figure 6: *Contour reduction of outer voices of chords in Feldman's* Last Pieces *(1959).*

We now turn to the contour of the outer voices of the 43 chords. Figure 6a, called pass 0, extracts these notes from the chords. The frequent zigzag motion of the extracted voices tends to make sure adjacent chords are not connected by anything like traditional voice-leading. The rest of the example shows the application of a version of my contour-reduction algorithm. Figure 6a, is gradually reduced to simpler and less turbulent contours by the alternative application of two types of rules. One first removes what we recognize as passing and repeated tones from each of the two lines; this shown in Figure 6b called "result of pass 1a." Then one removes internal tones, those that are not the maxima of the upper voice or minima of the lower voice. The result of this is Figure 6c called "result of pass 1b." We continue until the rules cannot be applied any longer. The result is a prime contour, shown in Figure 6d. As the algorithm is applied, various chords are omitted until the resulting prime references only four of the chords, #1, #23, #31, #43. The fact that it takes eight applications of the algorithm to fully reduce the chords to the prime testifies to the complexity of the local changes of the outer voices. To see this, consider a descending series of notes; it takes only one pass of the algorithm to reduce it to a prime. We can ask however how relevant it is to apply this tool to the Feldman piece, as it shows that there are multiple and hierarchical levels of contour change going on in a piece that attempts to assert local particularity above all. But such levels may function to group events so as to assert greater degrees of suchness after clusters of similar events, as in the third of my three hypothetical principles. On the other hand, the events in the prime contour do assert singularity for they are the piece's first and last chords and the chords with the lowest and highest text pitch; moreover, the single low D of #31 further asserts its unique contour function.

Up to now, I have treated the notes of this piece without attending to the exact intervals between and among them. An important question arises when we contemplate intervals and sets of notes: Should we consider this piece to assert octave-equivalence or not? While there are many simultaneous octaves and adjacent octave intervals in the piece, we could regard them as ways to produce various timbral effects as in chord #33, shown in Figure 7a, in which the double-octave A♮ are the only tones articulated because the others are tied from the previous chord. Another striking chord with octaves is the very next one, #34. Yet, #33 and #34 have a very different "determinant feel," which is induced in part by where the octave is located in each chord. From the sonic effect of such examples, we see that the presence of octaves need not recommend that the pitches in this piece—or of any twentieth-century composition—be regarded as pitch-classes; after all, the octave is just one interval out of many, each of which have different qualities and weights. And conversely, an absence of octaves need not imply that it is inappropriate to use pitch-classes and their sets in analysis; in fact, a good deal of atonal music and twelve-tone music without simultaneous or successive octaves makes sense mainly from a pitch-class point of view. In any case, as I have argued elsewhere, it is good practice to separate out the aspects of tone relations that depend on pitch versus pitch-class.

Octave relations are implicated from the second chord of the piece. Figure 7b, shows chords #2-#7. It is interesting to ask oneself if one hears an octave pitch-class relation from the single F♮ in chord #2 to the inner voice F in chord #3. If one does, the two Fs are still distinct because they occur in different locations in each chord. If one does not, it might be because the low F is connected to the F♯ in the bass of chord #3 by the smallest interval between the two chords. We can also ask about the E♯ and G♯ in the chord #4. Do we hear them as connected by repetition and octave transfer from #3? Feldman's enharmonic notation suggests that this connection may not be intended, but the pianist could bring out an octave connection by voicing the chords to that end. The rest of Figure 7b shows other octaves and pitch repetitions. Figure 7c indicates that chord #21 beginning the third section of the piece exchanges the notes of the previous chord sustained by the fermata. Here again the pianist can support or suppress this connection. The three chords in

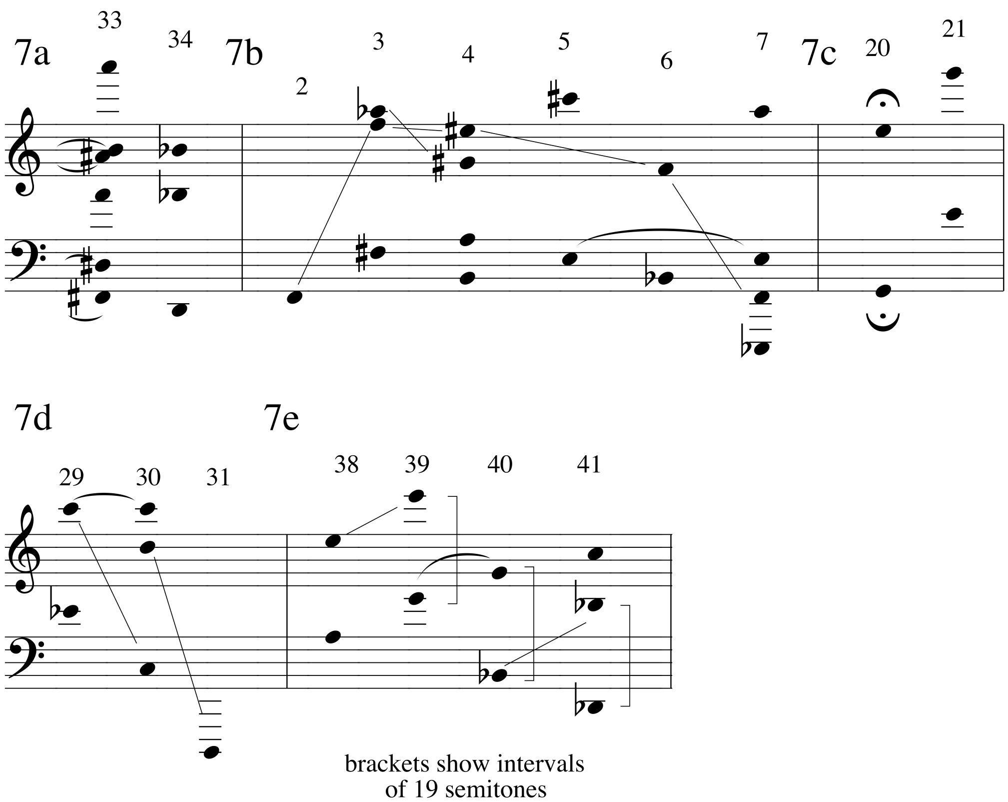

Figure 7: *Octaves and pitch repetitions in Feldman's* Last Pieces *(1959).*

Figure 7d present five pitches and three pitch-classes. Chord #30 repeats the high C of #29 but also launches a leap of three octaves down from #29 to #30. Then the D enclosed by the Cs in #30 leaps down four octaves to the singular low D in #31. Or do we hear Cs and Ds in different registers, or different colors of the same pitch-class set? Obviously the judicious use of the right piano pedal can help underscore one or more of these different hearings. In Figure 7e , the sequence of chords #38 to #41, projects some octave relations among the notes E, G, and B♭. The connection between #39 and #40 is unique because the lowest note of a chord becomes the highest note of the next. And there is an additional feature here; the bracketed notes are all of the same pitch interval of 21 semitones. Once again pitch and pitch-class relations are intertwined

The intersection of pitch and pitch-class spaces is addressed by the adaptation of a traditional analytic tool, figured bass. Here the figures are the numerals from 1 to 12. The number n written under the bass of a chord indicates the presence of chord tones n semitones plus any number of octaves above the bass. Figured bass equivalence mediates between pitch and pitch-class set-classes. Figure 8a shows chords #3, #27, and #41. While each chord is from a different pitch set-class, all three are members of the pitch-class set-class 3-2[013], However, the last two chords have the same figured bass, <9 11>. Feldman frequently constructs chords with identical figured basses in Last Pieces. Two adjacent chords with the figured bass are illustrated in Figure 8b, and another, but non-adjacent, pair occurs in Figure 8c. The figured bass is now <148>, which includes the figures in Figure 8b. This implies that there is pitch-class set-class inclusion relation between the chords of Figure 8c and 8d.

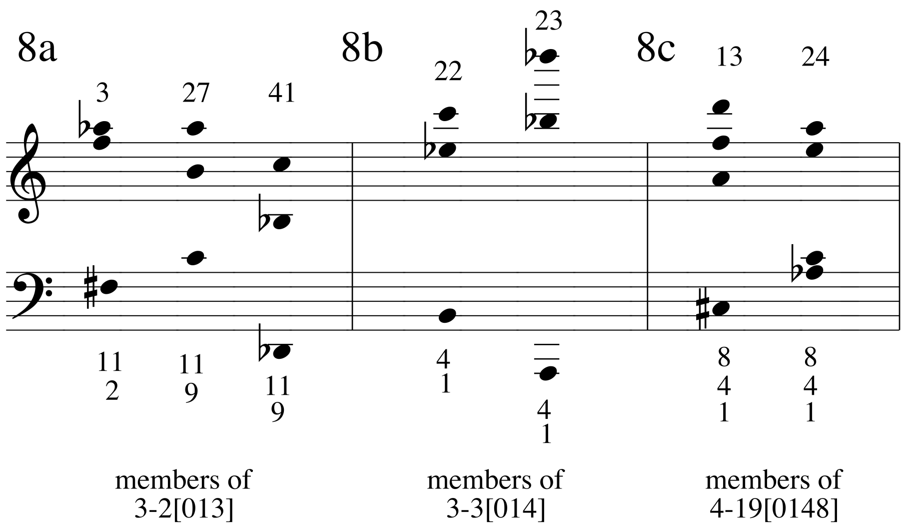

Figure 8: *Figured bass equivalence in Feldman's* Last Pieces *(1959).*

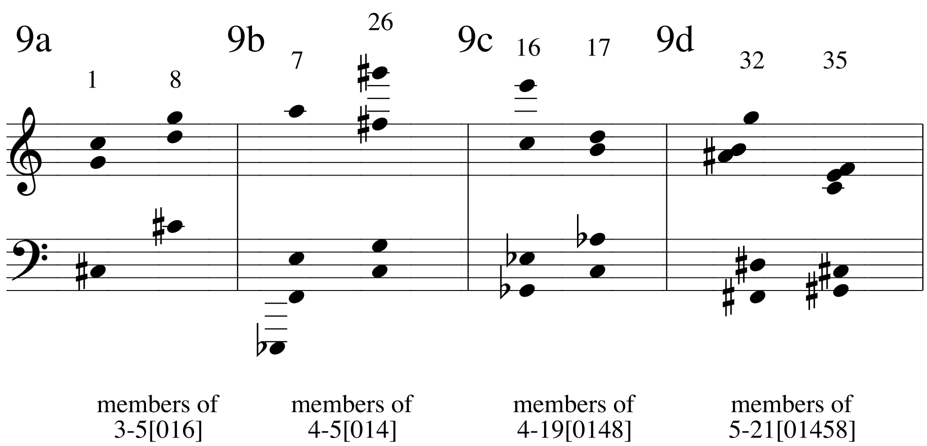

Figure 9: *Pitch-class equivalence in Feldman's* Last Pieces

The relevance of considering pitch-class relations as important in this piece is supported not only by equivalence under figured bass, but the presence of multiple instances of pitch-class set-classes. For instance, the chords #1 and #8 are given in Figure 9a; these are both members of set-class 3-5[016]. It's particularly easy to hear this relation because both chords sustain a perfect fourth over a C♯. Chords #7 and #26 are members of a set-class that contains 3-5. These chords in Figure 9b may sound somewhat different due to the register and order of their pitch intervals from low to high, but they are also similar because they are the only chords in the piece to have the inverse-overtone spacing. Feldman also places two chords from the same set-class adjacently as shown in Figure 9c. And in 9d, the two five-note chords #32 and #35 are of the set-class 5-21.

When we examine all the set-classes in the piece we notice a fairly tight setcomplex structure. See Figure 10. There are two strains of embedded setclasses; those embedded in set-class 5-21 (generalizing ic 4) and 5-38 (generalizing ic 3). These two set-classes are Rp-related. All

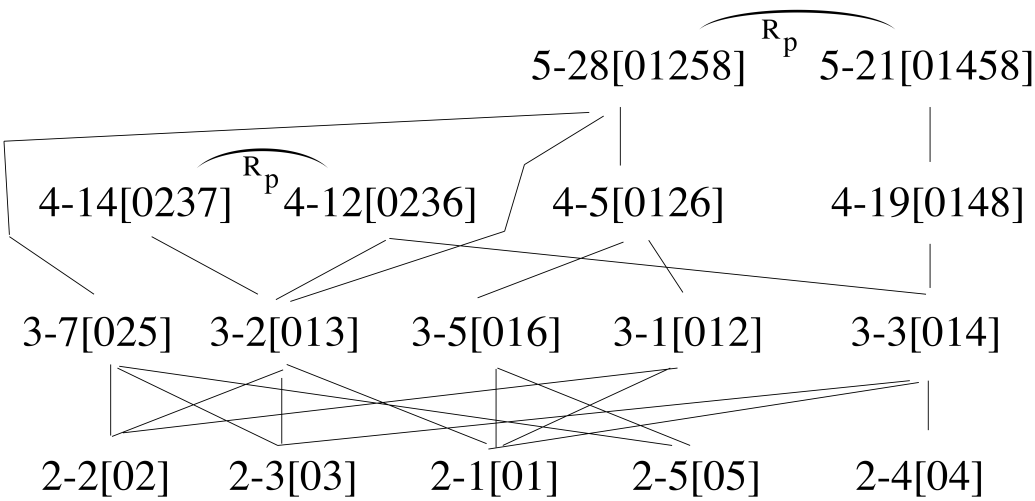

Figure 10: *Set-class network in Feldman's* Last Pieces *(1959).*

the trichordal set-classes are in one or the other strain, while set-classes 4-12 and 4-14 are not. Nevertheless, set-classes 4-12 and 4-14 are Rp-related. It is perhaps significant that the last chord of the piece presents ic 1, for that interval is found within every set-class except 3-7. Ic 1 figures also in the pitch-classes of the prime contour of the piece, a member of the chromatic set-class 5-1, and in chord #36, whose multiply-represented pitch-classes are literally included in the set of the prime.

The success we have had in discovering coherent pitch-class relations suggests that aggregate completion might also play a role in the composition. Looking at the first 8 chords in Figure 2, pitch-class turnover progress rapidly except for few repetitions and the retention of F♮, which occurs in five of the chords. At chord #8, with the introduction of the tone D, all twelve pitch-classes have been introduced. This point of saturation is significant since we noted a strong association of chord #8 with #1 in Figure 9a. Moreover, the D replaces the C in chord #1. A similar advance to the aggregate begins quickly at the opening of the third section with chord #21, but at chord #24 it slows down and halts at #27. The only pitch-class not sounded is D♮, which occurs finally with two C♮ in chord #30 and then by itself in chord #31. The parallel with the opening aggregate progression implicating pitch-classes C and D is remarkable.

Having examined some connections and contrasts between the chords, we return to the question of performance. How should one present the chords to the listener? Since the network of interrelations of the chords is complex, we might just let the music play itself and keep out of its way. But if Feldman wanted this kind of performance he might have noted the piece in traditional rhythmic notation, and there are pieces from the same period that do just this. In Last Pieces Feldman allows room for choice in duration and dynamics, with the compositional proviso that each chord cancels out the previous ones. As I implied above, this suggest that each chord should be played with attention to bringing out its suchness, but not in spite of the other chords, but within their context.

I have shown many ways in which a particular chord is either locally or globally singular and the opposite, how sets of adjacent chords share similar characteristics. One performance strategy would be to highlight the singularities and play down the local connections. But the singularities do not need performer support; they are "already" singular. It's the local continuities that need to be broken up, so they do not become generic. For instance, chord #4 might be played differently from chord #3, thereby minimizing the shared pitch and pitch-class. (And as I mentioned before, in this case, Feldman's notation suggests this approach.) In contrast, chords #31 and #32 might be played similarly since the move from a single low tone to a middle-range barbell-spaced chord with a grace note dyad is relatively large in the context of the composition. Of course, sticking slavishly or automatically to a strategy of minimizing local continuity would not be in the spirit of responsible performance, where attention must be constantly focused on the moment by moment flow of the piece as one unfolds it with all the resources of piano performance, including timing, voicing, and pedaling. Relying only on habits of performance, traditional or not, will only reveal the lapses of taste that Feldman has so carefully tried to avoid in his compositional method. As in other performance practices, the more one knows about a piece, the more one can bring to do it justice.

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Philosophical Sketches on Category Theory Applied to Music-Mathematical Polar Semiotics

Gabriel Pareyon
CENIDIM-INBA, Mexico City,
& CUCSH-UDG, Guadalajara
cenidim.gpareyon@inba.edu.mx
Orcid: 0000-0002-3471-0102
DOI: 10.46926/musmat.2020v4n2.41-51

Abstract: *This is an attempt to combine Matthai philosophy (of Heraclitan inspiration) and Category Theory using the Yoneda Lemma as a means for harmonizing the traditionally opposite values and conceptions dissociated between the Euclidean tradition and Heraclitus thought. The text is divided in three sections: general background and description of Yoneda, a contextualization on Heraclitan aesthetics and* polar semiotics *(a notion firstly intuited by I. M. Lotman and Th. Sebeok), and an experiment suggested for the revision of the grounds of music theory, with the purpose of conciliate extremely dissociated notions of music (Euclidean vs. Heraclitan) however making part of a common musical experience and knowledge. Conclusions are addressed to hypothesize that Yoneda lemma may support a robust philosophy of music within the field of Category Theory where any group is isomorphic to a subgroup of a permutation, with one-to-one paired correspondences.*

Keywords: *Yoneda embedding. Matthai postulate. Category theory. Polar semiotics.*

I. Introduction

In addition to exposing the cohomology between the Yoneda lemma and Matthai postulate ("All thoughts are truthful"), not so much for its demonstration as a logical statement –already satisfied by Grothendieck logic–, a main interest in this article is to discuss major implications of that cohomology over the philosophy of music and describe logical operators for systematic musicology in practice,[1] particularly in their theoretical expansions of *intersemiotic continuum* and *intersemiotic synecdoche* as explained below.

The concept of *intersemiotic continuum* is formulated in [14, pp. 102–114], and then adapted into a variety of disciplines.[2] It consists of a succession of symbolic associated systems, transforming

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[1] As *practice* here I assume a main task for musicologists (for analytic, synthetic and heuristic purposes), as well as for music philosophers and composers (creation and transformation of music and musical ideas).

[2] Since February 2011, when *On Musical Self-Similarity* came out from the imprint in Helsinki, the concept of *intersemiotic continuum* have had a successful adaptation into the fields of information and communication sciences (L. Isaeva, N. Komina; S. Krestinski, 2012), literature analysis (A. Sousa, 2011; N. M. Dusi, 2014; F. Saggini, 2015), and cognitive science (Iu. Shamayeva, 2016). It comes as a refinement of the concept of *semiotic continuum* coined by Iu. M. Lotman (1984/1989).

any object however without losing its main features admitted and needed for its ontology or *ontological truth*. In other words, *intersemiotic continuum* involves a process that preserves an "informational track" to be transported through a multidimensional space of n-meanings (a set isomorphic to m-interpretations) in a process known as *semiosis*.[3]

Perhaps the *intersemiotic continuum* concept can be easily understood if we suppose,[4] that a musical pitch interval named $[C, A]$ equals the relative value of $[261.63, 440]$ (Hz), or the absolute ratio of $[1 : 2, 3 : 5]$ translatable as a rhythmic/metrical pulsation; but also equals the duration interval of $[120 : 100]$, and also equals the visible light pair of colors $[\textit{white}, \textit{yellow}]$, according to Barlow's interpretation.[5] This strain of continuous translation would intertwine an intrasemiotic system of translation if this occur exclusively within the "musical" domain. But music seems to be a complex set of worlds within worlds; thus, although presumably composing a same "domain", we should think of an *intersemiotic continuum* even when we do not trespass the cultural field of "music".

Now, the latter presupposition may lead us to a conceptual cul-de-sac, in the jargon of logics known as a *Frege chain* –in allusion to Gottlob Frege–,[6] with the theoretical cancellation of language and code functioning. However this actually does not happen in natural language, neither in the core of Category Theory as represented by the Yoneda embedding. Therefore, in order to adapt this mathematical concept to musicology, let's firstly recall the Yoneda lemma in its classical form:[7]

$$X \to h_A = \mathrm{Hom}_C(A, X). \tag{1}$$

As literally expressed in [12]: for C a locally small category, every object X of C induces a presheaf on C: the representable presheaf h_X represented by X. This assignment extends to a functor $C \to [C^{op}, Set]$ from C to its category of presheaves. The Yoneda lemma implies C as a full subcategory inside its category of presheaves. The presheaf represented by an object X of C is the functor $h_X : C^{op} \to Set$, which sends each object U to $\mathrm{Hom}_C(U, X)$ and each morphism $\alpha : U' \to U$ to the function

$$h_X\alpha : \mathrm{Hom}_C(U, X) = \mathrm{Hom}_C(U', X), \tag{2}$$

then, for $f : X \to Y$ a morphism in C, induces a natural transformation $h_f : h_X \to h_Y$ (for its corresponding graph see [12]), that develops into the commuting diagram in Figure 1.

This is a basic Category Theory description for the Yoneda embedding. Henceforth, let's constrain Matthai postulate "All thoughts are truthful" as a partial feature (logical and philosophical) of Cayley's theorem from group theory, and complementary to Yoneda lemma:

$$\{U' \to U[\text{よ}]\} \ni \{X \to Y[\mathfrak{M}]\}, \tag{3}$$

where Yoneda embedding is conventionally formalized よ, and its Matthai cohomology is formalized $\mathfrak{M}$. This allows us to subsume the latter "philosophical category" under a wider and more complex Set (よ).[8] Furthermore, if we define $\mathfrak{m}$ as the subset of "musical thoughts", then

$$\{\mathfrak{M}\} \ni \{\mathfrak{m}\}. \tag{4}$$

[3] See [9].

[4] As [6, p. 97] echoing [4, p. 100].

[5] Barlow's interview at the University of California Television (UCSB, 1 feb. 2008, segment 12:16" to 13:36"): "C was always white [...] F was always green, A was yellow, D was royal red, and for a reason I don't know, A flat was a wonderfully blue colour [...] If you look at the colours of Brahms and Scriabin, they have totally different [colours] to each other, and different to me; words also have colour for me, words and even single letters and single digits have colour".

[6] By association to Frege's paradox, see [14, p. 193].

[7] The Yoneda lemma says that the set of morphisms from a representable presheaf $y(c)$ into an arbitrary presheaf X is in natural bijection with the set $X(c)$ assigned by X to the representing object c.

[8] This symbol corresponds to the katakana kana *Yo* (here for Yoneda) in Japanese phonetic representation.

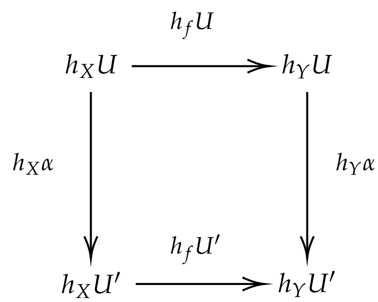

Figure 1: *Commuting diagram summarizing the Yoneda lemma.*

In consequence, "All [musical] thoughts are truthful". This actually matches very well with Peircean semiotics, as noticed by [17, p. 42]: "truth and communication in Peirce's view are completely isomorphic because the inferential character of argumentation is always dialogic–not between two different people who are in *communication* but between two different moments of the same mind in which the unity of the semiotic continuum is realized". Here we obviate the concept of *musical thought* as defined in [2] with especial attention paid to the symmetry principles, and later adapted by [6] to the context of the continuum-discontinuum theory of music.

Now, for the case $\{\mathfrak{M}\} \ni \{\mathfrak{m}\}$, simple membership is trivial. What is useful for musicology is qualitative membership for translatability and systematic coherence among distinct kinds of "objects" $(\alpha, \beta, \ldots)$ and "complexities" $(Y, Z, \ldots)$ in Category Theory, as represented by Figure 2, below.

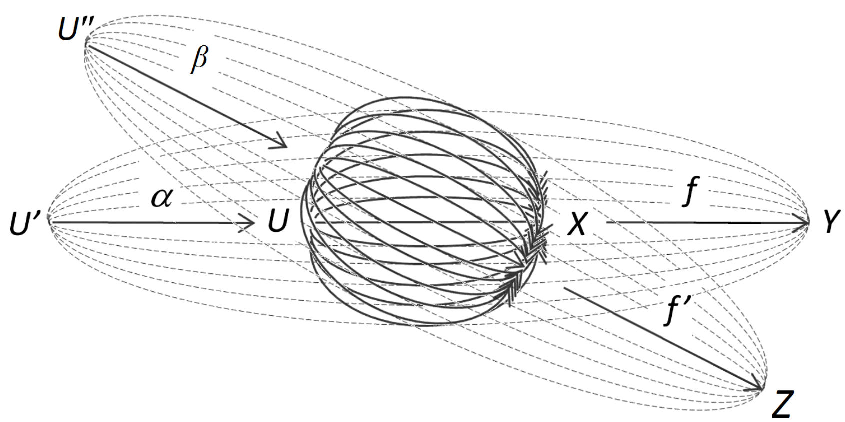

Figure 2: *A self-overlapping of the graph borrowed from [12]; A non-trivial adaptation on Yoneda embedding, where a positional variation of the strands polarization suggests the first of n^{th} correspondences for cohomology, a notion here used for expressing intersemiotic continuum (i.e. parallel universes of signs reciprocally intertwined in degrees of translatable signification).*

A subsequent concept is *intersemiotic synecdoche*,[9] a powerful tool for the science of signs, which main ability is nothing but *jumping from a sign domain to another one*, preserving a *key component* that allows a synecdochic discontinuum (see Figure 3), however consistent and coherent within the network of *intersemiotic continuity* (with economical virtues in comparison to Figure 2).

[9] A notion also introduced in *On Musical Self-Similarity*. See [14, pp. 101–102, 110–111].

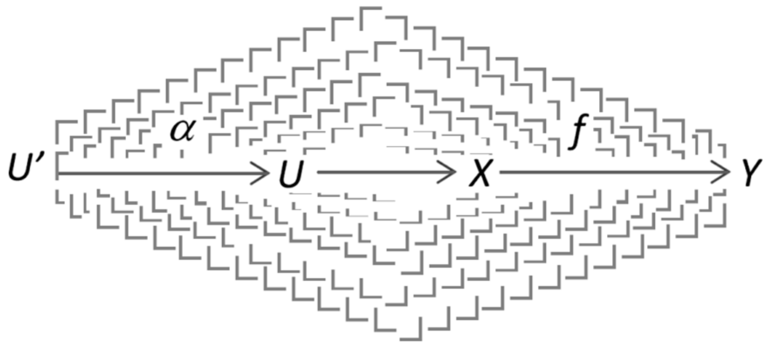

Figure 3: *An extension and reinterpretation of the previous scheme, with* Γ *shape segments used to represent non-direct connectivity of intersemiotic (dis)continuity, a segmented lattice that represents a plot of intersemiotic synecdoche. Self-overlapping is also a feature to be* n^{th} *polarized-varying.*

The figure of "jumping" in this symbolic space is referred to any case where, according to the given example, a pitch interval $[C, A] = [261.63, 440] = [1:2, 3:5] = [120:100] = [white, yellow]$, and then a different color interval, $[purple, green]$ would correspond to $[\Gamma_\alpha]_X \rightarrow [\Gamma_\beta]_Y \rightarrow [\Gamma_\gamma]_Z$, and so on, without symbolic gaps and where coherency is defined by the properties of a topological space.

II. A Heraclitan point of listening (philosophical discussion)

During the 4th International Congress of Music and Mathematics (UFRJ, October 2019) I offered a couple of lectures,[10] both philosophically centered on the problem of Western musicology as an axiological tradition. A common topic of both lectures was discussing how this axiology contributed to bias and obscure music theory; especially in musical cultures different to the Western European "classical" one and its derivatives. Then I recalled Heraclitus legacy to suggest that such axiology was not that relevant in Greek pre-Parmenidean times, although Platonism gained the political battle for axiologically depict "good" and "bad" musical ideas and practices, with specific social usages and cultural implications.

How the Yoneda lemma and Heraclitus philosophy are necessarily related, is a component that I could not clarify enough during the meeting; a task to be accomplished here. Let's simplify the case as follows: the Heraclitus motto *panta rhei*, "everything flows", also known by the metaphor "No man ever steps in the same river twice", has deep semio-epistemic implications. Namely, if the *river* (or the *world*) does constantly change and never "is the same" (to itself), then the world is impossible (or at least its communication). This equals Frege's paradox of lack of consistency between object and concept. However, the Yoneda lemma facilitates solving the paradox under the assumption that *any object is the coordinative sum of all its necessary definitions and even of all its truthful definitions*.

The "problem" with the latter is, then, that no any dictionary or grammar could hope to rule natural language. But languages, as indeed also happens with music, are much more complex than the highest complexity of functionalizing lexicons and syntax. Usefully, what Yoneda lemma teaches us on this, is rather that such a "problem" does not exist; and our mechanical-analytic

[10] "Non-Linear Approach to Popular Musicalities through a Non-Western Mathematical Understanding of Rhythm, Intonation and Locally-Generated Harmony" (Oct. 23th, 2019) & "Peircean Mathematics for Musicology: Transcending Heraclitus Melancholy through Harmonic Synechism" (Oct. 25th, 2019).

focus on languages was either adequate to deal with semiotics of music from that starting point.

Under this clarification, we may assess the relationship of an object and its conceptualization, as the self-construction of *conceptual clouds* (i.e. cumuli of concepts) that actually allows us to identify the object; even when the manifold construction of these clouds may partially fall into local and apparent contradiction (e.g. the object as a "cloud of meanings" and simultaneously as concrete specificity). These contradictions, logically harmless ones, may even be described as extremely far allocated from each other, however making part of a same wholeness of sense. Then the notion of *polar semiotics* enters the scene, connecting Category theory to musical semiotics.

In this context, I employ the conceptual "polarization" between Euclidean principles and the Heraclitan philosophy with the aim of unveiling a hidden continuum between these foci, presumably opposite as models of thought and imagination. A good start would be the *point* itself as undefined or blurry defined by the Heraclitan model. Then I pose the question whether this human comprehension of the world, could rather be biased by our own way to build our models of comprehension (a conceptual hesitation that actually modern physics do reflect in its manner, along the discussion of *particle and/or wave* as a correct characterization for *dynamical systems/objects* behavior). This blurry opposition between contraries and somehow complementary semiotics are studied below under the proposed label of *polar semiotics*.[11]

Contrary to what is established by the Euclidean *Elements*, Heraclitus did not emphasize a definition of point and straight line, as key concepts before any arithmetic, geometrical or algebraic development. In contrast, Heraclitus conceives an absolute impermanence for any state of mind or matter, as the crucial knowledge for philosophy and mathematical truth–a non-Euclidean one. This conceptualization, less intuitive than the stability of the point extended as a pure line, straight or curve, was condemned by a cultural system that needed to satisfy certainty in order to preclude ambiguity and to banish any doubt on the aesthetic, religious and political establishment of the Greek republics.

The fundamental concept of the point as "that which has no part" is analogous to the Democritan atom as "that which has no partition", and these is closely related to the logical definition of unity and nothingness, which, in their turn, are the fundamental "bricks" for the building of Euclidean space and commensurability. Thus, in one hand, we gain a logically deterministic axiomatics, and an absolutist notion of mathematical truth and certainty. But, on the other hand, we lose a charming intuition of complexity, irregularity, uncertainty and impermanence, fundamental to assess our notions of probability. We gained a perfect geometry Platonic paradise in our minds, but we lose focus for understanding what happens in our bodies and ecologies.

When Roman politicians understood the net weight of this conceptual unity, Constantine The Great adopted monotheism and created the most powerful empire of the globe, weaving a religious, military and political network which had a decisive impact in what we now name as Western culture. Then the concepts of purity of the regular polygons and Platonic solids became politically useful, and the indivisible point, the pure line, straight or curve, the circle and noticeably the sphere, became symbols for political and religious usage, to the extent that the Christian church, monopolized by Rome, identified the sphere, the circle and the point, with the absolute image of God.

Thinkers like Giordano Bruno and scientists like Galileo suffered serious threats for questioning the supposed purity of the Euclidean-Aristotelian geometry, and for doubting the geocentric model attributed to Ptolemy. The philosophical arguments for this symbolic association between moral, politics, aesthetics and music, were already formulated by Plato, definitely refusing the conception of Heraclitus, of a cosmos based on change and constant deviation under the concept of *clinamen*

[11] Although not yet formalized, the notion of *polar semiotics* is already traceable in [19, p. 29]: "In the web of nature, plants are, above all, producers [...] The polar opposites of plants are the funghi, nature's decomposers".

("the part that is partition itself").

The scholastic tradition of Platonic-Aristotelian influence, cursed Heraclitus ideas and he passed to be mockingly described as The Obscure or The Weeping Philosopher. Even to this day a Roman characterization of Heraclitus portrays him as–quoting Sotion of Alexandria–"overtaken by tears, [and] Democritus by laughter". Throughout the Middle Ages, Heraclitus ideas were also ridiculed, supposedly for being related to the Epicurean tradition that look for an equilibrium between pleasure and moderation.

Now, my purpose on revising this story on Greek and Roman philosophy, must be directed to the question of what happens if we remove Heraclitus' anathema, attempting to assess his philosophy from the understanding of modern Non-Euclidean mathematics, and which consequences would have this regarding music.

First of all, we have to confront Heraclitus most known argument on impermanence, towards the thought of Parmenides, his contemporary. Heraclitus is supposed to say that "No man ever steps in the same river twice". Parmenides answers with a fully deterministic logical sentence: "Whatever is, is, and what is not cannot be". However we notice, from a logical rigorous perspective, that this Parmenidean answer has no falsifiability. In fact, it works as a tautology and redundant self-consistency, not because of a lack of axiomatic truth, but because of triviality.

A similar judgment can be applied confronting Heraclitus' clinamen to the Euclidean tautological depiction of the point as "that which has no part", also related to the Platonic-Aristotelian conception of *purity* as a necessary characteristic of the point, the circle and the sphere. Perhaps the consequences of defending Heraclitus conceptions on the impermanency of a non-Euclidean point, line, atom and space would change our understanding on the sense and intuition of space, distance, shape and time.

A necessary step back in this way, would be revising the ancient Greek concept of "chaos", because of the relevance of this notion in Pre-Socratic mathematics. A glimpse into the most studied ancient sources, allows us to identify the concept of chaos in an Epicurean interpretation, as "the first state of the universe", and in Lucretius as the "unformed matter". No surprise. In contrast, the Pythagorean interpretation of chaos as "the name for [number] one" is striking. This last conceptualization leads us to think on Heraclitus impermanence of the unity. Indeed, this thesis was defended by the Epicureans, and by the followers of Lucretius' *De rerum natura*, a famous epicurean treatise on existentialism, ethics and aesthetics.

Up to this point of history, or rather up to this history of the point, there is no mathematical proof for any of Heraclitus postulates on spatial and numerical impermanence. If so, a proof may come from recent times, although through an unexpected way, after the computational discovery of deterministic chaotic behavior from simple non-linear dynamical equations, such as the logistic map: a polynomial mapping which chaos strongly depends on the initial conditions for the function iteration, originally designed by Robert May (1936–2020) with the purpose of modeling demographic growing (a property also observable, of course, within many other chaotic non-linear mappings). Next, third and closing section, is yet far to systematize this feature for musical definitions. Nevertheless, the actual discussion is whether this kind of polar semiotics may lead musicology to its own finding for a possible systematization on this route.

III. Polar Semiotics: Edges of Parallel that Meet at Infinity

Whether an aphorism can logically be synthesized as an axiom and further sprout into a morphism (a structure-preserving map from one abstraction to another analogous one), the *intersemiotic continuum* described at the first section of this text proposes the Yoneda embedding for connecting any categorical subtleties to this morphism (see Figure 4).

The following experimental, enunciative exercise consists of a search for a common lineage bonding music theory and mathematics (or poetry and philosophy), where [よ] does represent a Yoneda subset (a segment of the Yoneda embedding), both as a musical definition and an abstract proposition, "polarized" between the Euclidean tradition (here symbolized $\mathfrak{M}_a$) and its Heraclitan "complement" ($\mathfrak{M}_b$, where the letter $\mathfrak{M}$ stands for Matthai logic based on the postulate "All thoughts are truthful"). The symbol $\circ$ denotes information-thermodynamic equilibrium (*cf.* [22, 20, 5]). Following examples are provided on purpose from sources that somehow make evident their membership to a semiotic polarization, nevertheless comprised by a Yoneda embedding (formally represented by symbol よ) to be built as a morphism.

Infinity of poles may constitute the fuzzy edges of intersemiosis composing a simple object embedded in a Yoneda complex. The following examples are just few cases of simplification, useful for music theory revision and its possible transformation within this context. In the left column a postulate is posed, following the Euclidean tradition both in formal and imaginary qualities. The right column, instead, shows postulates rather empathetic to Heraclitus attributed thought.

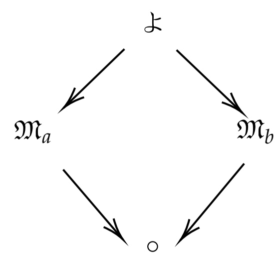

Figure 4: *Flux map for the diagram below (following pages): a morphism model for a Yoneda cycle representing equilibrium between to semiotic "poles" ($\mathfrak{M}_a \leftrightarrow \mathfrak{M}_b$). Symbol よ (katakana kana Yo) stands for Yoneda; symbol $\mathfrak{M}$ for Matthai election (i.e. at least a partial, yet context-coherent validation of his postulate), and symbol $\circ$ for information-thermodynamic equilibrium between the semiotic poles (left – right).*

Point

[よ$_1$]

Euclidean 1 [$\mathfrak{M}_a$]

"The geometric point is an invisible thing. Therefore, it must be defined as an incorporeal thing. Considered in terms of substance, it equals zero."

[8, p. 19]

Heraclitan 1 [$\mathfrak{M}_b$]

"In a basic two-dimension carbon structure, straight lines may represent precondition for the Euclidean straight line intuition, and their vertices represent precondition for the Euclidean point."

[16, p. 223]

$\circ$

Line

[⅄$_2$]

Euclidean 1 $[\mathfrak{M}_a]$

"The geometric line is an invisible thing. It is the track made by the moving point; that is, its product. It is created by movement-specifically through the destruction of the intense self-contained repose of the point."

[8, p. 51]

Heraclitan 1 $[\mathfrak{M}_b]$

"The border zone is the reality, and the dividing line, the abstraction of it [...] [by its polarization] is the membrane that separates two media of different density"

F. Ratzel (*apud* [10, p. 182])

○

Rhythm

[⅄$_3$]

Euclidean 1 $[\mathfrak{M}_a]$

"Repetition is a potent means of heightening the inner vibration and is, at the same time, a source of elementary rhythm which, in turn, is a means to the attainment of elementary harmony in every form of art."

[8, p. 19]

Heraclitan 1 $[\mathfrak{M}_b]$

"There is no space and time, but space-times in which natural phenomena and human events sink, impregnating themselves with the qualities of each place and each moment."

[21, p. 174–175]
(also quoted in [10, p. 333])

○

Sound

[⅄$_4$]

Euclidean 1 $[\mathfrak{M}_a]$

[The symbol has an] "inner sound [...] The point belongs to the more confined circle of habitual everyday phenomena with its traditional sound, which is mute".

[8, p. 19]

Heraclitan 1 $[\mathfrak{M}_b]$

"We study the ear to know the nature of sound and the nose to know the nature of odors. When Heraclitus said that he searched for himself, he knew that to know the cosmos he must know himself because he was an exact microcosm of that cosmos."

[7, p. 146]

○

Melody

[⅄$_5$]

Euclidean 1 $[\mathfrak{M}_a]$

"Succession of sounds somehow ordered according to the laws of rhythm and modulation, that it forms a pleasant sense to the ear".

[18, p. 421]

Heraclitan 1 $[\mathfrak{M}_b]$

"Think that everything sings [...] bathe your melancholy in the rhyme of the falling water, and you will understand many things that are hidden from you today and then you will know this: that music is the soul of the Universe"

[13, p. 798]

○

Harmony

[よ$_6$]

Euclidean 1 [$\mathfrak{M}_a$]

"[Let's an] inventory:
Elements: two points + plane.
Result:

1. the inner sound of a point,
2. repetition of the sound,
3. double sound of the first point,
4. double sound of the second point,
5. sound of the sum of all these sounds".

[8, p. 32]

Heraclitan 1 [$\mathfrak{M}_b$]

"These two poles are so far apart from each other that between them there is room for a whole world of sensations of infinite beauty [...] When trees resemble waterfalls of color! [...] It also fits between one pole of beauty and another."

[1, p. 101]

○

Duration

[よ$_7$]

Euclidean 8 [$\mathfrak{M}_a$]

"The notion of *interval of duration* would be the product of a measurement carried out between two points located on different moments of the temporal evolution of our [spatio-temporal] macrospectrum."

[6, p. 133]

Heraclitan 8 [$\mathfrak{M}_b$]

"In the infinite duration, the hour is indistinguishable from the day; the day of the year; the year of the century, the century of the moment; because moments and hours are not more numerous than centuries, and the proportion of some is not less than that of others"

Giordano Bruno (*De la causa, principio, et uno*, 1584; *apud* [10, p. 286])

○

Composition

[よ$_8$]

Euclidean 1 [$\mathfrak{M}_a$]

"A composition is nothing other than an exact law-abiding organization of the vital forces which, in the form of tensions, are shut up within the [spatio-temporal] elements"

[8, p. 86]

Heraclitan 1 [$\mathfrak{M}_b$]

"What, in fact, is the compositional field? Should it be characterized as a theoretical field, a field of practices, a field of studies, of learning, of creations, of all this together? How is it internally articulated? Is there a discernible outline–definition, borders, intersections, internal segmentations (parts), gradations of difficulty, lines of connection between what is elementary and what is advanced? Can it be thought of as a static set, or necessarily as a process? What kind of process?"

[3, p. 9]

○

↓

n^{th} intersemiotic [$\mathfrak{M}_a \leftrightarrow \mathfrak{M}_b$] continuity
(Euclidean – Heraclitan dissipative self-transformation).

IV. Conclusion

In a recent past the academy and its formalized knowledge disapproved any interpretation of musical concepts that did not adhere to the orthodoxy modeled by the rational principles celebrated by Enlightenment and Positivism. Now we see that notions so apparently contradictory or dissociated from each other can actually be part of a rich totality, benefiting our manifold understanding of music.

The presented conceptualization for polar semiotics may contribute to studying music as the body/mind–individual/social thermodynamic equilibrium that Ilya Prigogine (1917–2003) used to identify as *dissipative structures*; in this particular case, epistemes (at least musical ones) whose intertwined relationships can be observed through the Yoneda method.

The more important a concept does appear in the human mind and in social history, the more heterogeneousness of thoughts around it. This, which is a "problem" for classical philosophy,[12] is the greatest virtue for the concepts of philosophy behind polar semiotics. Yoneda lemma facilitates this recognition for philosophy in general, and for musicology in particular.

For the provided examples in the third section of this essay, I could not omit the historical skilfulness of Kandinsky to transform the Euclidean postulates and to synthesize human feelings and reasoning translatable into a helpful intermediation that allows us to communicate among our mind polarizations. May this help to a common desired harmonization of our critical moment as a self-questioning society.

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Time and Reversal in Birtwistle's *Punch and Judy*

Robert W. Peck
Louisiana State University
rpeck@lsu.edu
Orcid: 0000-0002-6226-1631
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Abstract: *We examine the occurrence of peripeteia in Harrison Birtwistle's 1967 opera* Punch and Judy*, as manifest in a reversal of cyclic time. Specifically, we extend a metaphorical association between the passage of cyclic time in the opera and discrete rotation in the complex plane generated by the imaginary unit* i*. Such a rotation moves alternately between the real and the imaginary axes, as scenes in the opera pass correspondingly through sacred and profane orientations. The instance of peripeteia results in a counter rotation, a dramaturgical inversion. To bring this reversal into the metaphor, we extend it from its situation in the complex plane to one in the space of Hamilton's quaternions, wherein such negation is obtained through the product of upper-level imaginary units. The scene that contains the reversal and that which consists of the opera's comic resolution epitomize the drama and occupy the highest level of dramatic structure.*

Keywords: *Birtwistle. Inversion. Peripeteia. Complex numbers. Quaternions.*

I. Introduction

The operation of inversion is of central importance in many music theories, dating at least to Guido d'Arezzo, who, in the eleventh century, likened melodic inversion to a reflection: "Note that when a neume traverses a certain range or contour by leaping down from high notes, another neume may respond similarly in an opposite direction from low notes, *as happens when we look for our likeness confronting us in a well*" ([9, emphasis added]).[1] Like Guido, modern music theories have generally regarded inversion as a reflection (e.g., pitch-class inversion is often depicted as a reflection through an axis on a mod-12 clock face). More recently, Guerino Mazzola [16, p. 44] describes inversion not as a reflection, but in terms of a gesture that is a 180° rotation in the complex plane. Mazzola depicts this gesture as "leafing" (as in turning a page in a book). He writes, "Leafing turns the original figure to its mirrored version. The point is that instead of mirroring x to $-x$...lift it into a new dimension and rotate the point until it comes down to $-x$." In regard to the complex plane, such a gesture passes through an imaginary dimension.

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[1] In the original text: "*Uel lineam neuma una facit saliendo ab acutis. Talem altera inclinata eregione opponat. Respondendo a grauibus. sicut fit cum in puteo nos imaginem nostram contra spectamus.*"

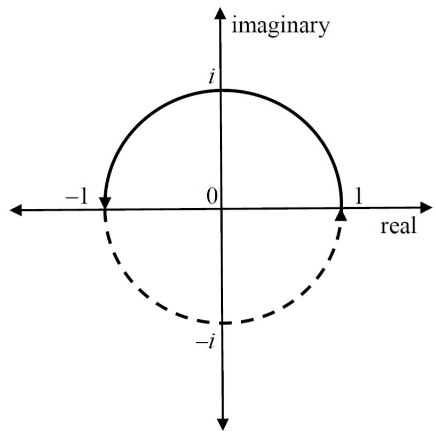

Figure 1: *Inversion as a half rotation in the complex plane.*

Mazzola's gestural interpretation situates inversion—or, more generally, negation—in the context of a cycle. Figure 1 illustrates this process: starting at the unit 1 on the real axis, the cycle passes through the imaginary unit i on its way to the negative unit -1, therewith obtaining inversion (negation). It continues through the negative imaginary unit $-i$, returning to 1 and completing the cycle (thus negating the negation). Hence, i is a square root of -1, which is in turn the square root of 1. Similarly, $-i$ is also a square root of -1. It generates the reverse [inverse] cycle of i. Such cycles may be continuous, as suggested by the circular path in Figure 1, or discrete, consisting of a finite number of points (e.g., 1, i, -1, and $-i$).

One may accordingly model musical inversions in terms of cycles, either continuous or discrete.[2] By not limiting ourselves to operations in pitch and pitch-class spaces, a more general interpretation of inversion as a process of negation yields a richer set of musical experiences for analysis. One such approach applies a cyclic interpretation of inversion to a dramaturgical context, specifically to that of Aristotle's concept of *peripeteia* (reversal of a situation). In *The Poetics*, Aristotle [2, p. 72] describes peripeteia as an integral part of tragic plot. "Peripeteia is a change by which the action veers round to its opposite, subject always to our rule of probability or necessity,"[3] the quintessential example's being Oedipus's reversal of fortunes in *Oedipus Rex*. Peripeteia, Aristotle notes, is attended by *anagnorisis* (or, recognition): "Recognition, as the name indicates, is a change from ignorance to knowledge, producing love or hate between the persons destined by the poet for good or bad fortune."[4] In Sophocles' play, it is Oedipus's recognition that his wife, Jocasta, is his own mother that precipitates the peripeteia.

In this study, we examine peripeteia as manifest in Harrison Birtwistle's 1967 opera *Punch and Judy*, with libretto by Stephen Pruslin,[5] in terms of a metaphorical association with the processes of negation described above. The opera—"a tragical comedy or a comical tragedy"[6]—is replete with cycles: cycles of seasons and zodiacal signs, times of day and hours on the clock,

[2]For instance, [18] studies pitch-class inversion in terms of discrete cycles.

[3]"Ἔστι δὲ περιπέτεια μὲν ἡ εἰς τὸ ἐναντίον τῶν πραττομένων μεταβολὴ καθάπερ εἴρηται, καὶ τοῦτο δὲ ὥσπερ λέγομεν κατὰ τὸ εἰκὸς ἢ ἀναγκαῖον."

[4]"Ἀναγνώρισις δέ, ὥσπερ καὶ τοὔνομα σημαίνει, ἐξ ἀγνοίας εἰς γνῶσιν μεταβολή, ἢ εἰς φιλίαν ἢ εἰς ἔχθραν, τῶν πρὸς εὐτυχίαν ἢ δυστυχίαν ὡρισμένων· καλλίστη δὲ ἀναγνώρισις."

[5]At the time of *Punch and Judy*'s composition, Pruslin was pianist in Birtwistle's and Peter Maxwell Davies's new-music ensemble The Fires of London.

[6]The opera's subtitle, "a tragical comedy or a comical tragedy," is adapted from the title to an 1828 script by John Payne Collier: *The Tragical Comedy or Comical Tragedy of Punch and Judy*, illustrated by George Cruikshank [5].

compass directions, colors on a color wheel, and so on. It also features a large-scale instance of dramaturgical inversion. The character Punch undergoes an Ebenezer Scrooge-like transformation from villain to hero—or, perhaps Punch remains essentially the same, and it is the world he inhabits that becomes so-inverted.[7]

Birtwistle's music incorporates a number themes, many of which are in evidence in *Punch and Judy*. Several authors have discussed the roles of dramaturgy, symmetry, cyclic structures, spatial imagery (motion), geometrical interpretation, ritual and sacrifice, and order and randomness throughout his oeuvre [12], [7], [8], [1], [3], [4]. Among these topics, the treatment of time—especially regarding the recurrence of time and the reversal of time—is particularly salient. Among Birtwistle's works that engage significantly with aspects of time are *Refrains and Choruses* (1957), *Précis* (1960), *Tragoedia* (1965), *Chronometer* (1971–72), *The Triumph of Time* (1972), *The Mask of Orpheus* (1983), *Pulse Field* (1977), *Bach Measures* (1996), *Pulse Shadows* (1996), and *Exody '23:59:59'* (1997).

II. *Punch and Judy*

Punch and Judy is a one-act opera of approximately 100 minutes. It is based loosely on the popular glove-puppet shows, which have remained a vital part of British seaside culture since at least the eighteenth century. These puppet dramas feature the antics of the principal character, Mr. Punch, in his (usually violent) interactions with various other puppets. In the opera, the principal characters are as follows:

- Punch, a puppet, and an infantile, sadistic mass murderer.
- His wife Judy, who is Punch's first murder victim. She appears later in the opera in the guise of a Fortune Teller.
- The pair of characters Doctor and Lawyer, who always appear together. The original Punch and Judy puppet shows occasionally included a lesser character named The Doctor; the Lawyer here seems to be Birtwistle and Pruslin's invention.
- Pretty Polly, a minor character and Punch's mistress in the classic Punch and Judy puppet dramas, and the object of Punch's infatuation in the opera. She also appears as a Witch in one scene.
- Perhaps most significantly, a character named Choregos, a Greek term that refers to a choral impresario in Athenian theatre. In *Punch and Judy*, Choregos fills the role of the Greek chorus; he is also the one character in the opera who is not a puppet. Choregos also appears later in the opera as Jack Ketch—a lesser character from the original Punch and Judy puppet dramas, patterned on a real-life, infamously brutal executioner in England in the seventeenth century by the same name. Jack Ketch's role in the opera is ambiguous: is he a villain, an instrument of the law, or both?

By including the character of Choregos, Birtwistle and Pruslin appear to be interpreting Aristotle's word in *The Poetics* literally: "The Chorus too should be regarded as one of the actors; it should be an integral part of the whole, and share in the action, in the manner not of Euripides but of Sophocles" [2, p. 92].[8]

[7] Another interesting example of inversion in the context of puppet drama appears at the end of Heinrich Kleist's "On the Marionette Theatre" [14]. In an allegory for deification (transformation from puppet to god) at the end of this essay, Kleist describes the process of inversion that occurs as one approaches a concave mirror: the reflected image appears inverted at a distance, stretches into infinity at the focal point, and reappears upright up at a closer distance.

[8] "Καὶ τὸν χορὸν δὲ ἕνα δεῖ ὑπολαμβάνειν τῶν ὑποκριτῶν, καὶ μόριον εἶναι τοῦ ὅλου καὶ συναγωνίζεσθαι μὴ ὥσπερ Εὐριπίδη ἀλλ᾽ ὥσπερ Σοφοκλεῖ."

The plot of the opera derives only loosely from the original puppet dramas, which were not scripted, but often formulaic.[9]. It might be summarized as follows:

- Punch throws his baby into fire and murders Judy.
- Punch begins stalking Pretty Polly.
- Punch murders Doctor and Lawyer.
- Punch continues stalking Pretty Polly.
- Punch murders Choregos.
- Punch has a nightmare in which all the characters he has murdered thus far intend to murder him.
- Punch awakens and resumes stalking Pretty Polly.
- Punch murders Jack Ketch, the evil alter ego of Choregos.
- Punch wins Pretty Polly, and they live happily ever after.

As we will see, the superficial aspects of the plot as described here are only part of the story that Birtwistle and Pruslin are telling. Friedrich Nietzsche writes in *The Birth of Tragedy*: "Everything which comes to the surface in the Apollonian part of Greek tragedy, in the dialogue, looks simple, translucent, beautiful"[10] [17, p. 67]. Another dimension to this opera exists—the Dionysian part, as typically characterized by the chorus in Greek drama—that is even more significant. As Nietzsche says: "We must understand Greek tragedy as the Dionysian chorus which over and over again discharges itself in an Apollonian world of images" [17, p. 58]. He continues, "Those choral passages interspersed through tragedy are thus, as it were, the maternal womb of the entire dialogue so-called, that is, of the totality of the stage world, the actual drama."[11] Pruslin writes: "[*Punch and Judy*] is an opera in quotation marks" [19, p. 7]. As such, the real drama of *Punch and Judy* takes place not on the stage, but rather in the opera's dramaturgical context.

Literary theorist Gabriel Josipovici describes *Punch and Judy* as "ancient Greek drama in the guise of popular puppetry" [13]. Jonathan Cross explores the ancient Greek basis for *Punch and Judy* in greater detail, including its incorporation of peripeteia as a structural device [7, 8]. Figure 2 is reproduced from [7].[12] Specifically, reading left-to-right and top-to-bottom, it demonstrates how the opera contains a number of passes through various recurring scenes: Melodrama, Passion Chorale, and Quest for Pretty Polly. The scenes appear in that order in the initial iterations, but their sequence inverts in the final iteration, in which Punch's fortunes also reverse. Cross gives the Nightmare scene as the work's instance of structural peripeteia. Cross goes further to suggest that the moment of anagnorisis that occasions this peripeteia is Punch's recognition in the nightmare of Judy—whom he has purportedly already murdered—disguised as a Fortune Teller. "At the height of the tarot game Punch literally recognizes Judy—she reveals herself; she is unmasked" [8, p. 76].

British conductor and music critic Michael Hall offers another account [12], which is somewhat more in keeping with the dramaturgical basis we are establishing here. Hall invokes Nietzsche's dualism between Apollonian and Dionysian forces in relation to the characters of Punch and Choregos. Nietzsche writes:

> Let us think about our own surprise at, and unease with, the chorus and the tragic hero of those tragedies, both of which we did not know how to reconcile with what we

[9]Collier presents an account of a typical Punch and Judy show "[a]s told to John Payne Collier by Giovanni Piccini in 1827"[5]

[10]"*Alles, was im apollinischen Theile der griechischen Tragödie, im Dialoge, auf die Oberfläche kommt, sieht einfach, durchsichtig, schön aus.*"

[11]"*Nach dieser Erkenntniss haben wir die griechische Tragödie als den dionysischen Chor zu verstehen, der sich immer von neuem wieder in einer apollinischen Bilderwelt entladet. Jene Chorpartien, mit denen die Tragödie durchflochten ist, sind also gewissermaassen der Mutterschooss des ganzen sogenannten Dialogs d.h. der gesammten Bühnenwelt, des eigentlichen Dramas.*"

[12]Our Figure 2 appears as Figure 1 in [7].

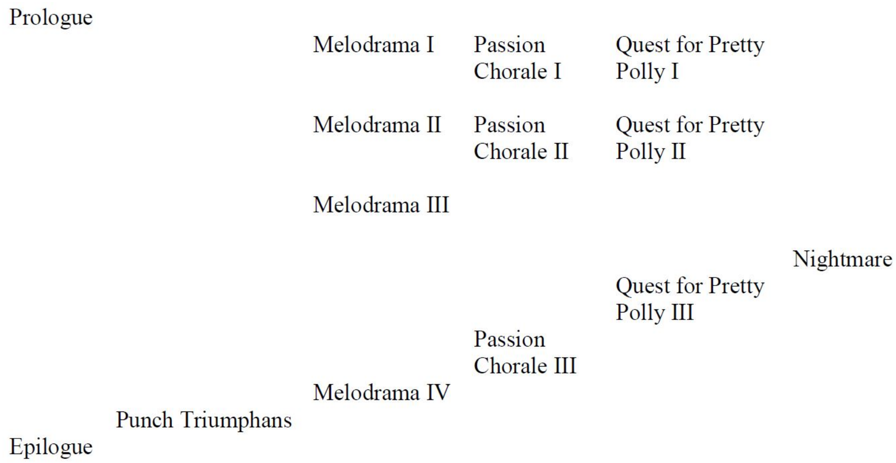

Figure 2: *Punch and Judy, Overall Design [7, p. 204].*

> are used to, any more than with the tradition—until we again recognized that duality itself as the origin and essence of Greek tragedy, as the expression of two artistic drives woven together, the Apollonian and the Dionysian.[13] [17, p. 75]

According to Hall, Punch, the tragic hero, represents Apollonian individualism, whereas the chorus, embodied in the character of Choregos, represents Dionysian universalism. Just prior to his nightmare, Punch murders Choregos, which precipitates a nightmare; then, in the Nightmare scene, he envisions his own execution at the hands of all the characters he has murdered to that point. Hall suggests that Punch's killing of Choregos disrupts in a Jungian sense the balance of his own ego and the collective unconscious. To restore the balance, Punch *recognizes* unconsciously that he too must die, bringing on the substance of his nightmare. In Hall's interpretation, as in Cross's, the dream is the locus of the anagnorisis, the recognition that results in the peripeteia. For Hall, however, the peripeteia is manifest merely in terms of the plot—for instance, in the reversal of Punch's fortunes with Pretty Polly—not in the ordering of scenes.

Our reading here combines certain aspects of Cross's and Hall's analyses: in particular, Cross's idea that peripeteia has a structural manifestation with regard to the order of scenes, and Hall's Jungian notion that anagnorisis is associated with the murder of Choregos and Punch's resultant nightmare. However, we find a further, deep connection between the two aspects beyond those that these authors address.

III. Time in *Punch and Judy*

The repetitive, cyclic nature of Birtwistle's score, and particularly of Pruslin's libretto, situates the drama in what we might call *time-outside-time*. The passage of time in the opera is cyclical rather than linear, following solar, lunar, and similar patterns; plot events and their musical settings recur correspondingly. The Romanian anthropologist and historian of world religions Mircea Eliade likens such natural cycles to repetitions of the cosmogonic act, the original act that gave birth

[13] *"Denken wir an unsere eigene Befremdung dem Chore und dem tragischen Helden jener Tragödie gegenüber, die wir beide mit unseren Gewohnheiten ebensowenig wie mit der Ueberlieferung zu reimen wussten - bis wir jene Doppelheit selbst als Ursprung und Wesen der griechischen Tragödie wiederfanden, als den Ausdruck zweier in einander gewobenen Kunsttriebe, des Apollinischen und des Dionysischen."*

to the world, to the cosmos. "The creation of the world, then, is reproduced every year" [11, p. 62]. Eliade refers to this conception of time-outside-time as "*in illo tempore*" (a Biblical Latin term, meaning "in that time")—the continually renewable and renewing time of ritual, of sacred acts. He writes, "Any ritual...unfolds not only in a consecrated space...but in a 'sacred time,' 'once upon a time' (*in illo tempore*, *ab origine*)" [11, p. 21].

Eliade links the ritual of sacrifice with the ritual of creation; both are embedded in a culture's collective memory [11, pp. 74–75]. Among numerous examples in world religions, he notes that the Judeo-Christian site Golgotha (meaning "place of the skull," and also known as Mount Calvary) is simultaneously the place where Adam was created and buried (the skull in "place of the skull" is Adam's) and the location of Christ's crucifixion on the Cross. For Eliade, it is an example of "The Sacred Mountain—where heaven and earth meet...situated at the center of the world" [11, p. 12]. This sort of association does not appear to have escaped Birtwistle and Pruslin: at the end of the opera, Jack Ketch's hangman's gallows is transformed into a maypole. On the one hand, we have, in effect, a sacrificial altar—on the other, a symbol of vernal rebirth. These two poles, then, describe Eliade's *axis mundi*, the Center of the world.

The passage of time in the opera is discrete in terms of its global organization. The successions of sections are not segued dramatically; one does not proceed seamlessly into another. In certain sections, such as in the Passion Chorales, time stands still: a "continual present," which "completely ignores what is especially characteristic and decisive in a consciousness of time" [11, p. 86]. Within other sections, as in the Melodramas and Quests for Pretty Polly, is a greater sense of a temporal continuum—these sections are driven by their own internal dramas, but they are not linked to their surrounding material. The overall affect is similar to that of Karlheinz Stockhausen's concept of Moment Form [20], particularly as described by Jonathan Kramer [15], except that the non-continuous arrangement of sections in *Punch and Judy* does have a very specific function.

In his two published analyses of the opera, Cross notes the ordered repetition of the sections Melodrama, Passion Chorale, and Quest for Pretty Polly, and their ultimate reversal. However, his discussion omits certain sections that serve as integral parts of the picture: specifically, sections that Edward T. Cone might describe as elements of the frame [6]. Figure 2 above includes the framing sections Prologue and Epilogue—incidentally, both sung by Choregos—but it does not include the following instrumental interludes: the Sinfonia between Quest for Pretty Polly I and Melodrama II, and the interlude, "A Little Canonic Prelude to Disaster," between Quest for Pretty Polly II and Melodrama III. Adding these sections into the succession of scenes, we have the following sequence (Figure 3).

The sung and instrumental framing sections have a function similar to that of the Center in Eliade's world view. They are sacred places of origination—time-outside-time; they are the domain of Choregos, of universalism. Likewise, the Passion Chorales exist on the *axis mundi*. They are also the domain of Choregos: they are sung by him and by the rest of the cast, save for Punch and Pretty Polly—the "Chorus," as is indicated in the score. Further, from an aesthetic point of view, Pruslin writes that the presence of the Passion Chorales in the work is an indication of "an overt debt to Bach," and to the St. Matthew Passion in particular [19, p. 7]. One might argue that this allusion to Christ's sacrifice situates the Passion Chorales on the same axis as the sacred, framing places of origination—the Cross and Adam's skull, respectively: the hangman's gallows and the maypole. Moreover, the Passion Chorales are themselves framed by instrumental toccatas, linking them further with the framing material.

Prologue	Melodrama I (murder of Judy)	Passion Chorale I	Quest for Pretty Polly I	
Sinfonia	Melodrama II (murder of Doctor & Lawyer)	Passion Chorale II	Quest for Pretty Polly II	
Interlude	Melodrama III (murder of Choregos)			Nightmare
			Quest for Pretty Polly III	
		Passion Chorale III		
	Melodrama IV (murder of Jack Ketch)			
Punch Triumphans & Epilogue				

Figure 3: *Sequence of scenes (modified from [7]).*

Let us now examine the texts of the first two Passion Chorales:

PASSION CHORALE I (Choregos and Chorus):
Day murdered fame one game lost
Dreamer dread flaming lust
Deforming lameness
Deaf or nameless
Demon dared
Dam-ned
Dumb

PASSION CHORALE II (Choregos and Chorus):
Two times too lost four her sake
Totem stool for hearse ache
Tempest quicksilver
Tempts evil hearts
Tambour vile
Tumbril
Tomb

Both these chorales exhibit a subtractive syllabic process: they begin with lines of seven syllables, and decrease line-by-line to one syllable (one with a rather negative connotation). The third Passion Chorale, following the peripeteia, is different: it features an additive process, building from one syllable (one with a very positive intention) to seven. Structurally and aesthetically, it is the inverse of the previous two.

PASSION CHORALE III (Choregos and Chorus):
Love
Live on
Liven leaf

Life heaven feel
Lost haven unveil
Lust ever unavail
Last eve arun adumbrate

The additive process of the third Passion Chorale resembles remarkably the structure of the Adding-Song, which serves as the climax of the Nightmare scene. In the Adding Song, Choregos and the Chorus—in a call-and-response manner—enumerate various "tricks" that they intend to "treat" on Punch, as reparations for the treats he has tricked on them. Beginning with a single trick, they add tricks progressively until they reach seven, the same number as syllables involved in the additive and subtractive processes of the Passion Chorales.

ADDING-SONG (from NIGHTMARE SCENE) (Choregos and Chorus):
a fractured skull
a fractured skull, a bleeding face
a fractured skull, a bleeding face, a severed limb
a fractured skull, a bleeding face, a severed limb, an oozing eye
a fractured skull, a bleeding face, a severed limb, an oozing eye, a twisted neck
a fractured skull, a bleeding face, a severed limb, an oozing eye, a twisted neck, a gangrene foot
a fractured skull, a bleeding face, a severed limb, an oozing eye, a twisted neck, a gangrene foot, a burning sore

These treats with which you tricked us
We'll now treat as tricks on you!

Just as the Passion Chorales are associated with sacrificial death, the Adding-Song has a sacrificial aspect: Punch must die to restore balance, to realign Apollonian and Dionysian forces. If the Passion Chorales exist on the *axis mundi*, it would appear that so too does the Adding-Song, and, by extension, the entire Nightmare scene it epitomizes. Figure 4 incorporates these further modifications to Cross's original figure.

Rather than in tabular form, we can represent the cyclic passage of time suggested above in another way (Figure 5), using as a metaphor the illustration of negation as a half-rotational gesture in the complex plane from Figure 1. The discrete cycle of the imaginary unit i describes four points, passing alternately through the real and imaginary axes. Now, replace the real axis with the *axis mundi*. Map the unit 1 to the framing material—i.e., to the origin, to the bed of creation. Map the negative unit -1, also on that axis, to the Passion Chorales (including the Adding-Song)—to ritual sacrifice, to the paradox of life from death. These points lie on the sacred axis, the axis of true and enduring reality, of time-outside-time. It is the domain of Choregos, the one "real" character, who is not a puppet.

Likewise, replace the imaginary axis with the profane axis. This is the illusory axis of day-to-day existence; it is the puppet's axis. Map the imaginary unit i to the Melodramas, and map the negative imaginary unit $-i$ to the Quests for Pretty Polly. The surface, Apollonian elements of the plot describe paths that appear to direct the drama through the profane dimension. They are driven initially by Punch's obsessive-compulsive needs: the drive to murder (in direct contrast to regenerative sacrifice), alternating with his lustful ambition for Pretty Polly (a mockery of nuptial sacrament). Following the peripeteia, everything veers round to its opposite, including

Axis mundi	Profane axis	*Axis mundi*	Profane axis
Prologue	Melodrama I (murder of Judy)	Passion Chorale I	Quest for Pretty Polly I
Sinfonia	Melodrama II (murder of Doctor & Lawyer)	Passion Chorale II	Quest for Pretty Polly II
Interlude	Melodrama III (murder of Choregos)	Nightmare (Adding Song)	
			Quest for Pretty Polly III
		Passion Chorale III	
	Melodrama IV (murder of Jack Ketch)		
Punch Triumphans & Epilogue			

Figure 4: *Sequence of scenes (modified from Figure 3).*

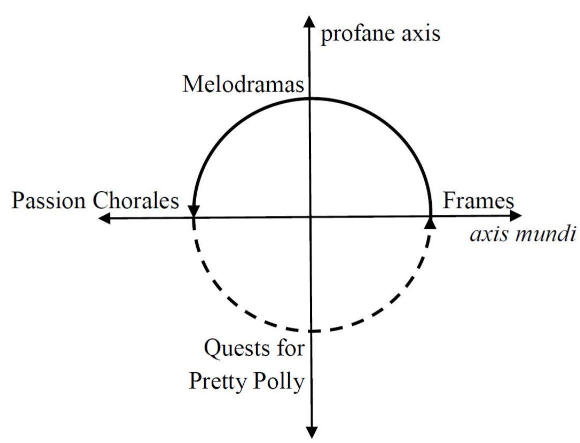

Figure 5: *Dramaturgical cycle.*

these seemingly ordered paths. The drive to murder becomes the need for justice, hence, Punch's execution of the villain Jack Ketch; and licentious pursuit becomes sincere courtship, with Punch's ultimately winning Pretty Polly.

IV. A Four-Dimensional Model

This metaphor, while assimilating the periodic passes through -1, does not yet address the hierarchical distinction between the Passion Chorales on the one hand, and the Adding-Song—which results in the peripeteia—on the other. Further, it does not account for the section Punch Triumphans, which appears in tandem with the final element of the frame, the Epilogue. Like the Adding-Song, Punch Triumphans seems to function on a higher architectonic layer. To bring these aspects into the metaphor, we add another level of structure that incorporates the space of the quaternions.

The quaternions, $\mathbb{H}$, discovered in 1843 by Irish mathematician William Rowan Hamilton, extend the idea of the two-dimensional complex numbers to a four-dimensional space: one real dimension and three imaginary dimensions. Their algebra is characterized by Hamilton's famous equation:

$$i^2 = j^2 = k^2 = ijk = -1 \tag{1}$$

in which i, j, and k are all imaginary units: they are all square roots of -1. (Further, each has its own unique negative, which is also a square root of -1.) As they lie in different dimensions from one another, the respective cycles of i, j, and k visit points on the circumferences of three mutually perpendicular circles that intersect in 1 and -1. Hence, the unit quaternions lie on the surface of a hypersphere. Further, the nodes of the discrete cycles generated by i, j, and k describe eight points on the hypersphere's surface: $\pm\, i$, $\pm\, j$, and $\pm\, k$ These eight points comprise a finite mathematical group, Q_8.

Each iteration of the dramaturgical cycle contains consecutive subsections titled "Travel Music," "Weather Report," and "Punch's Serenade." These subsections establish a temporal and spatial orientation for the iteration, as well as a section of a color wheel, such as appears in the following text of Travel Music I, Weather Report I, and Punch's Serenade I.

<u>TRAVEL MUSIC I</u>
Punch is seen traveling on Horsey in a picture frame on the Murder Altar.
Choregos (at his booth):
Suspended between Heaven and Earth.
Punch travels eastward to the land of eternal innocence.
Under the sign of the Crab.
Punch serenades his beloved on a shining summer afternoon.

<u>WEATHER REPORT I</u>
Punch descends and assumes a frozen serenade-pose at Pretty Polly's Pedestal.
Chorus:
3-o'clock east. 3-o'clock east.
A tempest swept by, then suddenly ceased.
3-o'clock east. 3-o'clock east.

<u>PUNCH'S SERENADE I</u>
A green spotlight reveals Pretty Polly dancing mechanically around her pedestal in disregard

of Punch. Toward the end of Punch's Serenade, Choregos approaches Pretty Polly as Punch's intermediary and offers her a huge sunflower.
Punch:
The world is blinded by lightening of green.
Silence and sounds and song of flaming green.
Greenness of sun and greenness of moon.
Green, how I long for you flaming in green.

Travel Music I, Weather Report I, and Punch's Serenade I situate the first iteration in the east (or in an eastward direction), under the astrological sign of the Crab (i.e., at 90° longitude), in the summer, at 3-o'clock, in the hue of green. The subsequent iterations of the dramatic cycle feature corresponding subsections with other coordinates, directions, and colors. It is as if the drama navigates a new, colored path in space/time in each of the iterations. We might say that these each of these paths visits a different imaginary (profane) dimension of our quaternion system: in dimension i, Punch murders Judy; in dimension j, he murders the Doctor and Lawyer pair; and in dimension k, he murders Choregos. In the reversed fourth iteration, he murders, or executes, Jack Ketch, Choregos's evil alter ego. This reversed iteration also moves through dimension k, but via the cycle of $-k$.

Let us call each of these moves into the respective imaginary dimensions a *hyper-operator* (as indicated with angle brackets, e.g., $\langle i \rangle$). Such hyper-operators are higher-level structures, middleground versions of the (imaginary) units that generate their constituent foreground cycles. They are not a feature of the algebra of the quaternion group; rather, they reproduce its structure on a higher architectonic layer. Essentially, a hyper-operator functions as a collective memory of the cycle's impetus, a myth of its purpose. The first few cycles are initiated by Punch's murders of various characters, by sinful motivations. Such collective memories become myths that ultimately require the expulsion of evil through sacrifice [11, pp. 74–75]. The final cycle is different: it leads to Punch's triumph over Jack Ketch, a heroic feat. These collective memories are linked with creation myths, where heroes (along with gods, etc.) are among the originators of celestial archetypes for profane activities [11, pp. 28–34].

In addition to the equivalence to -1 of the squares of the three imaginary units, Hamilton's equation provides an additional relation that characterizes the algebra of the quaternions: that the product of i, j, and k, in that order,[14] is equal to -1. The cycle of k in the third iteration is left incomplete; the Passion Chorale is not possible, because Choregos is now dead. Yet, we might say that the initiation of the cycle is sufficient to establish its trajectory, to imply its continuation. If so, we have an upper-level sequence of hyper-i, hyper-j, and hyper-$(-k)$, the quaternion product of which is hyper-(-1) (see Figure 6). It is this deep-middleground or background -1 that gives structural significance to the Nightmare scene, and in particular to the Adding-Song, which appears in the Passion Chorale's stead. However, this interpretation is predicated on the notion that the initiation of the cycle is sufficient to imply its completeness. To quote Ebenezer Scrooge [10, p. 70], "Are these the shadows of the things that Will be, or are they shadows of things that May be, only?" We next explore an alternate interpretation.

Consider the first two cycles in Figure 6, those that advance the drama into dimensions i and j, respectively. In quaternion algebra, the product of i and j (performed in that order) equals k. Accordingly, we may describe hyper-k as the product of hyper-i and hyper-j (Figure 7). Next, consider the final, reversed iteration, in which Punch executes Jack Ketch: this iteration

[14] To equal -1, the units i, j, and k must multiply in that order or in one of its cyclic permutations, jki or kij. Any other ordering, such as jik, equals 1, not –1, as $ji = -k$.

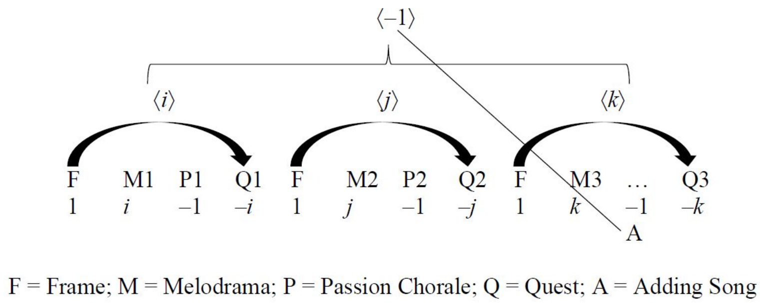

Figure 6: *$\langle i \rangle$, $\langle j \rangle$, and $\langle k \rangle$, and their product, $\langle -1 \rangle$.*

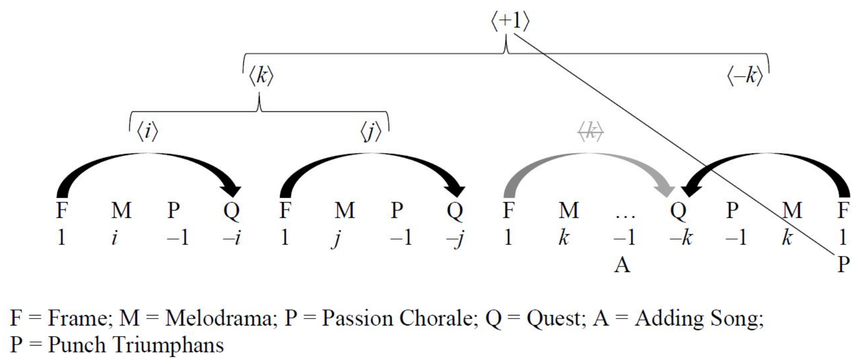

Figure 7: *Generation of hyper-$\langle +1 \rangle$.*

is generated by $-k$, the inverse of k. Taken together, the quaternion product of k and $-k$ equals positive 1 (i.e., if $k^2 = -1$, then $k \cdot -k = +1$). Likewise, the product of their hyper-operators equals hyper-$(+1)$. This hyper-$(+1)$, then, yields the framing section Punch Triumphans, which elevates this instance of the frame to the same upper level of structure as the Adding Song in the previous interpretation.

V. Conclusions

These two sections, then—the Adding Song and Punch Triumphans—embody the true spectacle of *Punch and Judy*. They are Nietzsche's "maternal womb of the entire dialogue...the actual drama." They exist on the *axis mundi*, on the real axis, the realm of "the Dionysian chorus which over and over again discharges itself in an Apollonian world of images." Nevertheless, the role of the chorus is not limited merely to such utterances. It, too, as Aristotle asserts, is a character, and contributes to the drama. Yet, its participation exists on a level higher than the surface plot, a level unaffected by the plot's cyclic time. Eliade writes, "The life of archaic man (a life reduced to repetition of archetypal acts...), although it takes place in time, does not bear the burden of time, does not record time's irreversibility" [11, p. 86]. Hence, time can change course on the *axis mundi*, in the Center.

Taken as a mathematical group, $Q_8 = \{1, i, j, k, -1, -i, -j, -k\}$ is noncommutative, meaning that the specific ordering of its elements as factors in a product is essential; for example, $ij \neq ji$. In our model, these elements represent factors in the dramatic structure—their ordering denotes the sequence of temporal events. Hence, i's happening before j is not equal to j's happening before i, suggesting that time in the metaphor is irreversible. In any mathematical group, however, there exists a subgroup that consists of the set of elements that commute with all members of the group. For the quaternion group Q_8, this subgroup consists of the two elements 1 and -1, the two points that lie on the real axis.

Above, we use these very points to represent the rituals of origin and sacrifice, as embodied in the framing material and the Passion Chorales—time-outside-time. “Through the paradox of rite, profane time and duration are suspended” [11, p. 53]. Further, each of these elements that is situated on the real axis commutes with those that appear in the various imaginary dimensions. For instance, $-1 \cdot -k = -k \cdot -1$, as in the elision of the end of the third iteration with the beginning of the fourth.

The consequences for this concept in our metaphor are significant: time, as an ordered sequence of temporal events, can “veer round to its opposite” in sacred time. Thus, it is not merely the presence of -1 in the Nightmare scene that engenders the peripeteia, but, more to the point, it is the situation of -1 in this particular subgroup. In mathematical group theory, such a subgroup is called the “center” of the group. The center of Q_8 is $\{+1, -1\}$; and, on that happy coincidence, “This comedy is at an end.”

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Voice Leading Among Pitch-Class Sets: Revisiting Allen Forte's Genera

PAULO DE TARSO SALLES
University of São Paulo (USP)
ptsalles@usp.br
Orcid: 0000-0001-7939-4521
DOI: 10.46926/musmat.2020v4n2.66-79

Abstract: *The theory of PC-set class genera by Allen Forte [5] was an important contribution to the understanding of similarity relations among PC sets within the tempered system. The growing interaction between the universes of PC-sets and transformational theories has explored the space between sets of the same or distinct cardinality, by means of voice-leading procedures. This paper intends to demonstrate Forte's method along with proposals by other authors like Morris [8], Parks [10], [9], [1], Straus [16], [15] Cohn [3], and Coelho de Souza [2]. Some analysis demonstrates such operations in passages picked from Heitor Villa-Lobos's works, like the Seventh String Quartet and the First Symphony.*

Keywords: *Genera. Voice leading. Similarity. Cardinality. Villa-Lobos.*

I. Introduction

Allen Forte's article, "Pitch-Class Set Genera and the Origins of Harmonic Species" [5], is a development of his previous theory on complexes of pitch-class sets [6, pp. 93–178]. Both are linked to a research field dealing with harmonic organization within the tempered system in the twentieth-century music, but its unfolding has taken different paths.).

Genera theory, as advanced by Allen Forte [5], proposes a kind of Darwinian parody on "harmonic species" built upon the smaller units, the trichords, and their division into "families". These families are Forte's idea to set a large-scale scheme of harmonic organization, displaying their homogeneity in post-tonal music contexts. In order to achieve a maximum coherence gathering those small units in increasingly large pc-sets, Forte adopted a system grounded on the interval vector similarity and inclusion among pc-sets.

Despite its cleverness, the theory of Genera presents some problems, whose solution is hard to find, maybe it's impossible. The most striking point is the inconsistence between what is told by similarity and inclusion against our perception. For instance, there is a considerable gap between what is not assigned as "tonal" according Forte's criteria and the real world of tonal music. Anyway, many other models have appeared with a similar end, like Parks [10], [9], Morris [8], Straus [16], [15], [14], Tymoczko [17], Cohn [3], and Coelho de Souza [2], among others.

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I have no ambition to design a model of my own. My intention is just to revisit some aspects of Forte's theory, along with some studies on voice leading developed in the last three decades. The fusion between the static model – proposed by Forte from the pc-sets and their interval vectors – with the dynamic model provided by transformational theory – based on voice leading – seems to be an interesting analytical approach to understand post-tonal harmony in some contexts. To fulfill this goal, I employ concepts often applied to art music in the first half of the twentieth century, such as cardinality change ([1], [15]) and voice leading ([8], [4], [16], [15], [2]). Thus, I intend to find a balance between theoretical consistence and perceptual data. However, the scope of this paper is concerned with some analytical insights, leaving a more comprehensive theory for a later time.

Villa-Lobos's music, with its complex layers of expanded tonal chords and lack of consistent functional progressions, offers an ideal opportunity to test the hypothesis on "harmonic species" and the transformational processes among them, like an enlarged concept of modulation. My methodology starts from the analyses of musical samples with transitional function between two well-defined tonal centers; chords and melody inside these passages are labeled and classified according to Forte's genera; it is followed by a study of the voice leading observed within the implied harmonic species.

II. Forte's Genera Theory

According to Forte, there are twelve genera, divided into four "simple" ones (based on a single trichord) and eight "complex" ones, which comes from two trichords sharing similar properties (Table 1).

Table 1: *Forte's twelve Genera (author's conception, after [5])*

Supra-Genus	Genus	Type	Trichord
	G1	Atonal	3-5
I	G2	Whole tone	3-8
	G3	Diminished	3-10
	G4	Augmented	3-12
II	G5	Chroma	3-1 & 3-2
	G6	Semi-Chroma	3-2 & 3-3
	G7	Chroma-Dia	3-2 & 3-7
	G8	Atonal	3-3 & 3-4
III	G9	Atonal-Tonal	3-3 & 3-11
	G10	Atonal-Tonal	3-4 & 3-11
IV	G11	Dia	3-7 & 3-9
	G12	Tonal	3-7 & 3-11

The Genera — except for G4 and G7 — are grouped into "Supra-Genera" in the following fashion: G1 + G2 + G3 = Supra-Genus I; G5 + G6 = Supra-Genus II; G8 + G9 + G10 = Supra-Genus III; and G11 + G12 = Supra-Genus IV.[1]

[1]Forte gathers genera into the supra genera, according to their common interval classes (ic) shared by their progenitor trichords. So, (012), (013), and (014) share ic 1 (half-tone), being grouped in Supra II. Supra I shares ic 6 (tritone); Supra III shares ic 4, and Supra IV shares ic 5.

III. A viewing from Genus 4: some properties

Forte sought to limit the number of supersets looking for the maximum consistence among their members, down to the basic trichords (progenitors). Thus, he eliminates all pc-sets whose complement is not a superset.

In Figure 1, Genus 4 is organized as a graph containing the 22 sets obtained from the augmented trichord (048), excluding those 11 ones with cardinality higher than 6. My graph has some additional features such as: the change of cardinality through SPLIT; and arrows indicating parsimonious movements between pc-sets. Straus [15, p. 56; 72] idealized a similar graph, displaying the offset from a chosen pc-set.

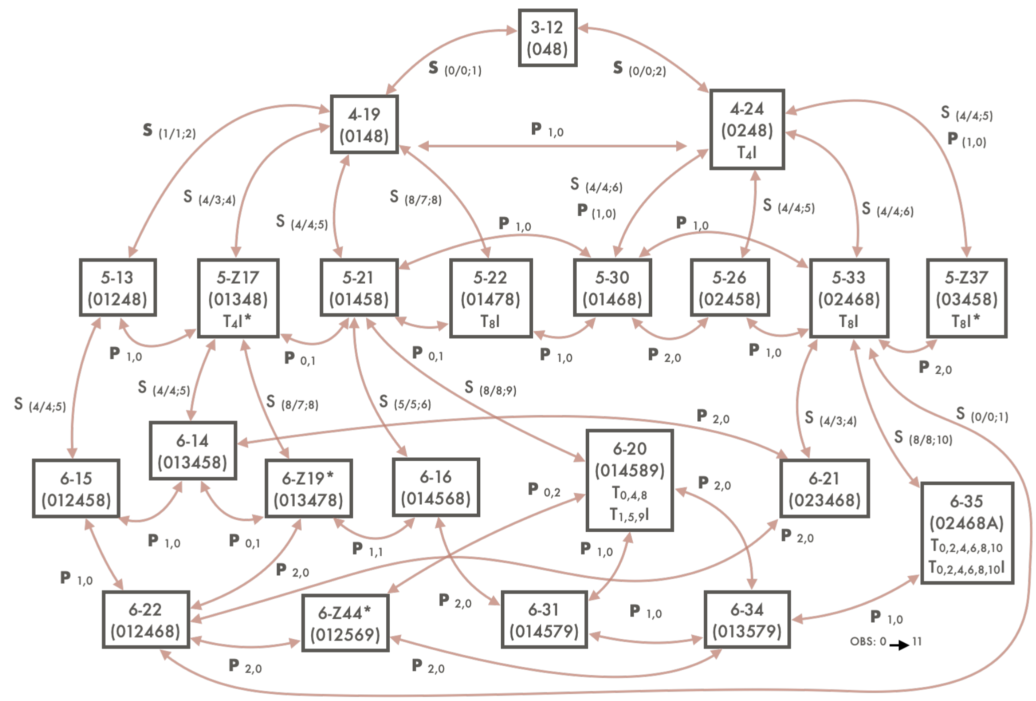

Figure 1: *Genus 4, starting from (048) (author's conception. Made after [5], [1], [4], and [16]).*

From the voice leading's graph on Genus 4, comes up the idea of generating a model of voice-leading zones, according to Richard Cohn's model [3, p. 102–4]. That circular representation reveals some curious properties due to the sum of pc-classes (Figure 2). All the 33 pc-sets belonging to Genus 4 are present, this time; pc-sets with cardinality 7, 8, and 9 have sums related with their complements in the following fashion:

a Pc-sets with cardinal 7 have "n − 1" sum compared with cardinal 5. The only exception is 7-z37 (sum n − 4), whose sum matches its z-related pair, 5-z17.

b Pc-sets with cardinal 8 have "m − 2" sum, compared to the "m" sum of cardinal 4.

c Pc-set 9-12 has "p − 3" sum against "p" sum of its progenitor, 3-12.

d The z-related hexachords 6-z19 and 6-z44 have the same sum.

The voice-leading zone with sum 10 is empty; on the other hand, the higher number of pc-sets with the same sum is on 6, 7, 8, and 11 voice-leading zones. The possible meaning of these data could be verified through similar graphs for all remaining Genera, a task beyond the scope of this work.

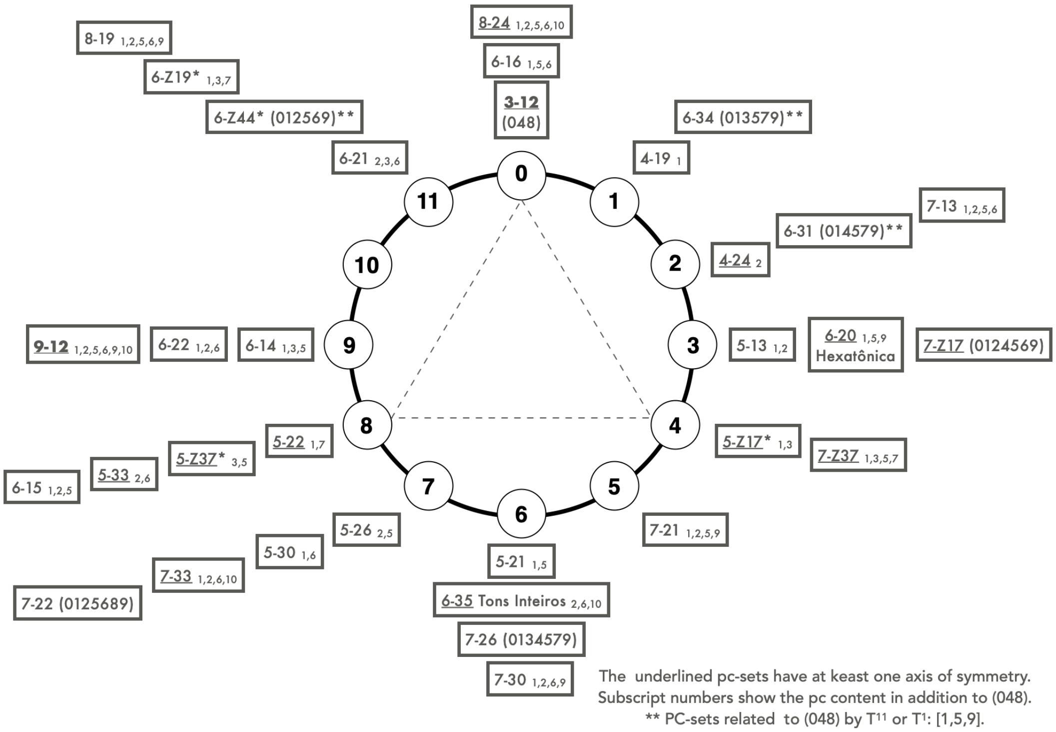

Figure 2: *Genus 4 set over the voice-leading zones (author's conception. Made after [5] and [3]).*

IV. Identifying harmonic genera and its transformations in the music of Villa-Lobos

Villa-Lobos's music is sometimes dismissed as "chaotic" because it wanders erratically between the triadic and non-triadic universes and because it oscillates between the banal and the genius.[2] When examining his music more closely, we see that this is not necessarily true – at least in relation to "chaos" – nor does it seem relevant to establish value judgments based on subjective concepts such as "banal" or "genius". The composer is quite concerned with his harmonic materials, although he never discloses his actual processes of tonal organization. In my study on his string quartets [12] I find a correlation between form and tonality, anchored on cadences whose formal function resembles the Classical style. Of course, Villa-Lobos's harmony is completely different; in his peculiar treatment of the dissonance, interval symmetry takes the role of traditional consonance.[3] Besides, the composer sets stable thematic areas in opposition to transitional or developmental areas. Thus, it seems possible to sketch a principle of tonal directionality in Villa-Lobosian music, employing genera of pc-sets in association with voice-leading mapping.

Villa-Lobos's Seventh String Quartet (1942) is a piece of music filled with neoclassical features as thematic areas in opposition to transitional or developmental sections, highlighted by textural

[2]Lisa Peppercorn claims that "is impossible to deal with all his work [. . .] because of their very uneven quality – works containing music which is at times banal, as well as music which has the stamp of genius" [11, p. 106].

[3]For a detailed study on Villa-Lobos's cadences in his string quartets, see [12, pp. 156–182].

changes. I start with the segmentation of melodic and harmonic elements starting from the primary theme, passing through the transition, and reaching the subordinated second group of themes (Figure 3). The exposition of the first theme is a sequential progression in descending fifths, and its main motif is the diatonic "minor" tetrachord (0135), FN 4-11; during the transition, the minor chord with minor seventh (FN 4-26) is recurrent, making parallel movements along with the thematic statements.

Despite Forte's careful criteria, is somewhat disturbing to find that (0135) does not belong to Genus 12 (Dia-Tonal), but to Genus 7 (Chroma-Dia). Surprisingly, a typical diatonic tetrachord[4] is labeled as a sort of mixture between chromatic and diatonic species; Forte's explanation is on the progenitor (025), shared by Genus 7 and Genus 12. Nevertheless, as most of its members are designated as primary members of Genus 12, the passage has considerable harmonic consistency.[5]

The recomposition process -– in which the thematic materials presented in the exposition are resumed in the recapitulation — has some interesting features, which reveals how meticulous Villa-Lobos could be, concerning his melodic-harmonic materials (Table 2).[6] At first sight, the sequential progression of the first theme seems to be overruled by a "major" diatonic tetrachord (C-D-E-F instead of the expected G-A♭-B♭-C) that pops up at mm. 154–156, demonstrating his awareness to the inversional property between these pc-sets, along with its ulterior parsimonious transformation. Thus, the treatment given to the recapitulation shows appreciation for the subtle transformation, for the balance between consistency and variety.

Table 2: *Villa-Lobos, SQ7, I, exposition vs. recapitulation. (author's conception).*

Measures	Exposition	Measures	Recapitulation
1–3	A–B♭–C–D	151–3	D–E♭–F–G
4–6	D–E♭–F–G	154–6	C–D–E–F
7–9; 13–15	G–A♭–B♭–C	159–161; 164–6	C–D♭–E♭–F

During the exposition, the transitional area "modulates" from the diatonic first theme to the octatonic second theme. Some tonal chords, dominant-seventh like (0358), triads (3-11), and the diatonic collection (7-35), punctuates the transition, along with some melodic reiteration of the motif on tetrachord (0135). The arrival to the second theme initially passes by the octatonic septachord (7-31) before affirming the unmistakable octatonic collection (8-28).

At the recapitulation, there is an additional cardinal 7 collection (7-34), which makes more smoothly the transformation from diatonic to octatonic (Table 3). Considering the chords involved in that passage, Genus 12 (Dia-Tonal) modulates to Genus 3 (Diminished, to which octatonic belongs). A remarkable feature is the transposition pattern within the second theme, that makes the "consequent" (mm. 205–8) matches the "antecedent" (mm. 32–4) phrase of the exposition, resulting in a sort of large-scale tonal resolution.

[4]If one put it in simple terms, (0135) is represented by the scalar fragment C-D-E-F or G-A-B-C; both are transpositions of a tetrachord that is strongly associated with the major scale.

[5]Richard Parks proposes four criteria to disambiguate between Genus: 1) Prefer those genera that contain as members as many as possible (ideally, all) of the scs represented in the musical object that is the subject of investigation. [. . .] 2) Prefer that Genus whose primary members or characteristic members embrace the largest number of scs from the musical object. 3) Prefer that Genus which contains the smallest number of members or which contains the smallest number of primary members. 4) Prefer that Genus whose cynosural and member scs evince the greatest similarity to familiar pitch constructs [9, p. 211]. In Park's terminology, "sc" stands for "set class" to mean "pitch-class set class" (207).

[6]I did a detailed account on recomposition in the Seventh String Quartet (SQ7, for short) [13, pp. 446–9], after Hepokoski and Darcy's sonata theory [7, pp. 239–280].

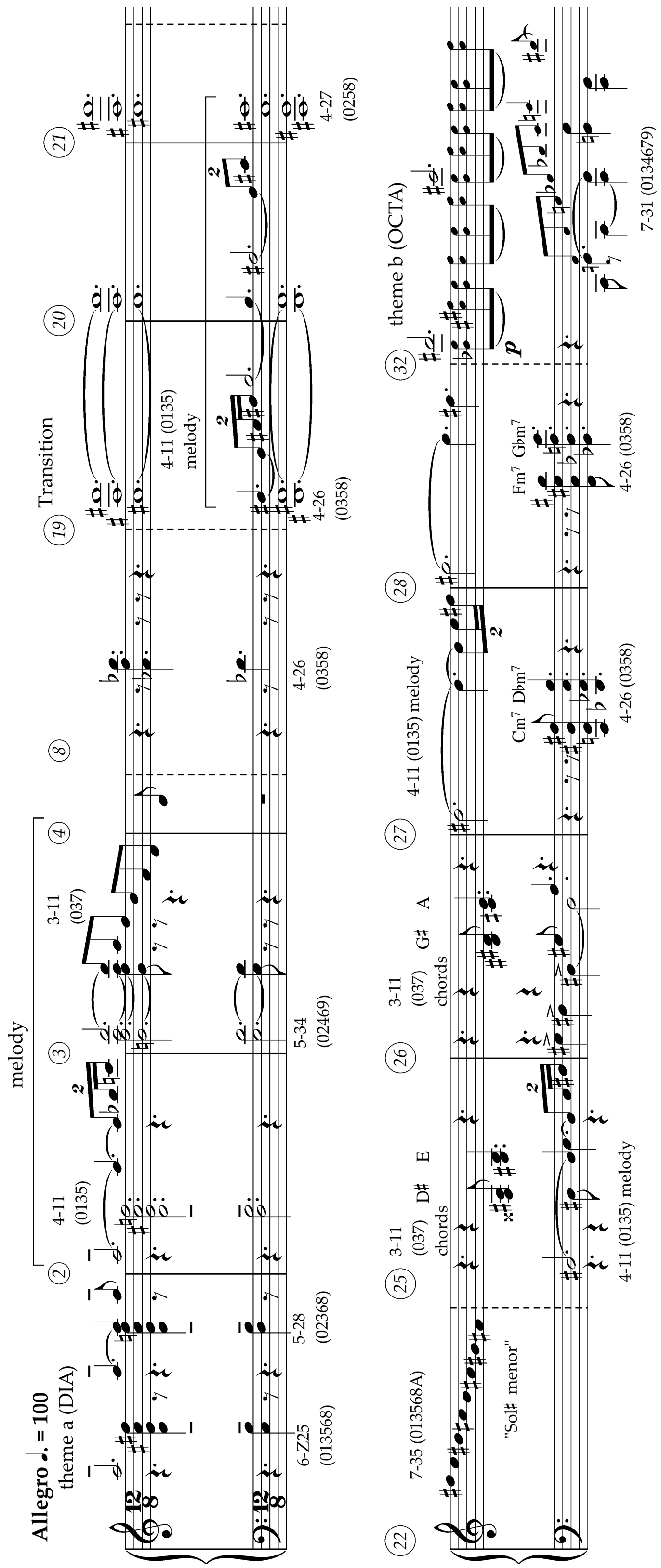

Figure 3: *Villa-Lobos, SQ7, I, bars 1-32, exposition (analytical reduction, author's conception).*

Table 3: *Villa-Lobos, SQ7, I, exposition vs. recapitulation (author's conception).*

168–9	176–7	201–4	205–8
[0, 1, 2, 5, 6, 8, 10] (013568A) 7-35	[9, 10, 0, 1, 3, 5, 7] (013468A) 7-34	[5, 7, 8, 10, 11, 1, 2] (0134679) 7-31	[0, 1, 2, 4, 6, 7, 9, 10] (0134679A) 8-28
TRANSITION		SECOND THEME	SECOND THEME, ORIGINAL PITCH
		Pivotal function Genus 12 $\rightarrow$ Genus 3 DIA$\rightarrow$OCTA	Genus 3 (Diminished) OCTA

Drawing a tree-like graph (Figure 4) creates a 'compositional space' among the pc-sets belonging to Genus 12. It makes a path to pc-set 7-31, which acts like a pivot between Genus 12 and Genus 3, preparing the arrival to the octatonic second theme. Tetrachord (0135) is directly connected to the progenitor 3-7. Thus, the post-tonal features in Villa-Lobos's string quartet are analogous to 18th-century Classical composers like Haydn and Mozart. The chords that support the initial theme (mm. 1–2) can be understood as members of the same harmonic "family", although they cannot be assigned with triadic labels.

The second movement ("Andantino vagaroso") from Villa-Lobos's Ninth String Quartet (1945) is quite different from the SQ7. The first theme (mm. 1–12) is mostly chromatic, with a distinct expressionist color, surprisingly common in some of Villa-Lobosian music in the 1940s (Figure 5).

The melodic and harmonic layers have pc-sets belonging to Genus 1 (Atonal) (Figure 6). On the melody level, just pc-set 8-2 does not belong to that Genus, but to Supra-Genus II (gathering Genus 5 – Chroma, and Genus 6 – Semi-Chroma); even so, that is the main superset to the remaining pc-sets (Figure 7).

On the harmony side (Figure 8), voice leading is not properly parsimonious, but keeps consistent with Genus 1, except for pc-set 5-21, that belongs to Genus 4 (Augmented) and Supra-Genus III (Genus 8, 9, and 10), confirming the label "atonal" (Figure 9). The inclusion related chords are displayed in normal form (Figure 10).

The thematic exposition (bars 1-30) in the first movement of Villa-Lobos's First Symphony (1916) has great harmonic diversity. The primary theme is made on the nonachord 9-4, which belongs to Genus 8 — "Atonal" and Genus 10 -– "Atonal-Tonal". That theme can be divided in three smaller units with distinct features: an almost diatonic septachord, a chromatic trichord, and a pentatonic collection (Figure 11). The sequential progression of that theme (Figure 12) results in an octatonic collection, considering their implied triadic formations (Figure 13).

Villa-Lobos does not preserve the octatonic cycle throughout the first movement. At the moment that the first theme abandons the sequential progression, its "atonal" or almost "freely tonal" profile is enhanced. No other allusion to the octatonic scale recurs during the rest of the first movement.

The transition to the subsidiary theme (mm. 13–30) shows up chords with variable density, oscillating between four and seven voices, while their cardinality bounces up and down from 4 to 6 (Figure 14).

During the transition, there is more diversity of pc-sets, according Forte's genera (Table 4). A criterion like the one developed by Parks [9] finds seven matches of pc-sets belonging to Genus 11 (Dia) and Genus 12 (Dia-Tonal). Its noteworthy to remember that the transition links to the diatonic theme B, on the Mixolydian mode, which also belongs to Genus 11 and 12.

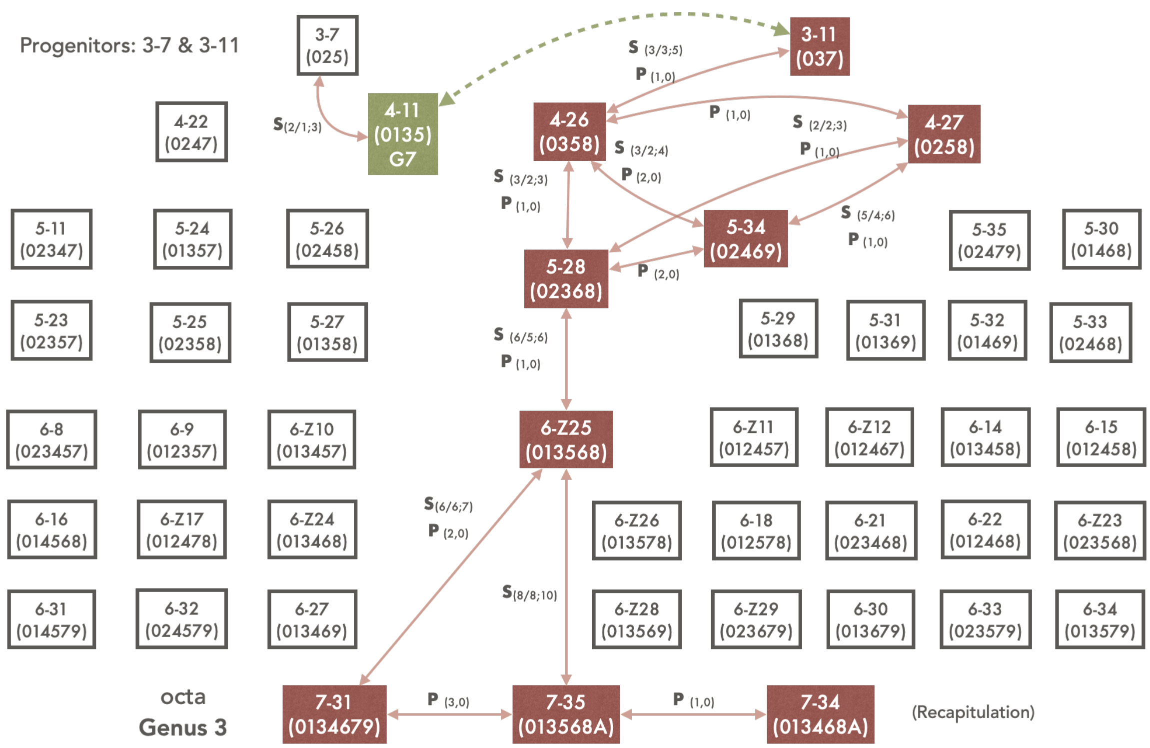

Figure 4: *Villa-Lobos, SQ7, I, exposition and recapitulation, viewed from Genus 12 (author's conception).*

Table 4: *Villa-Lobos, Symphony n. 1, I, "Allegro moderato", mm. 13–30, genera found in the transition (author's conception).*

PC sets	Amount	Genus
4-22	3	11 (Dia); 12 (Dia-Tonal)
5-29	1	1 (Atonal); 3 (Diminished); 7 (Chroma-Dia)
		10 (Atonal-Tonal); 11 (Dia); 12 (Dia-Tonal)
4-27	2	2, 3 & 12
5-35	2	11 & 12
4-26	1	12
4-z29	1	1 & 2
4-20	1	7 & 10
6-32	2	10, 11 & 12
6-33	1	10, 11 & 12
3-9	1	11
4-23	1	11
4-19	1	4, 8, 9 & 10

During the exposition, harmony can be simply described as transformational process from an "atonal" nonachord, which becomes progressively "tonal", while leading to the "diatonic" subsidiary theme (Table 5). That process is analogous to the classic modulation.

Figure 5: *Villa-Lobos, SQ9, II, "Andantino vagaroso", mm. 1-12 (author's conception).*

Table 5: *Villa-Lobos, Symphony n. 1, I, "Allegro moderato", mm. 13–30, modulation during the transition (author's conception).*

Theme A		Transition		Theme B
9-4	→		→	5-34/ 7-35
G8; 10		G11;12		G2, 3, 12/ G11, 12
Atonal/ Atonal-Tonal		Dia/Dia-Tonal		Whole-Tone, Diminished, Dia-Tonal/ Dia, Dia-tonal
		Referential collections/ Parks's Genera [10]		
Chroma		→		Dia

V. Conclusion

Forte's theory of pc-set genera [5] offers some elements to depict the main features of the so-called harmonic "families". Notwithstanding, it has some inconsistencies and entangled details; sometimes the genera criteria fails to recognize some distinct members of the diatonic collection — even when they are quite familiar by ear; besides, it creates some non-intuitive terminology such as "Atonal-Tonal" or "Semi-Chroma" – terms whose meaning he does not clarify. In that sense, Park's theory of pc-set genera [10], [9] is more intuitively related to the musical practice. In Park's

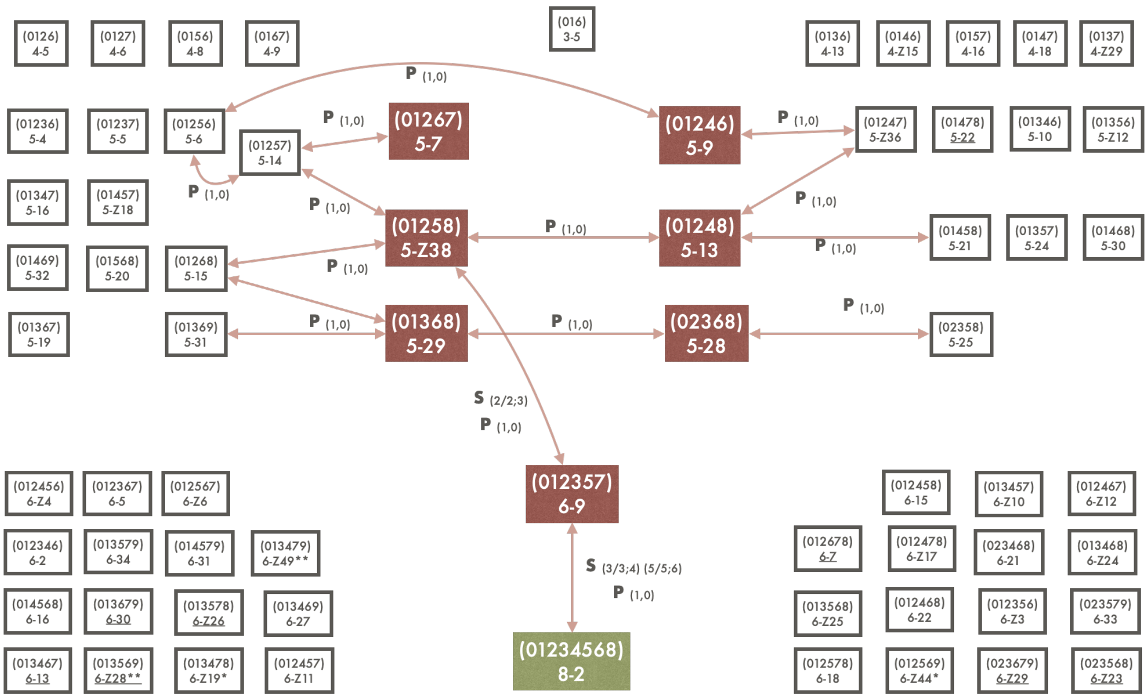

Figure 6: *Villa-Lobos, SQ9, II, melody, mm. 1–12, in Genus 1 (Atonal) (author's conception).*

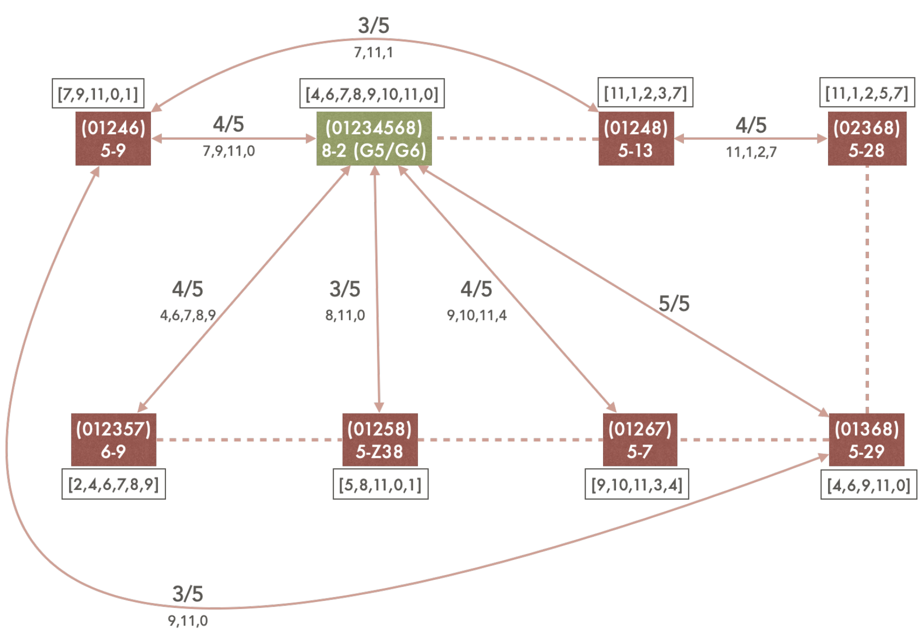

Figure 7: *Villa-Lobos, SQ9, II, melody, mm. 1–12, inclusion (normal form) (author's conception).*

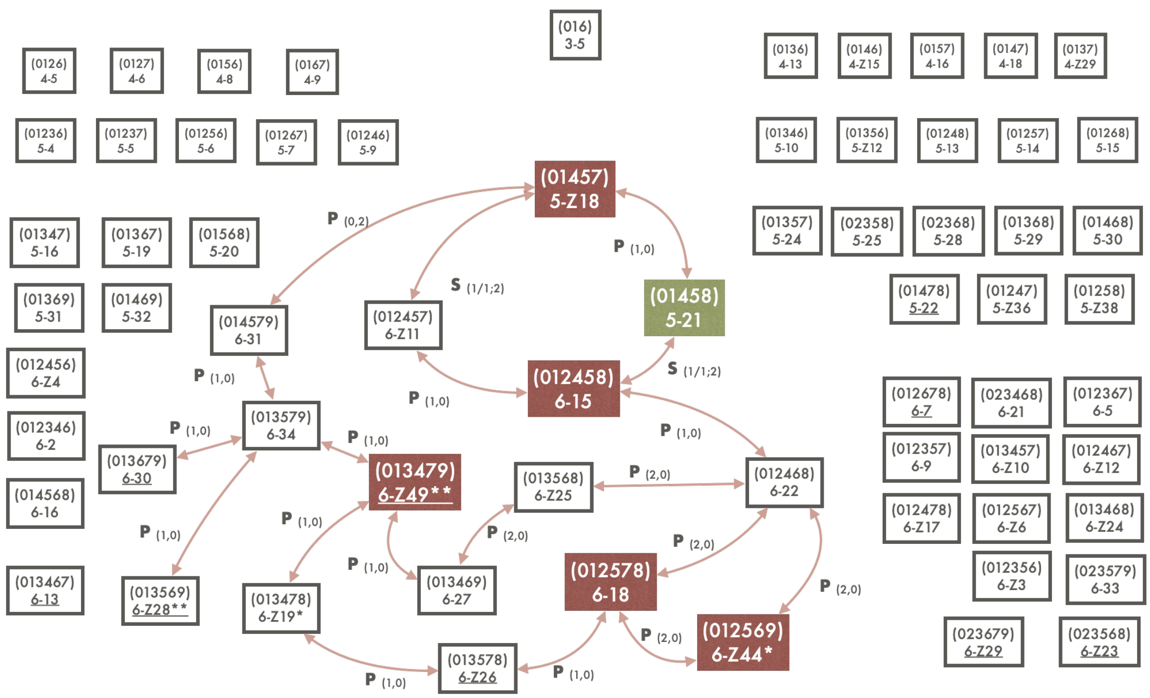

Figure 8: *Villa-Lobos, SQ9, II, chords, mm. 1–12, accompaniment layer, in Genus 1 (author's conception).*

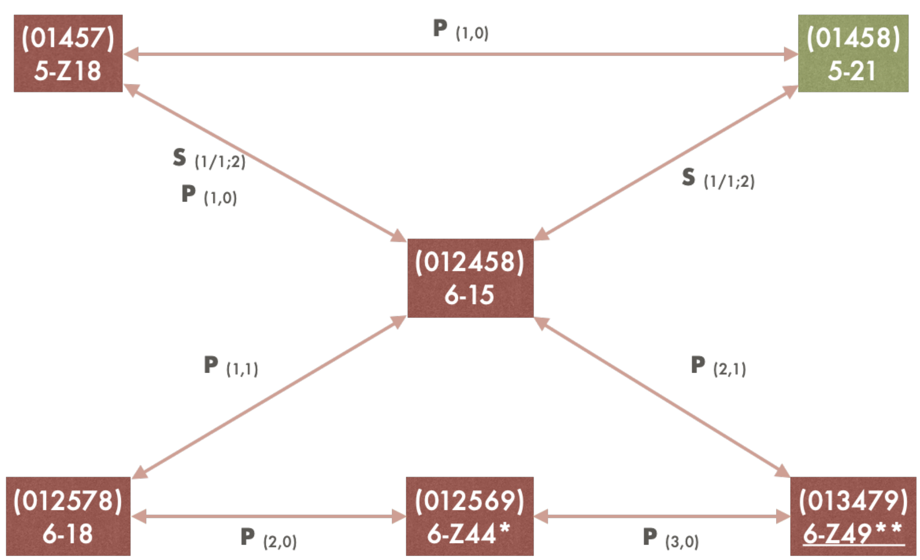

Figure 9: *Villa-Lobos, SQ9, II, mm. 1–12, chords, voice leading (author's conception).*

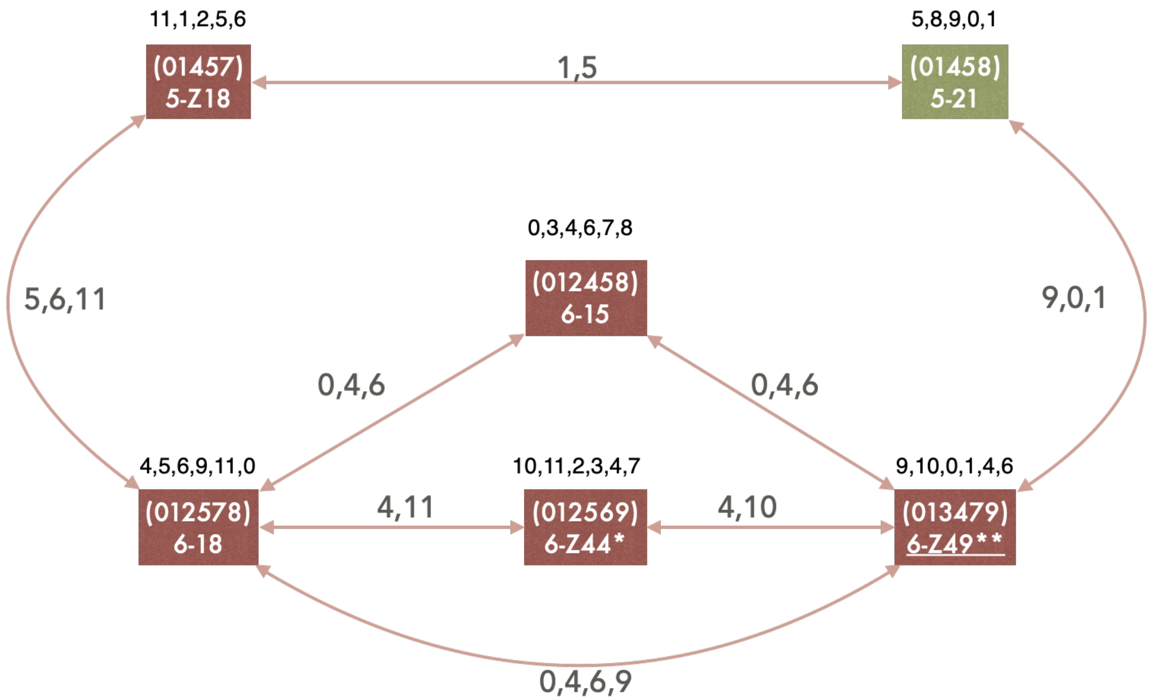

Figure 10: *Villa-Lobos, SQ9, II, mm. 1–12, chords, inclusion (normal form) (author's conception).*

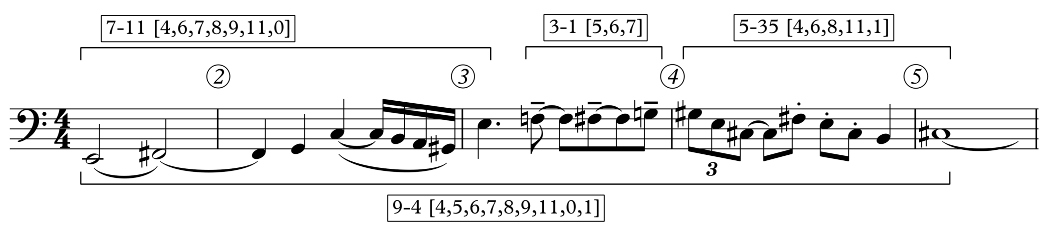

Figure 11: *Villa-Lobos, Symphony n. 1, I, "Allegro moderato", mm. 1–5, primary theme (author's conception).*

book on Debussy, he comes close to the concept of referential collections used frequently in more recent times [14, pp. 228–262]. However, the concept of referential collection is somewhat vague in relation to its subsets, which reinforces the value of Forte's initiative. Thus, the more the detail is sought, the more it is lost in relation to the general characteristics of the collections; on the other hand, the most comprehensive and comfortable definitions conceal the difficulty of establishing the kinship between groupings with greater precision. There is possibly a compromise between the referential option and the detailed genera option; however, I prefer to adopt a perspective that takes into account the musical context, therefore more analytical than theoretical.

The study of voice leading within these harmonic families demonstrates some important elements related to compositional processes and transformations, with a potential to enlighten some post-tonal practices. That allows us to see "genera" as "compositional spaces", where one can trace how directionality is perceived in the post-tonal repertoire. During the period of common practice, the harmonic thinking of European composers was based on the cycle of fifths, establishing the criterion of tonal distance around major and minor modes. Throughout this article, I have tried to demonstrate how Forte's genera, in association with the concept of Cohn's

Figure 12: *Villa-Lobos, Symphony n. 1, I, "Allegro moderato", mm. 1–13, sequence on theme a (author's conception).*

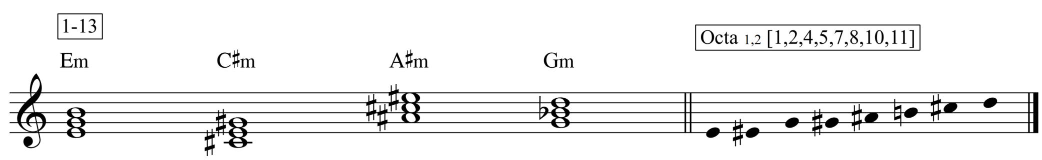

Figure 13: *Villa-Lobos, Symphony n. 1, I, "Allegro moderato", mm. 1–13, first theme's implied triads and resulting octatonic collection (author's conception).*

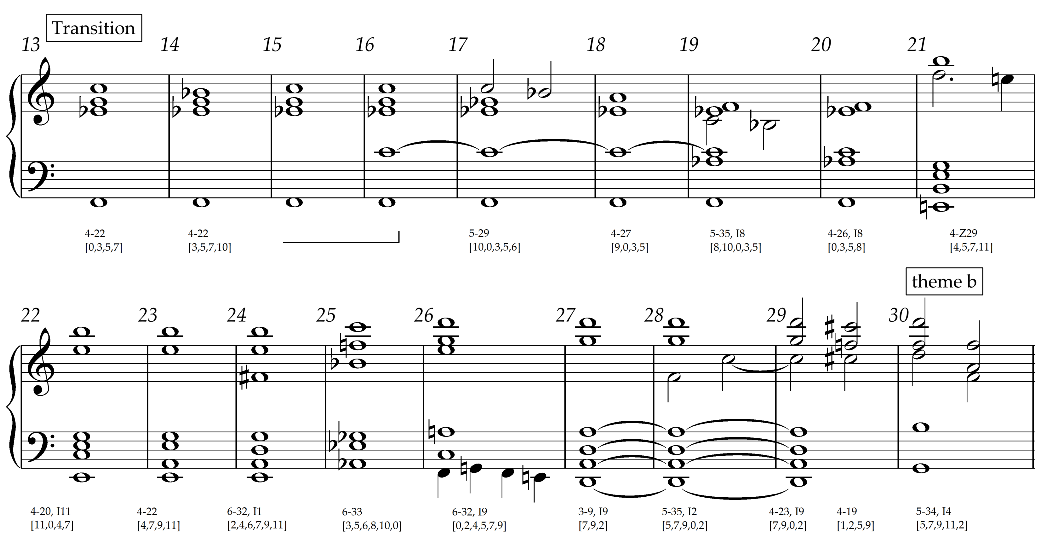

Figure 14: *Villa-Lobos, Symphony n. 1, I, "Allegro moderato", mm. 13–30, transition (author's conception).*

voice-leading zones, can generate an analogous mapping. In this model, changes in cardinality between pitch-class sets are incorporated into the flow within the genera, through SPLIT and FUSE operations.

Villa-Lobos's music illustrates some post-tonal processes, in which harmonic transformation provides internal coherence; the almost complete match with Forte's genera demonstrates how selective the composer can be, tracing his harmonic paths (by ear or by calculating, it does not matter how) to be capable of finding homogeneous intervallic patterns in such expanded tonal centers, in a way pretty much analogous to the Classical tonality.

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Reading Textural Functions, Instrumental Techniques, and Space Through Partition Complexes

*Pauxy Gentil-Nunes
Federal University of Rio de Janeiro (UFRJ)
pauxygnunes@musica.ufrj.br
Orcid: 0000-0003-1548-7694
DOI: 10.46926/musmat.2020v4n2.80-97

Abstract: Partitional complexes *are sets of discrete textural configurations (called shortly of* partitions *in* Partition Analysis*) that successfully interact to construct a global textural structure. This textural mode is called the* Textural Proposal *of a piece, where* referential partitions *(those that represent the main features of textural configurations in the excerpt) stand out. This conceptual environment, developed in musical texture formalization through observation and musical repertoire analysis, is now applied to musical practice. In the present work, we highlight three of these situations. The first one deals with the creative flow (compositional process) and its relation with textural planning. The second observes how these same textural functions condition the body's physical coupling to the instrument (fingers, hands, pedals, instrumentation). Finally, just as an introduction, we envisage some spatial relations, involving instrument distribution on stage, emphasizing historical concert music.*

Keywords*: Texture. Partitional Analysis. Textural Design. Performative Partitioning. Partitional Complex.*

I. Introduction

Many authors have discussed musical texture as a subject associated with hierarchical levels of the musical discourse, usually as a surface feature ([7], [18], [8]). The tension between the musical structure (commonly correlated to harmonic and formal units observed in large time spans) and the musical surface is one of the interests raised by some post-tonal theorists (for instance, [2]).

This question guides the research project called *Surface and Structure in Applications of Partitional Analysis (2018)* [16], developed in the Graduate Program in Music of the Federal University of Rio

*This work brings the content of a conference in the round table "Theories of the Texture" in the III Congress of the Brazilian Association of Musical Theory and Analysis (TeMA) and the IV International Congress on Music and Mathematics, held in 2019 in the city of Rio de Janeiro.

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de Janeiro. It came as a natural arrival of former projects, since the year 2003, all revolving around the formalization of texture through the application of the Theory of Integer Partitions, leading to a field called *Partitional Analysis* (henceforward, *PA* – [10], [13], [15]).

In the earlier years (2003-2009), PA gradually consolidates by developing proper musical analysis and composition tools. Since 2009, researchers have made some effort in applying these concepts to practical musical situations, internal and external to the theoretical realm, leading to new approaches and the expansion of the limits of the theory.

The more recent lines of research inside PA concentrate on three main fields:

1. *Textural Design* – the proposal by Daniel Moreira brings together the expanded application in the compositional process of PA tools combined with Compositional Design concepts by Robert Morris [29], as well as original tools. ([21], [22]).
2. *Performative partitioning* – formalizes the relation between textural configurations and the coupling of the body (fingers, hands, arms) and musical instruments. Bernardo Ramos ([34], [33], [35]), Pedro Miguel de Moraes [20] and the present author [14] work on this approach.
3. *Spacial partitioning* – as an introduction, we make in the present paper some preliminary observations about this application, foreseen in earlier publications ([12]).

Next, we will make a brief elaboration of each one of these applications, in order to draw some lines for the near future of the research.

II. Textural Design

Daniel Moreira has achieved some remarkable advances in the expansion of PA concepts in successive publications, using the *Countour Theory* ([9], [30], [19]) to enable the partitions a possibility of a linear organization ([27], [26], [25], [24], [23]). In 2019, Daniel defended his doctoral dissertation, with Robert Morris' supervision, where he proposes a combination of PA concepts with Morris' *Compositional Design* principles to arrive at the *Textural Design* theory, where he also brings original ideas and tools.[1]

One of the most striking features of Moreira's work is the expansion of the textural space in two opposite directions.

First, taking the partition level as a reference, Moreira proposes a generalization of partitional structure through a quality evaluation, reading the parts just as *lines* or *blocks*. This categorization was made earlier by Ramos [33], but Moreira deepened his reading, proposing a *Hasse diagram* for the textural class space — the *tc-space lattice* (Figure 1). In addition to organizing the inclusion relations, Moreira qualitatively evaluated the parsimonious operators between the elements (*shifting*, *linear layering* and *block layering*). Moreover, he provided numerous analytical examples and original related concepts, such as *monopart*, *polypart*, *isopart*, and *heteropart*.

As a second step, Moreira proposes instantiations of partitions in the musical surface direction through *ordered partitions* and *thread-words*. Ordered partitions are known in mathematics as *compositions* and consider each permutation of the parts inside the partition as a distinct element. On the other side, *thread-words* allow the evaluation of parts constituted by non-neighbor components (or threads), e.g., textural configurations with interpolated parts. The comprehensive set of textural spaces — *Textural class space*, *Unordered partitions space* and *Partition layout space* (Figure 2) — comprises a global taxonomy of textural codes.

[1] As Moreira has published a paper about Textural Design in the present journal ([21]), we highly recommend the reading, as he provides a much more detailed discussion for the subjects briefly explained in this subsection. His doctoral thesis is also available ([22])

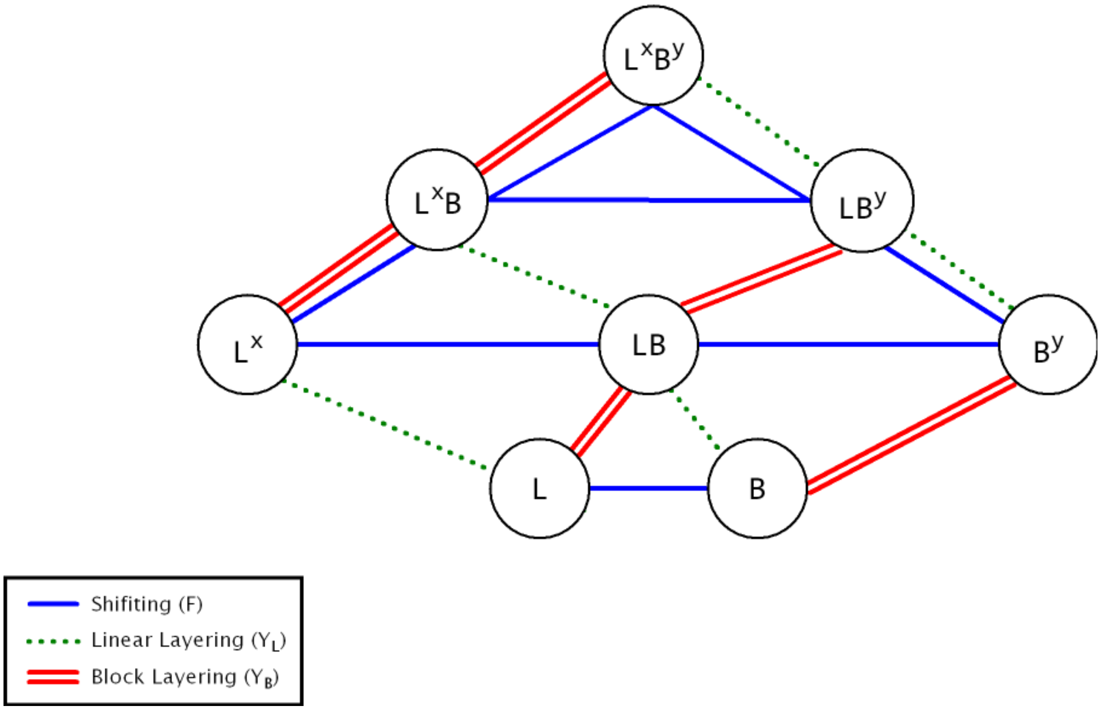

Figure 1: *Textural class lattice: all textural classes connected by layering (Y) and shifting (F) operations ([21], p. 26).*

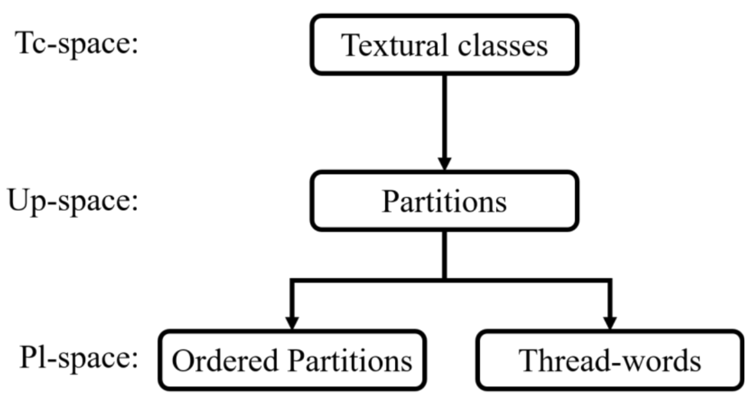

Figure 2: *Inclusion relations among textural codes within textural spaces ([21], p. 33).*

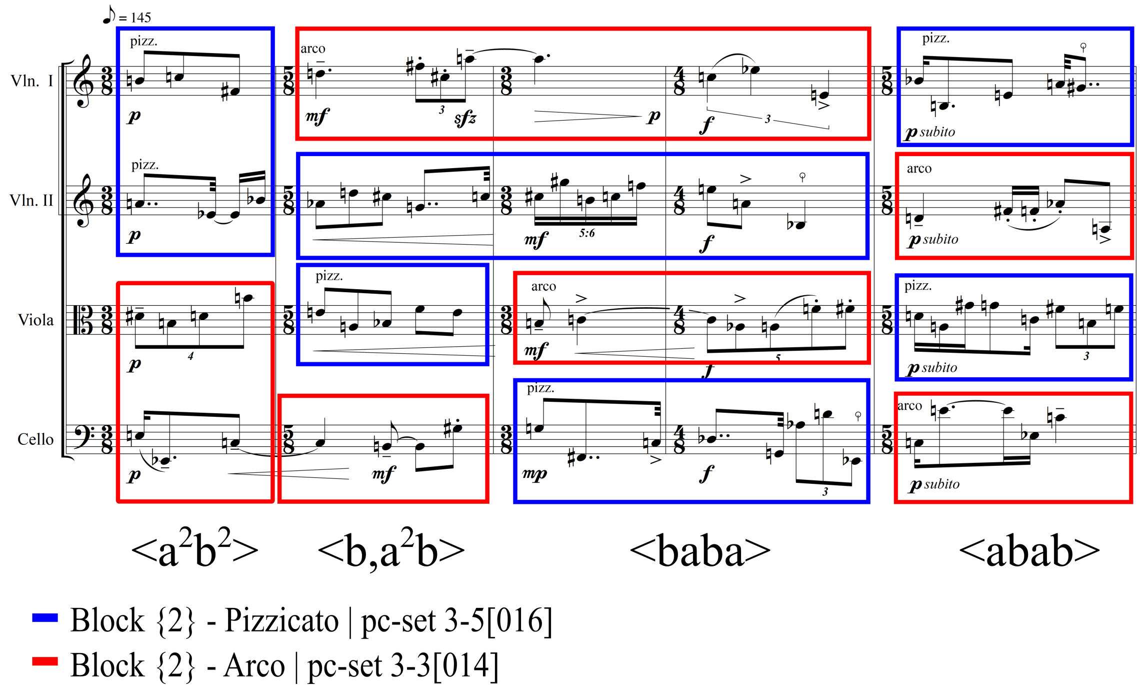

Figure 3: *Thread-words producing different organizations of threads within partition* $[2^2]$ *(adapted from [21], p. 32).*

One crucial point is that Moreira provides a series of musical examples (original and from repertoire) that use criteria that meet the composer's direct interest and point to the surface as well. For example, the articulation or timbre of the instruments (Figure 3).

In order to attend the specific needs of the compositional process, Moreira highlights five modes of textural realization, each representing a recurring procedure regarding the construction of musical textures: a) *Standard Mode*; b) *Partitional Complex*; c) *Evolving mode*; d) *Colorization*; e) *Montage* ([21], p. 153). These concepts are actual examples of a contribution from applying theory to musical practice.

III. Performative Partitioning

The centrifugal tendency of textural analysis led the analyst sight from inside to outside the score, where the contact between the body and the instrument causes the gestures that constitute the musical surface. Bernardo Ramos proposes this approach in his analysis of the 20 *Estudios Sencillos* for guitar by Leo Brouwer [33]. The main question was about the construction of the Etudes' progressiveness in terms of textural writing for the instrument. For instance, the textural configurations could become gradually more complex, or its number and degree of changes could just level up.

The conclusion is that the accrual grows towards more advanced technical demands with no replacement but accumulation. Still, the prevalence of some textural configurations follow a specific pattern (Table 1).

The partitions (1), (1^2), (1.2), (3) and (1^3) are the most used in the series. Some common features between them can explain their prominence. They easily demand the three main fingers of the right hand – the thumb, index, and middle (noted in the score as *p*, *i*, and *m*, respectively).

Table 1: *Distribution of partitions in Leo Brouwer's* 20 Estudios Sencillos *([3]) in order of occurrence in the series ([36], p. 21). The last line shows the number of Etudes where each partition occurs.*

	1	**2**	**1.2**	**1^2**	**1.3**	**3**	**1^3**	**4**	**2^2**	**5**	**1.4**	**$1^2 2$**	**6**	**1^4**	**$1^2 3$**
E1	•	•	•												
E2					•	•									
E3	•			•			•								
E4	•		•	•	•										
E5	•		•	•				•							
E6								•							
E7	•	•		•											
E8	•	•	•	•		•	•								
E9	•			•	•		•								
E10	•	•		•		•									
E11	•	•	•	•	•	•	•	•		•	•				
E12	•		•	•			•				•				
E13	•		•	•		•	•	•				•	•		
E14	•	•	•	•								•	•		
E15	•	•	•	•		•	•		•						
E16	•	•	•	•		•	•							•	
E17	•		•	•	•		•	•		•	•		•		•
E18	•	•	•	•	•	•	•	•		•		•			
E19	•				•	•		•			•	•	•		
E20	•	•	•	•	•	•		•	•						
(Occ.)	18	9	13	16	8	10	10	8	2	3	4	4	4	1	1

Almost all occurrences involve these fingers. The thumb is responsible for nearly every unitary part, notably when it combines with a block – in partition (1.2), for instance.

What emerges from the analytical conclusions is that the observations returned to the human hand's shape with its opposable thumb, bringing the ergonomy of the human body into the scene and the force of instrumentation in determining the textural discourse.

The first *Etude* shows this correlation with clarity. The alternation between the parts (1) – the thumb playing the melody – and (2) – the combination of index and middle finger performing a steady harmonic dyad as an accompaniment – is dominant throughout the piece (Figure 4). Eventually, each part articulates separately, and there could yet be a simultaneous attack, merging the parts. The set of all these combinations – in this case, partitions (1), (2), and (1.2) – expresses a type of textural mode that can define some specific instrumental writing. There is, too, one or more partitions that prevails over the others, establishing a hierarchic structure.

The addition of the technical procedures for performing the textural mode to the partition constitutes the *Textural Proposal* (*TP*) of a piece or excerpt. As each instrument always has specific technical demands derived from its physical and historical domain, their TPs tend to be exclusive.

The enumeration and assessments of TPs, based on the instrument's technique and practice, is now one of the goals of this area, called then as *Performative Partitioning*.

For instance, Ramos proposes the evaluation of all possible GTPs (*Guitar Textural Proposals*)

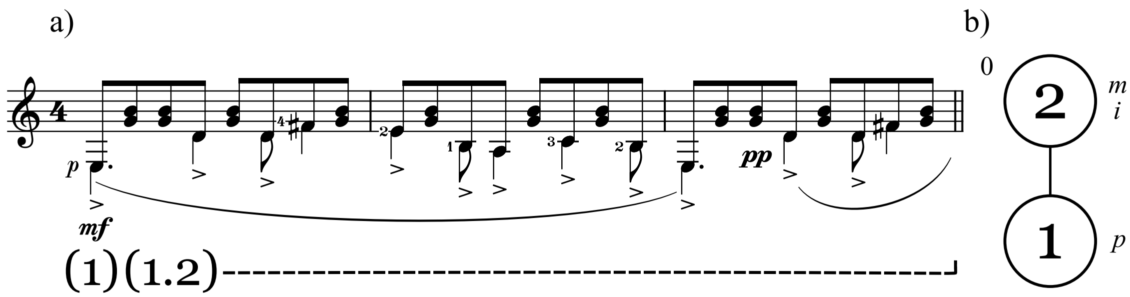

Figure 4: *a)* Etude No. 1, *excerpt with partitions* (1) *and* (1.2) *([3], mm. 1–2). b) Diagram of its GTP. The letters in italics indicate the fingers of the right hand that articulate the parts nearby. The numeral* 0 *on the upper left side of part* (2) *indicates that the part is performed using notes located on open strings ([35], p. 11).*

associated with partition (1.2). The reflection begins with the uses of this structure in Brouwer's *Etudes*. The Cuban composer resort to basic techniques, like open strings, inversion of registers, and left-hand slurs. But even in this restricted universe, with a single partition and elementary procedures, some combinations were not used. Ramos enumerates the missing ones and shows how they can form a net of parsimonious relations (Figure 5).

Pedro Miguel de Moraes follows the same path, applying PA's concepts to the performative partitioning in the piano writing. Unlike the guitar, where the left hand selects the pitches and the right produces the sound, both hands simultaneously perform two functions on the piano. From this point, the research splits into three large branches, following the work of Claude Cadoz [4]: assessing selection gestures (comprising *hand shapes*), excitation gestures (where are directly applied the partitions and partition complexes) and structural modification gestures (in the piano, the pedal, changing the textural modes in specific ways).

In the realm of the selection gestures, the thumb's opposition against the other fingers is present through the different treatment it receives in interval freedom. The maximum distance between thumb and index finger is the largest and makes the thumb a strategic element for the transition between hand shapes. Following a table elaborated by Richard Parncutt et al. [31], and using an original metric system based on the work of Matteo Balliauw [1], Moraes arrives at a list of 48 hand shapes (Table 2) arranged in parsimonious order ([20], pp. 13–15). This sequence generates a graph that shows how the hand extends and contracts as the shapes evolve through the register (Figure 6).

Excitation gestures comprise the ways the hand can distribute its movement between fingers to produce partitions. Research has already built a large amount of information about this subject that future works will cover.

Gentil-Nunes' formalization of melodic spaces is another proposal that concerns extending performative partitioning to monophonic instruments [14]. This research derives from another PA field, Linear Partitioning, that deals with the relations between lines inside melodic structures. A paper establishes the base for assessing the distribution of partitions in the flute's linear space.

IV. Partitional Complexes in *Portal do Sol*, from *Codex Troano*

A *Partitional Complex* is the set of all potential combinations that can express a partition in time. This set comprises the partition itself, its *subpartitions* (partial sets of parts), *subsums* (partial sums of the parts), and subsums of subpartitions. This structure also points to the instrumentation of

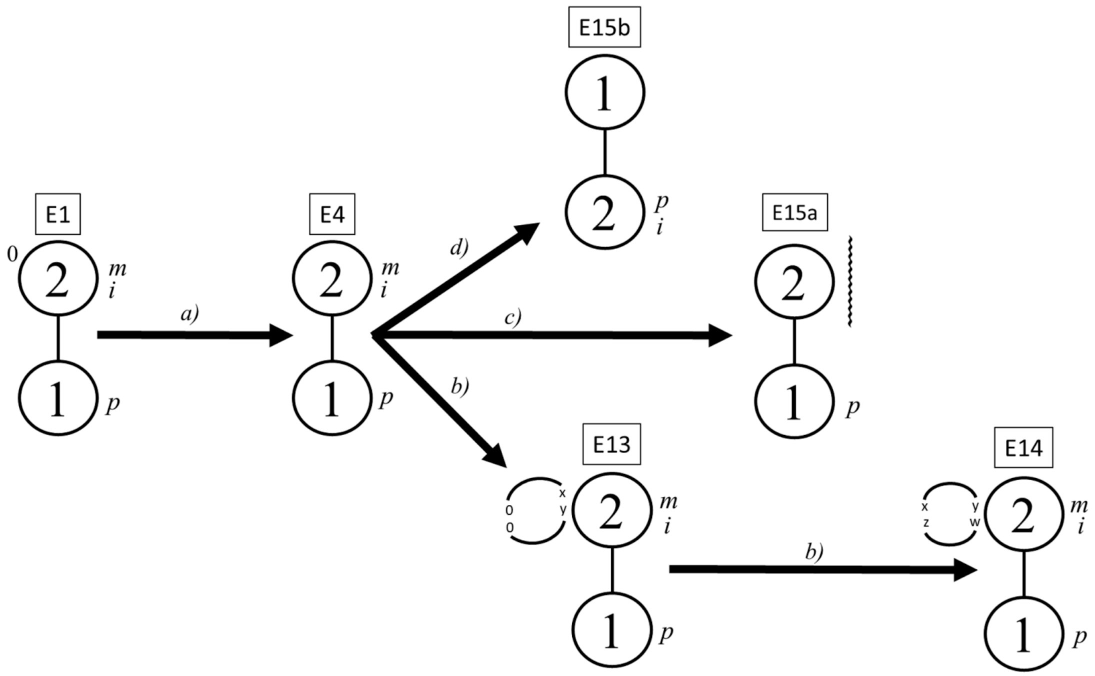

Figure 5: *Etudes 1, 4, 13, 14, and 15 ([3]): net of parsimonious variations in the scope of GTPs formed by partition* (1.2). *a) assignment of linear motion to one part; b) left-hand procedures (ascending and descending slurs); c) right-hand procedures (FPA); d) inversion of register between parts ([35], p. 17).*

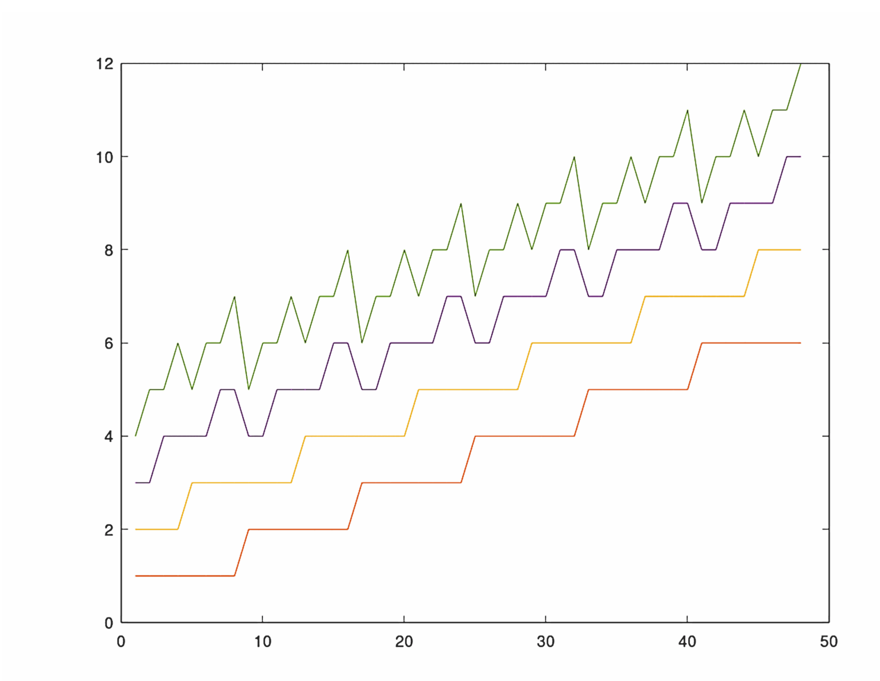

Figure 6: *Progression of the 48 hand shapes (measured in* balliauws*) inside Parncutt's ergonomic comfort zone [20].*

Table 2: *List of all hand shapes (measured in* balliauws*) inside Parncutt's ergonomic comfort zone [20].*

F_1	0	1	2	3	4
F_2	0	1	2	3	5
F_3	0	1	2	4	5
F_4	0	1	2	4	6
F_5	0	1	3	4	5
F_6	0	1	3	4	6
F_7	0	1	3	5	6
F_8	0	1	3	5	7
F_9	0	2	3	5	5
F_{10}	0	2	3	5	6
F_{11}	0	2	3	5	6
F_{12}	0	2	3	5	7
F_{13}	0	2	4	5	6
F_{14}	0	2	4	5	7
F_{15}	0	2	4	6	7
F_{16}	0	2	4	6	8
F_{17}	0	3	4	6	6
F_{18}	0	3	4	6	7
F_{19}	0	3	4	6	7
F_{20}	0	3	4	6	8
F_{21}	0	3	5	6	7
F_{22}	0	3	5	6	8
F_{23}	0	3	5	7	8
F_{24}	0	3	5	7	9
F_{25}	0	4	5	6	7
F_{26}	0	4	5	6	8
F_{27}	0	4	5	7	8
F_{28}	0	4	5	7	9
F_{29}	0	4	6	7	8
F_{30}	0	4	6	7	9
F_{31}	0	4	6	8	9
F_{32}	0	4	6	8	10
F_{33}	0	5	6	7	8
F_{34}	0	5	6	7	9
F_{35}	0	5	6	8	9
F_{36}	0	5	6	8	10
F_{37}	0	5	7	8	9
F_{38}	0	5	7	8	10
F_{39}	0	5	7	9	10
F_{40}	0	5	7	9	11
F_{41}	0	6	7	8	9
F_{42}	0	6	7	8	10
F_{43}	0	6	7	9	10
F_{44}	0	6	7	9	11
F_{45}	0	6	8	9	10
F_{46}	0	6	8	9	11
F_{47}	0	6	8	10	11
F_{48}	0	6	8	10	12

the piece. We bring next an example of application in a composition for percussion ensemble.

André Codeço ([5], [6]) developed in 2014 a partitional analysis of *Codex Troano*, for percussion ensemble, by Roberto Victorio. Codeço shows that the first movement, *Portal of Sun*, is organized in a modular way, based on four textural gestures that appear throughout the piece in varied and recombined versions. The author defines the limits of each gesture based on its recurrences, which support his interpretation consistently. The four gestures presented in succession constitute the first syntagmatic unit of the piece (Figure 7).

As the piece has no bold melodic or harmonic features, the progressions between the simultaneous combinations of the instruments are among the leading forces in organizing the musical discourse.

Each gesture has a sequence of *rhythmic partitions* deduced from the vertical matchings between attack points and the duration of the individual notes:[2]

[2] The exponents indicate the part's multiplicity, and the dots only separate the parts that could mislead in some cases. For example, partition $(1^3 2)$ should read as (1.1.1.2). Partitions (1.3) and (13) can also be differentiated.

Figure 7: *Codex Troano, I - Portal do Sol, by Roberto Victorio ([37], mm. 1–5): first occurrence of the four main textural gestures of the piece (namely,* a, b, c, *and* d*).*

1. Gesture *a* - $<(1^2 2)\ (1^4)\ (1^3 2)\ (1^2 2^2)>$
2. Gesture *b* - $<(1.2.3)\ (1^2 2^2)\ (1.2^2)\ (1^2 2^2)\ (1.4)\ (1.2^2)>$
3. Gesture *c* - $<(1^2)\ (1.3)\ (13)\ (2)>$
4. Gesture *d* - $<(1)\ (1^2)\ (1^2 2)\ (1^5)\ (1^4 2)\ (1^3 2.5)\ (1^2 2.3.5)>$

As each partition have its *agglomeration* and *dispersion* indexes,[3] the delineated curves form what we call *bubbles*, e.g., arcs of textural growth and decline. In this particular case, the gestures *a* and *b* share the same bubble; this occurs as the recurrences show independence of thematic treatment during the piece (see [5], p. 31).

We visualize these arcs in the *indexogram* of the excerpt. Here we read the dispersion (delineated by the superior distance from the middle axis) and agglomeration curve (read by the inferior distance from the middle) both against the timeline of the excerpt, expressed by time points or beats (Figure 8).

Gestures *a* and *b* have more stable curves, while gesture *c* and *d* have extreme focal points in opposite directions. Gesture *c* has two similar agglomeration peaks near time point 8 representing the more massive blocks of the excerpt (in this case, 13 instruments playing conjointly). On the other hand, gesture *d* has a growing curve towards the maximum peak of dispersion (top level of polyphony or diversity of rhythmic profiles) near time point 14. These two complementary peaks seem to balance each other and give a dramatic intensity for the first expository unit's closure, in contrast with the first two gestures, which seem to perform an accrual role. Considering the median line between the dispersion and agglomeration curves, one can then describe the textural profile's overall contour as <1 2 0 3>. There are many other methods for extracting such patterns, which can be a methodology on its own.[4]

Partitional complexes can be visualized here as the result of the overall interaction between partitions inside a delimited section. The idea is that a textural configuration is not an atomic event. Instead, it is a construction, put into play by the gradual presentation of its constituent parts through the flow of time ([16], [21], [22]).

In Figure 8 we present partitional complexes above each associated gesture. They are, respectively, $(1^4 2^2)$, $(1^3 2^2)$, $(1^2 3.8)$, and $(1^5 2.5)$. The evaluation of each part as a *subpartition*, a *subsum*, or a combination of both leads to the *referential partition*. The last represents the overall textural mode of the gesture. This process can recur, forming a nested structure, organized in levels, like a schenkerian reduction, but departing from the score and pointing to the instrumentation. Instead of a centripetal movement, leading to the deep structure, the evaluation of partitional complexes points to the outermost surface, located in the external world (instrumentation, performativity, body).

[3] The *agglomeration* and *dispersion* indexes are the simple counts of the number of collaboration or contraposition binary relations between textural configuration elements (here, individual notes). In the case of *rhythmic partitioning*, collaboration means the convergence of attack point and duration and contraposition the mismatch between these two parameters ([13], pp. 33–57).

[4] Daniel Moreira developed this type of evaluation of texture by contour with great detail in [21].

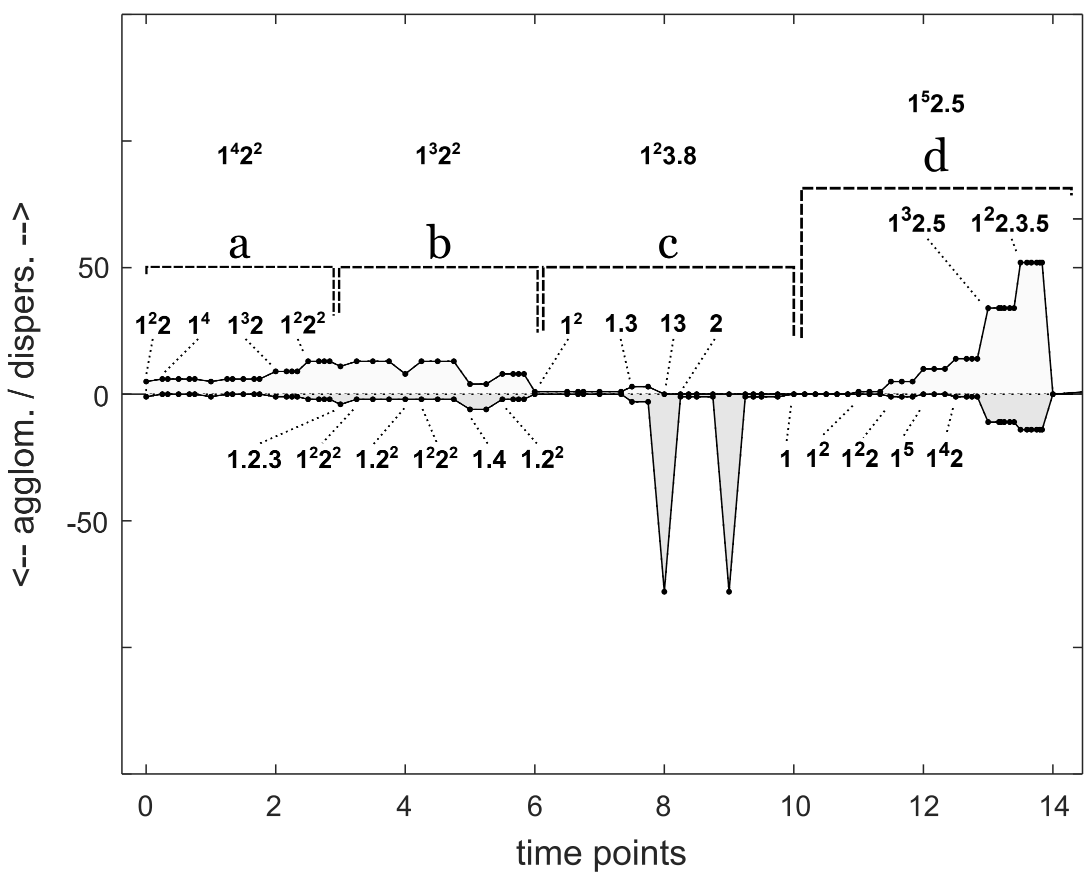

Figure 8: Codex Troano, I - Portal do Sol, *by Roberto Victorio ([37], mm. 1–5): indexogram showing the four main textural gestures of the piece (namely,* a, b, c, *and* d*) and associated partitions. We indicate referential partitions above the the delimitation of the gestures. Graphic produced by Parsemat® [11], adapted from [5].*

In this sense, the relations between the parts' distribution in partitions and complexes can give us some insights. For instance, in the gestures *a*, *b*, and *d*, there are some instruments that play a single role continuously, apparently promoting a function of stabilization and continuity. In gesture *a*, the cymbal and bells perform this role; in gesture *b*, this is performed by the tam-tam and marimba; and in gesture *d*, bass drum and timpani start this feature, and roto-toms and temple-blocks conclude it (Figure 9). Gesture *c* then stands out as the prominent location of contrast in the section, with an actual rupture between partitions (1.3), (13) and (2), the last pair with no instrumental intersection at all (full contraposition).

On the other side, the remaining instruments are responsible for the most noticeable textural changes. The interplay between temple-blocks, roto-toms, timpani, and bass drums dominate gesture *a*. In gesture *b*, the shift in instrumentation to mallets is remarkable: xylophone and vibraphone articulate the progression. In gesture *d*, the textural game consists of a pyramidal structure - additive process and further agglomeration forming blocks.

Mallets, in this case, contribute to individual parts (1), (2), and (4). These parts happen to be the natural mode of the performativity of these instruments, as there are two arms, each with two mallets. Part (1) occurs with the alternation of the arms, striking one note at a time, eventually

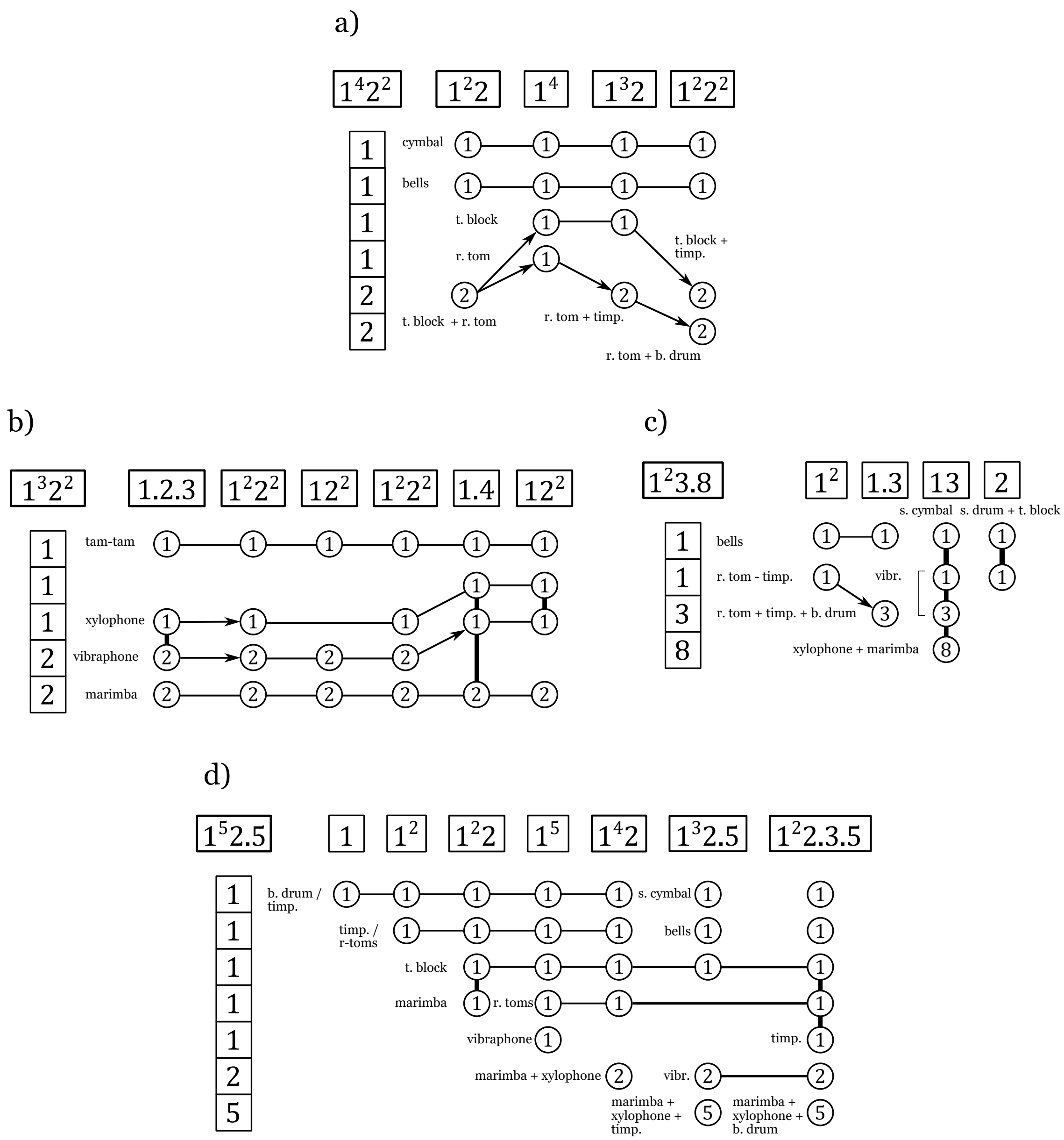

Figure 9: *Codex Troano, I - Portal do Sol, by Roberto Victorio ([37], mm. 1–5): partitions and instrumentation of the four main textural gestures of the piece (namely,* a, b, c, *and* d*). The first column shows the partitional complex of each gesture. The remaining ones are the partitions and indicate how the parts and instrumentation fit into the referential partition. Arrows indicate the changing of an instrument from one part to another. Vertical bars mean subsums. The bracket in gesture* c *shows how we understand the part [4] as a subsum in one of the mallet percussion instruments (in this case, the vibraphone). Original conception of the present author.*

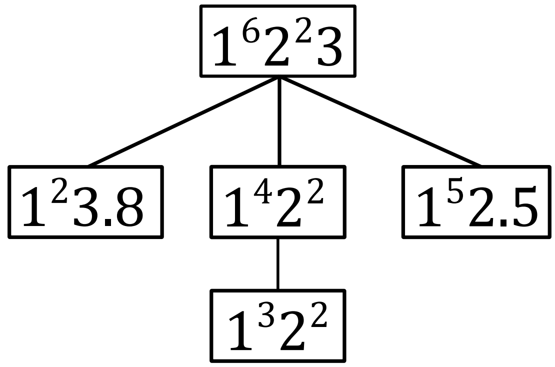

Figure 10: *Codex Troano, I - Portal do Sol, by Roberto Victorio ([37], mm. 1–5): global partitional. Original conception of the present author.*

articulating two successive notes with the same hand but another mallet. Part (2) implies a more obligatory alternation for a more fluid result, and mode (4) necessarily involves the two arms. Mallets can play three-note chords, but it is a more rare situation, as it implies in a slightly more unbalanced proposal.

All other eight instruments articulate only mode (1). Some of them have alternate use and can be played as a unitary instrument with variable timbre (for instance, the suspended cymbal and tam-tam could be played by one musician as a kit; the same observation for roto-toms and snare drum). This arrangement leads to the constitution of six unitary parts.

Part (3) in gesture *c* is a subsum of three unitary instruments (roto-tom, timpani and bass drum), as part (5) in gesture *d* is the subsum of two parts (2) and a single part (1) – marimba, xylophone and timpani. Finally, part (8) is the most massive and constitutes a compound unity, as it is the only part where the mallets act as a single massive instrument, each articulating a part (4) (in turn, this parts (4) are subsums of parts (2) and (1), as stated before). Here we are considering the two already existent parts, (1) and (3), subsumed to compose one of the parts (4) – we attribute it to vibraphone just for clarity, as it is only a conceptual option.

The next step is evaluating the interactions between the partitional complexes of each gesture to arrive at the partitional complex of the entire section. As we consider each partition as a set of subpartitions, subsums, or combinations of both, we conclude that the referential partition is $(1^6 2^2.3)$ (Figure 10).

V. Spatial Partitioning and conclusions

As an epilogue, we bring some preliminary observations about *Spatial Partitioning*, an application foreseen in previous works. As the analyst can choose a partitioning criterion, and the research directs towards contexts linked to the surface, applying partitioning to space became a natural outcome.

Fleeing from a platonic or phenomenological approach, we intend to start the work from the study of the historical use of spatial dispositions in concert music.

Concert music develops under the project of Italian Renaissance theater, which is based on the experience of perspective, that is, the most important dimensions in the pictorial field, which are width and depth (x and z dimensions). We in fact cannot see the development of a truly vertical spatial sense of sound on the Italian stage (y dimension) as it was abandoned after Monteverdi

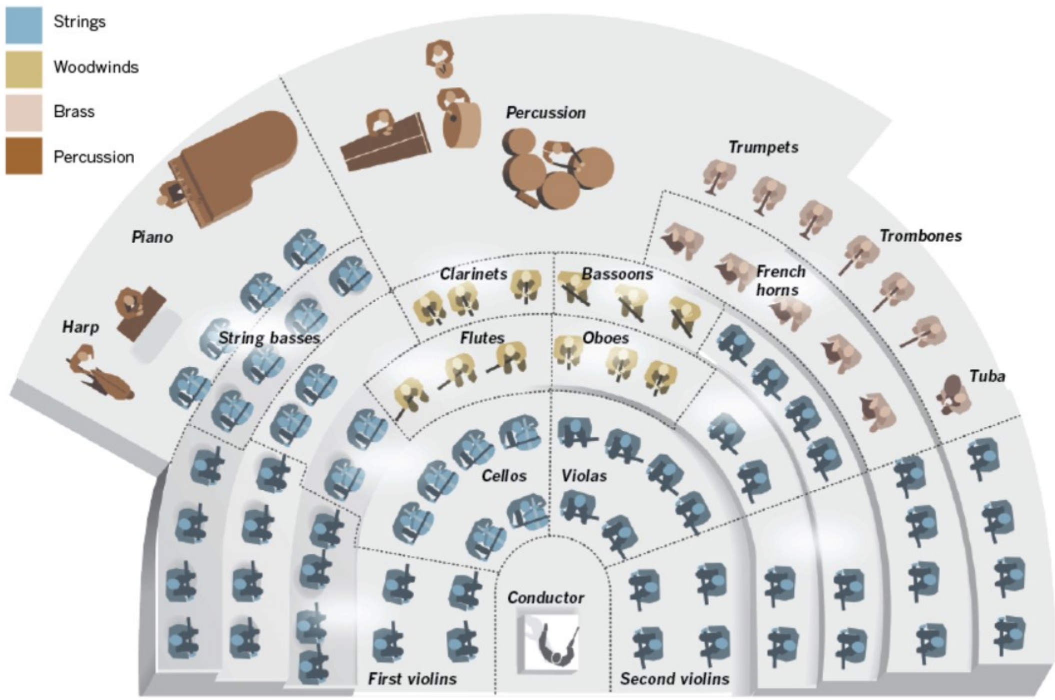

Figure 11: *Concentric spatial layout of the classical orchestra ([32]).*

and Gabrielli's experiences of more immersive spatialization within churches. Even today, the arrangement of speakers always favors the horizontal arrangement, with important but rare exceptions like multichannel speaker arrays.

Width is translated generically today by the term *panoramic*, with right and left channels being the basis of a stereo listening. This listening has as reference to the binaural nature of our auditory perception.

The element of depth translates in two ways, one more concrete and another, more virtual, related to the technology of music recording.

On stage, instruments that carry more individual and foreground expressions are positioned, usually, in the proscenium (i.e., as close as possible to the audience). This is the place of the soloist in concert with orchestra, or the singer in the opera arias. This position allows the sound content of the source to prevail over its reverberation – a drier sound.

Otherwise, instruments that perform background functions tend to be positioned at the rear of the stage. For example, the orchestra's choir or percussion instruments, which have a less tonic sound, as a rule, and additionally are wetter.

This dimension is translated into music technology as reverberation. Of course, there are other elements involved, such as the spectral profile, whose internal relationships change as the spatial recession is applied. Using too little reverb leads to a virtual approximation of the source, while a higher level of reverb tends to cause the sensation of withdrawal.

The orchestra organization seems to replicate this structure, with the orchestral sections being organized by concentric circles, which are successively occupied by strings, woods, brass and percussion. There are also some concerns about the relative volume of instruments – louder instruments that are allocated in the background, where they can thus balance with more delicate instruments in the proscenium (Figure 11).

Thus, it is hoped that an analysis can be developed from the placement of instruments on the stage, where the partitions would take place in two dimensions – in the groups that are established by the distribution of musical achievements in the horizontal range, in terms of the panoramic; and by the relative depth of the instruments on stage, both in terms of physical position and the driest or wettest level in the source/reverberation relationship.

Spatial partitioning can arise as a complement to the research about textural configurations. The study of the grounding of texture on the physical world is just beginning, and there is much room for expansions and new proposals. We hope that this research can fulfill a progressive departure from the domain of traditional music theory (mainly harmony and form) to an encounter (from a point of view derived from the linguistic-pragmatic turn) with music's social foundations. Along with all the proposals described in this article, a path is emerging to meet with other musical traditions where the body and physicality are already part of the daily musical experience.

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David Alves
Associate Director of Undergraduate Education

Marcelo Jardim
Director of the Cultural Artistic Sector

Maria José Bernardes Di Cavalcanti
Director of Public Outreach

João Vidal
Coordinator of the Graduate Program in Music

Aloysio Fagerlande
Coordinator of the Graduate Program Professional in Music

MusMat Research Group

COORDINATORS

Carlos Almada (leader)
Daniel Moreira
Hugo T. de Carvalho
Liduino Pitombeira

CURRENT MEMBERS

Alexandre Ferreira (Doctoral student)
Ana Miccolis (Doctoral student)
André Pinto (Undergraduate Research Project)
Ariane Petri (Doctoral student)
Bernardo Ramos (Doctoral student)
Carlos Mathias (Prof. UFF)
Claudia Usai (Master student)
Eduardo Cabral (Undergraduate Research Project)
Érico Bonfim (Doctoral student)
Filipe de Matos Rocha (Doctoral student)
João Penchel (Undergraduate Research Project)
Marco Antônio Ramos Feitosa (Pos-doc)
Mariane Batista dos Santos (Undergraduate)
Max Kühn (Master student)
Natanael Luciano de Matos (Undergraduate Research Project)
Nei Rocha (Prof. UFRJ)
Pedro Miguel de Moraes (Doctoral student)
Pedro Proença (Master student)
Pedro Zizels Ramos (Doctoral student)
Petrucio Viana (Prof. UFF)
Roberto Macedo (Doctoral student)
Rodrigo Furman (Master student)
Sérgio Ribeiro (Doctoral student)
Stefanella Boatto (Prof. UFRJ)
Vinicius Braga (Undergraduate Research Project)

ALUMNI

Adriel Viturino (Undergraduate Research Project)
Alexandre Avellar (Undergraduate Research Project)
André Codeço (Doctoral Student)
Desirée Mayr (Doctoral Student)
Fabio Monteiro (Master student)
Gabriel Mesquita (Master student)
Helder Oliveira (Doctoral student)
Jorge Santos (Master student)
Leandro Chrispim (Master student)
Marcel Castro Lima (Master student)
Rafael Fortes (Master student)
Rafael Soares Bezerra (Undergraduate Research Project)
Rodrigo Pascale (Undergraduate Research Project)

COLLABORATORS

Alexandre Schubert (Prof. UFRJ)
Fábio Adour (Prof. UFRJ)
Francisco Erivelton Aragão (Prof. UFC)
Pauxy Gentil-Nunes (Prof. UFRJ)
Raphael Sousa Santos
Rubem Medeiros