Musical Quasigroups

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Abstract: *A musical quasigroup is a musical groupoid in which all its left and right translation mappings are permutations. Some quasigroups of chords with a left (right, middle) identity element have been investigated. It is noted that the left (right, middle) identity element of a musical quasigroup is often associated with the root note of a musical chord. It is shown that chord inversions can be displayed by quasigroups. Examples of musical sequence of triads are constructed by using quasigroups. It is shown that a twelve-tone matrix can be created by using a quasigroup. Some examples of an n-tone composition chart using quasigroup are constructed. In particular, some charts showing a circle of fourths and fifths have been obtained by musical quasigroup. Some examples of ascending, descending, disjunct, and conjunct motions respectively have been described using quasigroups. Also, some examples of contrary, strict contrary, oblique, similar, and parallel motions have been given using quasigroups. The bass, treble, and grand staves have been described by a quasigroup. Examples on motion of a single melody is given having both conjunct and disjunct motions. Also, an example of an oblique motion in which one of its melodies is static while the other moves into conjunct and disjunct motions is demonstrated by a quasigroup. Some examples of a subquasigroup for pitch classes are constructed and verified with some musical examples. By the concept of a normal subquasigroup, a melodic motion which is disjunct is described to have a sub-melodic motion which is conjunct. It is shown that there are paired melodies which are not in contrary motion to each other but have paired sub-melodies which are in contrary (or strict-contrary) motion.*

Keywords*: Quasigroup. Subquasigroup. Normal subquasigroup. n-tone composition chart. Chord inversion. Melodic motion.*

I. Introduction

Mathematics in a number of ways has been applied to music theory in the past centuries. One of such interdisciplinary studies on music was carried out by Morris [10]. The relationship exhibited between mathematical reasoning and musical creativity has gained attention in recent years due to an increasing human interest in both subjects. Ben [4] studied the integral relationship between frequencies of tones, while Lewin [9] applied group theory to music by the concept of transformation theory using music intervals, and Wright [13] discusses on some abstract group relationship of musical structures. There are many research results showing the application of group theory to music, but when quasigroup theory and its relationship in music is

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mentioned, little or no direct result is obtained. From this study, we have seen that some musicians have been applying some concepts of quasigroup in one way or the other in their compositions. Therefore, this study seeks to examine a formal approach to music through quasigroups.

i. Preliminaries

Let Q be a set, then by binary operation $*$ on Q, we mean a mapping

$$* : Q \times Q \longrightarrow Q. \tag{1}$$

Then, the pair $(Q, *)$ is called a groupoid or a Magma. In this paper, we use the terminology adopted in Bruck [5].

Let $(Q, *)$ be a groupoid. If, for all $x, y \in Q$, $x * y = y * x$, then $(Q, *)$ is called a commutative groupoid. A groupoid $(Q, *)$ has an identity element $e \in Q$ if

$$\text{for every } x \in Q,\ x * e = e * x = x. \tag{2}$$

The order of a groupoid $(Q, *)$ denoted by $|Q|$ is the cardinality of Q. A groupoid $(Q, *)$ is said to be of finite order (that is, a finite groupoid) if $|Q|$ is a finite number, otherwise it is called an infinite groupoid.

We write xy instead of $x * y$ and designate that $*$ has a lower priority than juxtaposition among factors to be multiplied. For instance, $x * yz$ represents $x(yz)$. Let $(Q, *)$ be a groupoid, and let x be a fixed element in $(Q, *)$. Then the left and right translation maps of Q, L_x, $R_x : Q \longrightarrow Q$ are defined respectively by

$$yL_x = x * y \ \text{ and } \ yR_x = y * x, \tag{3}$$

for all $x, y \in Q$. In the above definition, we adopt the definition of left and right translation mappings used in Jaiyeola [8]. Let $(Q, *)$ be a groupoid. An element $a \in Q$ is said to satisfy the left (right) cancellation law if, for all $x, y \in Q$, $x = y$ if and only if

$$a * x = a * y \ (x * a = y * a). \tag{4}$$

In a groupoid $(Q, *)$, the left and the right translation mappings need not be permutations. A groupoid $(Q, *)$ is said to be associative if for all $a, b, c \in Q$, $(ab)c = a(bc)$. An associative groupoid is a semigroup. For more studies on binary operation, integers and groupoids, readers may check [1, 6, 7, 5]. Let $(Q, *)$ be a groupoid. If each of the equations

$$a * x = b \ \text{ and } \ y * a = b \tag{5}$$

has unique solutions in Q for x and y respectively, then $(Q, *)$ is called a *quasigroup*. Thus, a groupoid $(Q, *)$ is a quasigroup if its left and right translation mappings are permutations. An element u is a left identity element of a quasigroup $(Q, *)$ if for all $x \in Q$, $u * x = x$. An element v is a right identity element of a quasigroup $(Q, *)$ if for all $x \in Q$, $x * v = x$. An element w is a middle identity element of a quasigroup $(Q, *)$ if for all $x \in Q$, $x * x = w$. For more on Quasigroup, readers may check [2, 3, 8, 12].

Let $\mathbb{Z}$ be the set of integers and consider $x, y, z \in \mathbb{Z}$. The positive integer z is called the greatest divisor of integers x and y if any divisor of x and y is also a divisor of z, and z is a divisor of both x and y. An integer $p > 1$ is called a prime if its divisors are $\pm p$ and ± 1 only.

Two integers x and y are said to be relatively prime if their greatest common divisor is 1. As a consequence, the equation

$$1 = ax + by \tag{6}$$

holds for some integers $a, b \in \mathbb{Z}$.

Let $\mathbb{Z}$ be a set of integers and let $m, n \in \mathbb{Z}$. By modulo operation, abbreviated as *mod*, on $\mathbb{Z}$ we mean, n divides m and has a remainder $r \in \mathbb{Z}$, written as $m \ mod \ n = r$. For a fixed $n \in \mathbb{Z}$, the set of all r satisfying $m \ mod \ n = r$ for all $m \in \mathbb{Z}$ is denoted by $\mathbb{Z}_n$. The notation $\mathbb{Z}_n$ is called the set of integers modulo n. Also, the product of two elements g and h in $\mathbb{Z}_n$ can be defined as $gh \ (mod \ n)$. On a music scale, modulo operation is a computation which restarts the music notes once a certain value is attained.

We denote the ordered sequence of pitch classes in the angle brackets $\langle \ \rangle$. The function of a musical note can be changed by the use of flats and sharps. When two notes sound the same but are named as different notes, they are said to be enharmonic equivalent to each other. Scales, chords, keys and intervals can as well be enharmonically named.

II. Main Results

In this section, some concepts and examples of quasigroups are applied to music. Some quasigroups of chords with a left (right, middle) identity element are investigated. It is noted that the left (right, middle) identity element of a finite quasigroup is often associated with the root note of the musical chord. It is obtained that quasigroups with a left (right, middle) identity element provide options for harmonizations and progressions. Using finite quasigroups, triads, tetrads, pentads, and hexads with their inversions can be described respectively. Examples of musical sequence of triads are constructed by quasigroups. It has been shown that an n-tone row chart can be obtained by a musical quasigroup. The chart showing a circle of fourths and circle of fifths have been obtained by a musical quasigroup. Some examples describing the descending, ascending, disjunct and conjunct motions respectively have been given. By the motion of a single melody, the structure of a musical staff using quasigroups is investigated. Similarly, some examples of contrary, strict contrary, oblique, similar, and parallel motions have been described using quasigroups. Examples of subquasigroups have been constructed for pitch classes and are verified with some musical examples. Also, the concept of a normal subquasigroup is applied to motion of melodies. It is shown that there are paired melodies which are not in contrary motion to each other but have paired sub-melodies which are in contrary (or strict-contrary) motion.

i. Musical Quasigroups

In this subsection, some concepts of quasigroups are defined in music, and some examples are given. We demonstrated some orders of progressions involving I, IV, and V using quasigroups.

A set is a collection of well-defined objects. An object of a set is called a member or element of the given set. If a set is a collection of well-defined musical objects then these objects are also called musical elements or members of the set. Examples of sets are F-major scale and C-major chord. Each object in F-major scale and C-major chord is a musical element and it is also called a note.

Definition 1. Let us consider Q a set of musical notes. If the product of any two musical notes of Q under an operation is also a member of Q, then such an operation is called a binary operation.

For an example, consider a set of musical notes of G-major chord. An operation on G-major chord will be called a binary operation if the product of any two members of G-major chord is also a member of G-major chord.

Definition 2. A set Q of musical elements on which a binary operation $*$ is defined is called a musical quasigroup if, for all $u, v \in Q$, $u * v \in Q$ and there exist unique $x, y \in Q$ satisfying $u * x = v$ and $y * u = v$.

We denote the ordered sequence of musical elements in the angle brackets $\langle\ \rangle$. As applied below, if $\{I, IV, V\}$ is a set of pitch classes from a given scale, then $\langle I, IV, V\rangle$ is an ordered sequence from $\{I, IV, V\}$. We also assigned a sequence of pitch classes to musical elements of a given set. As used in Example 1, $\langle I, IV, V\rangle = \langle 0, 1, 2\rangle$ implies that I, IV and V represent $0, 1$ and 2 respectively, without a change in position.

Example 1. Let $Q = \{0, 1, 2\}$ be a set of musical elements. Define the binary operation $*$ on Q as $a * b = 2a + b\ (mod\ 3)$ then $(Q, *)$ is a musical quasigroup with a left identity element (Table 1).

Table 1: *A musical quasigroup with a left identity element 0.*

$*$	0	1	2
0	0	1	2
1	2	0	1
2	1	2	0

Let $Q = \{I, IV, V\}$ be the set of notes of a major scale in a progression, and let $\langle 0, 1, 2\rangle = \langle I, IV, V\rangle$. Then we have (Table 2):

Table 2: *A quasigroup of progression with a left identity element I.*

$*$	I	IV	V
I	I	IV	V
IV	V	I	IV
V	IV	V	I

The order of progression from left to right of the first, second and third rows are $I - IV - V$, $V - I - IV$, and $IV - V - I$ respectively.

Example 2. Let $Q = \{0, 1, 2\}$ be a set of musical elements. Define the binary operation $*$ on Q as $a * b = a + 2b\ (mod\ 3)$ then $(Q, *)$ is a musical quasigroup with a right identity element (Table 3).

Table 3: *A musical quasigroup with a right identity element 0.*

$*$	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Let $Q = \{I, IV, V\}$ be the set of notes of a major scale in a progression, and let $\langle 0, 1, 2\rangle = \langle I, IV, V\rangle$. Then we have (Table 4):

Table 4: *A quasigroup of progression with a right identity element I*

$*$	I	IV	V
I	I	V	IV
IV	IV	I	V
V	V	IV	I

The order of progression from left to right of the first, second and third rows are $I - V - IV$, $IV - I - V$, and $V - IV - I$ respectively.

Example 3. Let $Q = \{0, 1, 2\}$ be a set of musical elements. Define the binary operation $*$ on Q as $a * b = 2a + b + 1 \ (mod\ 3)$ then $(Q, *)$ is a musical quasigroup with a middle identity element (Table 5).

Table 5: *A musical quasigroup with a middle identity element 1*

$*$	0	1	2
0	1	2	0
1	0	1	2
2	2	0	1

Let $Q = \{I, IV, V\}$ be the set of notes of a major scale in a progression, and let $\langle 0, 1, 2\rangle = \langle I, IV, V\rangle$. Then we have (Table 6):

Table 6: *A quasigroup of progression with a middle identity element IV*

$*$	I	IV	V
I	IV	V	I
IV	I	IV	V
V	V	I	IV

The order of the progression from left to right of the first, second and third rows are $IV - V - I$, $I - IV - V$, and $V - I - IV$ respectively.

ii. Quasigroups of triads

In this subsection, we present some triads using a quasigroup of order three. The quasigroup tables obtained show some triad inversions for major triad when viewed in rows or columns. We also present the order in which a minor, diminished and augmented triads and their inversions can be displayed respectively by a quasigroup of order three.

Let $Q = \{I, III, V\}$ be the set of notes of the major scale in a triad. By Example 1, let $\langle I, III, V\rangle$ represents $\langle 0, 1, 2\rangle$. Then the Table 7 of major triad inversions is obtained.

Table 7: *C-Major chord from a Quasigroup with a left identity element*

$*$	C	E	G
C	C	E	G
E	G	C	E
G	E	G	C

Remark 1. The left identity element of a musical quasigroup of triads is assigned the root note of the triad. From Table 1, $(Q, \cdot)$ is a quasigroup with a left identity element and Table 7 is such that the root note occupies the left identity element position whenever $(Q, \cdot)$ has a left identity element. Example, for C-Major triad, C is the root note.

Let $Q = \{I, III, V\}$ be the set of notes of the major scale in a triad. By Example 2, let $\langle I, III, V\rangle$ represents $\langle 0, 1, 2\rangle$. Then the Table 8 of major triad inversions is obtained.

Table 8: *C-Major chord from a Quasigroup with a right identity element.*

$*$	C	E	G
C	C	G	E
E	E	C	G
G	G	E	C

Remark 2. The right identity element of a musical quasigroup of triads is assigned the root note of the triad. From Table 3, $(Q, \cdot)$ is a quasigroup with a right identity element and Table 8 is such that the root note occupies the right identity element position whenever $(Q, \cdot)$ has a right identity element.

Remark 3. Let $Q = \{I, III, V\}$ be the set of notes of the major scale in a triad. Examples 1 and 2 can also be applied for a minor, diminished, and augmented triads as follows:

For a minor triad, let $\langle I, III^{\flat}, V\rangle$ represents $\langle 0, 1, 2\rangle$.
For a diminished triad, let $\langle I, III^{\flat}, V^{\flat}\rangle$ represents $\langle 0, 1, 2\rangle$.
For an augmented triad, let $\langle I, III, V^{\sharp}\rangle$ represents $\langle 0, 1, 2\rangle$.

Remark 4. Quasigroup of triads with a middle identity element is similarly deduced.

iii. Quasigroup with examples of musical sequences from triads

Let $Q = \{I, III, V\}$ be the set of notes of the major scale in a triad. By Examples 1 and 2, let $\langle I, III, V\rangle$ represents $\langle 0, 1, 2\rangle$. Consider Tables 9, 10, and 11 of major triad inversions:

Table 9: *C-Major.*

$*$	C	E	G
C	C	E	G
E	G	C	E
G	E	G	C

Table 10: *F-major.*

$*$	F	A	C
F	F	C	A
A	A	F	C
C	C	A	F

Table 11: *G-Major*

$*$	G	B	D
G	G	D	B
B	B	G	D
D	D	B	G

In Figure 1, examples (i), (ii), and (iii) are obtained from Tables 7, 10, and 11, by using their 1st row-1st column-3rd column respectively, then the process is repeated in each case. Example (iv) is obtained from Tables 7 by using its 1st row-3rd column-1st row, then the process is repeated.

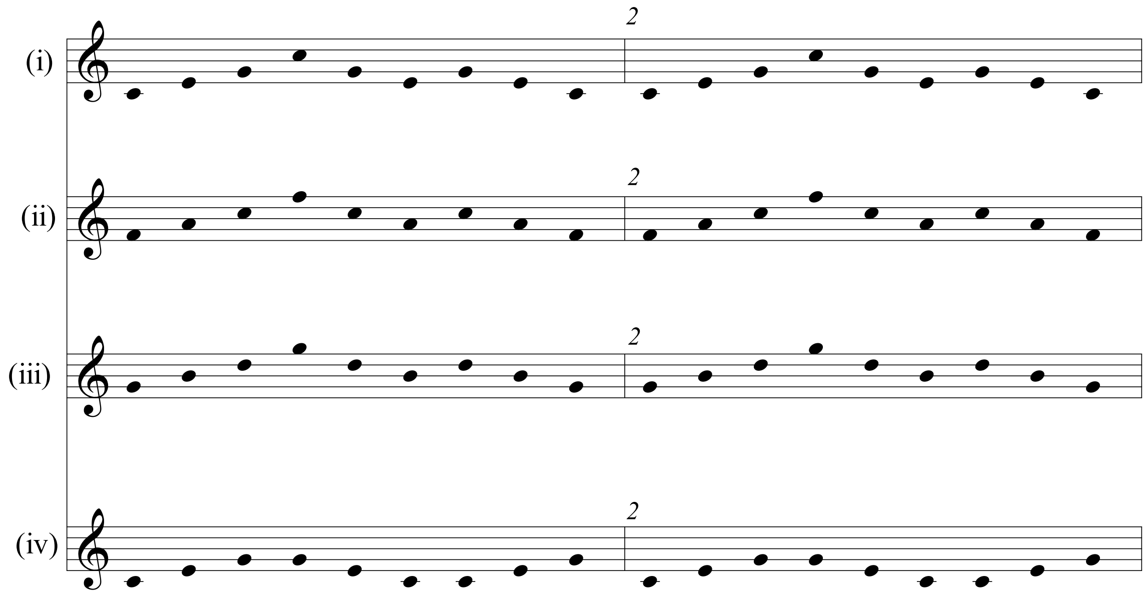

Figure 1: *Examples of musical sequences obtained via quasigroups.*

iv. A Twelve-Tone Matrix

In this subsection, we present the connection between n-tone composition chart with quasigroup. It is shown that a twelve-tone matrix can be created by using a quasigroup.

Example 4. (Wright [13]) Let $a_n \in \mathbb{Z}_{12}$ and let the ordered pair (i, j) be the position at row i and column j, such that the entries of the original row are labelled: $a_1 = [0]; a_2 = [3]; a_3 = [2]; a_4 = [5]; a_5 = [4]; a_6 = [8]; a_7 = [1]; a_8 = [10]; a_9 = [11]; a_{10} = [9]; a_{11} = [7]$; and $a_{12} = [6]$. Then the first column inversion in $\mathbb{Z}_{12}$ are given as : $-a_1 = [0]; -a_2 = [9]; -a_3 = [10]; -a_4 = [7]; -a_5 = [8], -a_6 = [4]$ and each position of the chart with the elements of $\mathbb{Z}_{12}$ corresponding to the appropriate note class is given as $a_j - a_i$. Then a twelve-tone matrix is obtained.

Remark 5. Example 4 is a known construction to musicians. It is called a quasigroup if it is presented in the form of a Cayley table. See Table 12 of Example 5 for an example.

Example 5. Let $a_n \in \mathbb{Z}_{12}$ and let the ordered pair (i, j) be the position at row i and column j. We assign the subscripts $a_0, a_1, a_2, \ldots, a_n$ of the entries thus: $a_0 = [0]; a_1 = [3]; a_2 = [2]; a_3 = [5]; a_4 = [4]; a_5 = [8]; a_6 = [1]; a_7 = [10]; a_8 = [11]; a_9 = [9]; a_{10} = [7]$; and $a_{11} = [6]$. Then

(i) the first column inversion in $\mathbb{Z}_{12}$ are given as :$-a_0 = [0]; -a_1 = [9]; -a_2 = [10]; -a_3 = [7]; -a_4 = [8], -a_5 = [4]; -a_6 = [11]; -a_7 = [2]; -a_8 = [1]; -a_9 = [3]; -a_{10} = [5]; -a_{11} = [6]$; and

(ii) the pair $(\mathbb{Z}_{12}, *)$ such that $*(i, j) = a_j - a_i \in \mathbb{Z}_{12}$ for all $a_i, a_j \in \mathbb{Z}_{12}$ is a quasigroup. The entry in the position $(9, 6)$ is $*(9, 6) = a_6 - a_9 = [1] - [9] = -[8] = [4]$. Therefore, $9 * 6 = 4$.

For more details, see Tables 12 and 13.

Table 12: *Multiplication Table for* $(\mathbb{Z}_{12}, *)$ *by Example 5.*

$*$	0	1	2	3	4	5	6	7	8	9	10	11
0	0	3	2	5	4	8	1	10	11	9	7	6
1	9	0	11	2	1	5	10	7	8	6	4	3
2	10	1	0	3	2	6	11	8	9	7	5	4
3	7	10	9	0	11	3	8	5	6	4	2	1
4	8	11	10	1	0	4	9	6	7	5	3	2
5	4	7	6	9	8	0	5	2	3	1	11	10
6	11	2	1	4	3	7	0	9	10	8	6	5
7	2	5	4	7	6	10	3	0	1	11	9	8
8	1	4	3	6	5	9	2	11	0	10	8	7
9	3	6	5	8	7	11	4	1	2	0	10	9
10	5	8	7	10	9	1	6	3	4	2	0	11
11	6	9	8	11	10	2	7	4	5	3	1	0

Table 13: *The chart of pitch classes from key F by Example 5.*

$*$	0	1	2	3	4	5	6	7	8	9	10	11
0	F	$G^{\sharp}$	G	$B^{\flat}$	A	$C^{\sharp}$	$F^{\sharp}$	$D^{\sharp}$	E	D	C	B
1	D	F	E	G	$F^{\sharp}$	$B^{\flat}$	$D^{\sharp}$	C	$C^{\sharp}$	B	A	$G^{\sharp}$
2	$D^{\sharp}$	$F^{\sharp}$	F	$G^{\sharp}$	G	B	E	$C^{\sharp}$	D	C	$B^{\flat}$	A
3	C	$D^{\sharp}$	D	F	E	$G^{\sharp}$	$C^{\sharp}$	$B^{\flat}$	B	A	G	$F^{\sharp}$
4	$C^{\sharp}$	E	$D^{\sharp}$	$F^{\sharp}$	F	A	D	B	C	$B^{\flat}$	$G^{\sharp}$	G
5	A	C	B	D	$C^{\sharp}$	F	$B^{\flat}$	G	$G^{\sharp}$	$F^{\sharp}$	E	$D^{\sharp}$
6	E	G	$F^{\sharp}$	A	$G^{\sharp}$	C	F	D	$D^{\sharp}$	$C^{\sharp}$	B	$B^{\flat}$
7	G	$B^{\flat}$	A	C	B	$D^{\sharp}$	$G^{\sharp}$	F	$F^{\sharp}$	E	9	$C^{\sharp}$
8	$F^{\sharp}$	A	$G^{\sharp}$	B	$B^{\flat}$	D	G	E	F	$D^{\sharp}$	$C^{\sharp}$	C
9	$G^{\sharp}$	B	$B^{\flat}$	$C^{\sharp}$	C	E	A	$F^{\sharp}$	G	F	$D^{\sharp}$	D
10	$B^{\flat}$	$C^{\sharp}$	C	$D^{\sharp}$	D	$F^{\sharp}$	B	$G^{\sharp}$	A	G	F	E
11	B	D	$C^{\sharp}$	E	$D^{\sharp}$	G	C	A	$B^{\flat}$	$G^{\sharp}$	$F^{\sharp}$	F

Remark 6. Table 12 is not associative. For instance, $4 * (5 * 6) = 4$ and $(4 * 5) * 6 = 9$. Thus, $4 * (5 * 6) \neq (4 * 5) * 6$.

Remark 7. Example 5 shows that a twelve-tone matrix can be obtained by using a quasigroup.

Remark 8. Wright [13] explored on creating an n-tone row chart using modular Arithmetic. From Examples 4 and 5 we note that, given an original row $a_0 = [0], a_1, a_2, \cdots, a_n$ from $\mathbb{Z}_n$ for some $n \in \mathbb{Z}_n^+$, the $n \times n$ chart constructed by taking the ordered pair entry (i, j) such that $*(i, j) = a_j - a_i$ where $a_i, a_j \in \mathbb{Z}_n$, is an n-tone musical quasigroup.

v. Construction of an n-Tone Composition Chart

In this subsection, we construct some examples of an n-tone composition chart using quasigroups. We apply Muktibodh [11] in the construction. In particular, some charts showing circles of fourths and fifths have been obtained by musical quasigroups.

Definition 3. Let $\mathbb{Z}_n = \{0, 1, 3, \cdots, n-1\}$ for $n \geq 3$, be the set of pitch classes. Define a binary operation $*$ on $\mathbb{Z}_n$ as $a * b = xa + yb (mod\ n)$ where x, y are two distinct elements in $\mathbb{Z}_n \backslash \{0\}$ which are primes such that $(x, y) = 1$, and $n = x + y$ and $+$ the addition of integers under mod n. An n-tone musical quasigroup $(\mathbb{Z}_n(x, y), *)$ is obtained.

Example 6. Let $Q = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$ be the set of pitch classes. Define the binary operation $*$ on Q as $a * b = 7a + 5b(mod\ 12)$ then $(Q, *)$ is a twelve-tone musical quasigroup (see Tables 14 and 15).

Table 14: *Multiplication Table for* $(\mathbb{Z}_{12}, *)$ *by Example 6.*

$*$	0	1	2	3	4	5	6	7	8	9	10	11
0	0	5	10	3	8	1	6	11	4	9	2	7
1	7	0	5	10	3	8	1	6	11	4	9	2
2	2	7	0	5	10	3	8	1	6	11	4	9
3	9	2	7	0	5	10	3	8	1	6	11	4
4	4	9	2	7	0	5	10	3	8	1	6	11
5	11	4	9	2	7	0	5	10	3	8	1	6
6	6	11	4	9	2	7	0	5	10	3	8	1
7	1	6	11	4	9	2	7	0	5	10	3	8
8	8	1	6	11	4	9	2	7	0	5	10	3
9	3	8	1	6	11	4	9	2	7	0	5	10
10	10	3	8	1	6	11	4	9	2	7	0	5
11	5	10	3	8	1	6	11	4	9	2	7	0

Table 15: *The chart of pitch classes from key F by Example 6.*

$*$	0	1	2	3	4	5	6	7	8	9	10	11
0	F	$B^\flat$	$D^\sharp$	$G^\sharp$	$C^\sharp$	$F^\sharp$	B	E	A	D	G	C
1	C	F	$B^\flat$	$D^\sharp$	$G^\sharp$	$C^\sharp$	$F^\sharp$	B	E	A	D	G
2	G	C	F	$B^\flat$	$D^\sharp$	$G^\sharp$	$C^\sharp$	$F^\sharp$	B	E	A	D
3	D	G	C	F	$B^\flat$	$D^\sharp$	$G^\sharp$	$C^\sharp$	$F^\sharp$	B	E	A
4	A	D	G	C	F	$B^\flat$	$D^\sharp$	$G^\sharp$	$C^\sharp$	$F^\sharp$	B	E
5	E	A	D	G	C	F	$B^\flat$	$D^\sharp$	$G^\sharp$	$C^\sharp$	$F^\sharp$	B
6	B	E	A	D	G	C	F	$B^\flat$	$D^\sharp$	$G^\sharp$	$C^\sharp$	$F^\sharp$
7	$F^\sharp$	B	E	A	D	G	C	F	$B^\flat$	$D^\sharp$	$G^\sharp$	$C^\sharp$
8	$C^\sharp$	$F^\sharp$	B	E	A	D	G	C	F	$B^\flat$	$D^\sharp$	$G^\sharp$
9	$G^\sharp$	$C^\sharp$	$F^\sharp$	B	E	A	D	G	C	F	$B^\flat$	$D^\sharp$
10	$D^\sharp$	$G^\sharp$	$C^\sharp$	$F^\sharp$	B	E	A	D	G	C	F	$B^\flat$
11	$B^\flat$	D	$G^\sharp$	$C^\sharp$	$F^\sharp$	B	E	A	D	G	C	F

Remark 9. Table 15 above displays a circle of fourths' progression when it is read from left to right horizontally, and displays a circle of fifths' progression, when it is read from right to left horizontally.

Definition 4. Let $\mathbb{Z}_n = \{0, 1, 3, \cdots, n-1\}$ for $n \geq 3$, be the set of pitch classes and let p be a prime number such that $\mathbb{Z}_p(x, y)$ is a groupoid and $x + y = p$, $(x, y) = 1$. A p-tone musical quasigroup $\mathbb{Z}_p(x, y)$ is obtained.

Example 7. Let $Q = \{0, 1, 2, 3, 4, 5, 6\}$ be the set of pitch classes. Define the binary operation $*$ on Q as $a * b = 2a + 5b(mod\ 7)$ then $(Q, *)$ is a seven-tone musical quasigroup (see Tables 16 and 17).

Table 16: *Multiplication Table for* $(Q, *)$ *by Example 7.*

$*$	0	1	2	3	4	5	6
0	0	5	3	1	6	4	2
1	2	0	5	3	1	6	4
2	4	2	0	5	3	1	6
3	6	4	2	0	5	3	1
4	1	6	4	2	0	5	3
5	3	1	6	4	2	0	5
6	5	3	1	6	4	2	0

Table 17: *The chart of pitch classes from key F by Example 7.*

$*$	0	1	2	3	4	5	6
0	F	$B^\flat$	$G^\sharp$	$F^\sharp$	B	A	G
1	G	F	$B^\flat$	$G^\sharp$	$F^\sharp$	B	A
2	A	G	F	$B^\flat$	$G^\sharp$	$F^\sharp$	B
3	B	A	G	F	$B^\flat$	$G^\sharp$	$F^\sharp$
4	$F^\sharp$	B	A	G	F	$B^\flat$	$G^\sharp$
5	$G^\sharp$	$F^\sharp$	B	A	G	F	$B^\flat$
6	$B^\flat$	$G^\sharp$	$F^\sharp$	B	A	G	F

Definition 5. Let $\mathbb{Z}_n = \{0, 1, 3, \cdots, n-1\}$ for $n \geq 3$, be the set of pitch classes. Define a binary operation $*$ on $\mathbb{Z}_n$ as $a * b = xa + yb(mod\ n)$ where x, y are elements in $\mathbb{Z}_n \backslash \{0\}$ and $x = y$. For a fixed prime n and varying x and y, an n-tone musical quasigroup $(\mathbb{Z}_n, *)$ is obtained.

Example 8. Let $Q = \{0, 1, 2, 3, 4, 5, 6\}$ be the set of pitch classes. Define the binary operation $*$ on Q as $a * b = 2a + 2b(mod\ 7)$ then $(Q, *)$ is a seven-tone musical quasigroup (see Tables 18 and 19).

Table 18: *Multiplication Table for* $(Q, *)$ *by Example 8.*

$*$	0	1	2	3	4	5	6
0	0	2	4	6	1	3	5
1	2	4	6	1	3	5	0
2	4	6	1	3	5	0	2
3	6	1	3	5	0	2	4
4	1	3	5	0	2	4	6
5	3	5	0	2	4	6	1
6	5	0	2	4	6	1	3

Table 19: *The chart of pitch classes from key F by Example 8.*

$*$	0	1	2	3	4	5	6
0	F	G	A	B	$F^\sharp$	$G^\sharp$	$B^\flat$
1	G	A	B	$F^\sharp$	$G^\sharp$	$B^\flat$	F
2	A	B	$F^\sharp$	$G^\sharp$	$B^\flat$	F	G
3	B	$F^\sharp$	$G^\sharp$	$B^\flat$	F	G	A
4	$F^\sharp$	$G^\sharp$	$B^\flat$	F	G	A	B
5	$G^\sharp$	$B^\flat$	F	G	A	B	$F^\sharp$
6	$B^\flat$	F	G	A	B	$F^\sharp$	$G^\sharp$

Definition 6. Let $\mathbb{Z}_n = \{0, 1, 3, \cdots, n-1\}$ for $n \geq 3$, be the set of pitch classes. Define a binary operation $*$ on $\mathbb{Z}_n$ as $a * b = xa + yb(mod\ n)$ where x, y are elements in $\mathbb{Z}_n \backslash \{0\}$ and $x = 1$ and $y = n - 1$. For a fixed integer n and varying x and y, an n-tone musical quasigroup $(\mathbb{Z}_n, *)$ is obtained.

Example 9. Let $Q = \{0, 1, 2, 3, 4, 5, 6\}$ be the set of pitch classes. Define the binary operation $*$ on Q as $a * b = a + 6b(mod\ 7)$ then $(Q, *)$ is a seven-tone musical quasigroup (see Tables 20 and 21).

Table 20: *Multiplication Table for $(Q, *)$ by Example 9.*

$*$	0	1	2	3	4	5	6
0	0	6	5	4	3	2	1
1	1	0	6	5	4	3	2
2	2	1	0	6	5	4	3
3	3	2	1	0	6	5	4
4	4	3	2	1	0	6	5
5	5	4	3	2	1	0	6
6	6	5	4	3	2	1	0

Table 21: *The chart of pitch classes from key F by Example 9.*

$*$	0	1	2	3	4	5	6
0	F	B	$B^\flat$	A	$G^\sharp$	G	$F^\sharp$
1	$F^\sharp$	F	B	$B^\flat$	A	$G^\sharp$	G
2	G	$F^\sharp$	F	B	$B^\flat$	A	$G^\sharp$
3	$G^\sharp$	G	$F^\sharp$	F	B	$B^\flat$	A
4	A	$G^\sharp$	G	$F^\sharp$	F	B	$B^\flat$
5	$B^\flat$	A	$G^\sharp$	G	$F^\sharp$	F	B
6	B	$B^\flat$	A	$G^\sharp$	G	$F^\sharp$	F

Definition 7. Let $\mathbb{Z}_n = \{0, 1, 3, \cdots, n-1\}$ for $n \geq 3$, be the set of pitch classes. Define a binary operation $*$ on $\mathbb{Z}_n$ as $a * b = xa + yb(mod\ n)$ where x, y are elements in $\mathbb{Z}_n \backslash \{0\}$ and $(x, y) = 1$, $x + y = n$ and $|x - y|$ is a minimum. For a fixed integer n and varying x and y, an n-tone musical quasigroup $(\mathbb{Z}_n, *)$ is obtained.

Example 10. Let $Q = \{0,1,2,3,4,5,6,7\}$ be the set of pitch classes. Define the binary operation $*$ on Q as $a * b = 5a + 3b (mod\ 8)$ then $(Q, *)$ is an eight-tone musical quasigroup (see Tables 22 and 23).

Table 22: *Multiplication Table for* $(Q,*)$ *by Example 10.*

$*$	0	1	2	3	4	5	6	7
0	0	3	6	1	4	7	2	5
1	5	0	3	6	1	4	7	2
2	2	5	0	3	6	1	4	7
3	7	2	5	0	3	6	1	4
4	4	7	2	5	0	3	6	1
5	1	4	7	2	5	0	3	6
6	6	1	4	7	2	5	0	3
7	3	6	1	4	7	2	5	0

Table 23: *The chart of pitch classes from key F by Example 10.*

$*$	0	1	2	3	4	5	6	7
0	F	$G^\sharp$	B	$F^\sharp$	A	C	G	$B^\flat$
1	$B^\flat$	F	$G^\sharp$	B	$F^\sharp$	A	C	G
2	G	$B^\flat$	F	$G^\sharp$	B	$F^\sharp$	A	C
3	C	G	$B^\flat$	F	$G^\sharp$	B	$F^\sharp$	A
4	A	C	G	$B^\flat$	F	$G^\sharp$	B	$F^\sharp$
5	$F^\sharp$	A	C	G	$B^\flat$	F	$G^\sharp$	B
6	B	$F^\sharp$	A	C	G	$B^\flat$	F	$G^\sharp$
7	$G^\sharp$	B	$F^\sharp$	A	C	G	$B^\flat$	F

vi. Motion of a single melody by Quasigroups

In this subsection, some motions obtained from three consecutive semitones in a chromatic scale are described by a qausigroup of order three. Some motions of a five-tone sequence are demonstrated by a quasigroup of order five. It is shown that each row and column of a finite musical quasigroup is a melodic motion on its own right. These examples of quasigroups of musical sequences considered, demonstrated the descending, ascending, disjunct, and conjunct melodic motions. We describe the order of the notes on the bass, treble, and grand staves using a quasigroup.

Melodic motions between three consecutive semitones in a chromatic scale are demonstrated by quasigroup table of order three as follows: We consider $C, C^\sharp, D$; $A, A^\sharp, B$ and $B^\flat, B, C$ as examples. By Example 2, let $\langle 0,1,2\rangle = \langle C, C^\sharp, D\rangle$, $\langle 0,1,2\rangle = \langle A, A^\sharp, B\rangle$ and $\langle 0,1,2\rangle = \langle B^\flat, B, C\rangle$ respectively. Then we have Tables 24, 25, and 26:

Table 24: *From Key C*

$*$	C	$C^\sharp$	D
C	C	D	$C^\sharp$
$C^\sharp$	$C^\sharp$	C	D
D	D	$C^\sharp$	C

Table 25: *From Key $B^\flat$*

$*$	$B^\flat$	B	C
$B^\flat$	$B^\flat$	C	B
B	B	$B^\flat$	C
C	C	B	$B^\flat$

Table 26: *From Key A*

$*$	A	$A^\sharp$	B
A	A	B	$A^\sharp$
$A^\sharp$	$A^\sharp$	A	B
B	B	$A^\sharp$	A

From the these tables we note the followings:

(i) The first column of each of these tables displays an ascending motion when read from the top to the bottom; and shows a descending motion when read from the bottom to the top.

(ii) Each of the columns and rows from these tables displays a conjunct motion.

Example 11. Let $Q = \{0, 1, 2, 3, 4\}$ be the set of pitch classes. Define the binary operation $*$ on Q as $a * b = 2a + 2b(mod\ 5)$ then $(Q, *)$ is a five-tone musical quasigroup (see Table 27).

Table 27: *A quasigroup table by Example 11*

$*$	0	1	2	3	4
0	0	2	4	1	3
1	2	4	1	3	0
2	4	1	3	0	2
3	1	3	0	2	4
4	3	0	2	4	1

Let $F, G, A, B^\flat, C$ be musical notes from key F and let $\langle 0, 1, 2, 3, 4\rangle = \langle F, G, A, B^\flat, C\rangle$. Then by Table 27, we have Table 28:

Table 28: *The chart of pitch-classes from key F by Table 27*

$*$	F	G	A	$B^\flat$	C
F	F	A	C	G	$B^\flat$
G	A	C	G	$B^\flat$	F
A	C	G	$B^\flat$	F	A
$B^\flat$	G	$B^\flat$	F	A	C
C	$B^\flat$	F	A	C	G

From Table 28, we note the followings:

(i) The main diagonal of the table shows a descending motion when read from the top to the bottom; and shows an ascending motion when read from the bottom to the top.

(ii) The table shows a disjunct melodic motion for each row and column.

Example 12. Let $Q = \{0, 1, 2, 3, 4\}$ be the set of musical notes in a five-tone sequence. Let $*$ be a binary operation defined on Q as in Table 29.

Table 29: *A five-tone quasigroup with a left identity element* 0

$*$	0	1	2	3	4
0	0	1	2	3	4
1	4	0	1	2	3
2	3	4	0	1	2
3	2	3	4	0	1
4	1	2	3	4	0

Then $(Q, *)$ is a quasigroup with a left identity element. Clearly, $(Q, *)$ is not a commutative groupoid and it is not a group. Let $F, G, A, B^\flat, C$ be musical notes from key F and let $\langle 0, 1, 2, 3, 4\rangle = \langle F, G, A, B^\flat, C\rangle$. Then by Table 29, we have Table 30:

Table 30: *A five-tone chart by Table 29 from key F*

$*$	F	G	A	$B^\flat$	C
F	F	G	A	$B^\flat$	C
G	C	F	G	A	$B^\flat$
A	$B^\flat$	C	F	G	A
$B^\flat$	A	$B^\flat$	C	F	G
C	G	A	$B^\flat$	C	F

From Table 30, we note the followings:

(i) Each row and column of this table is a melodic motion on its own right.

(ii) The first row displays an ascending melodic motion when read from left to right; and displays a descending melodic motion when read from right to left.

(iii) The first row and the last column of this table display a conjunct motion respectively.

(iv) The third row and the second column of this table display both conjunct and disjunct motions respectively.

Example 13. Let $Q = \{0, 1, 2, 3, 4, 5, 6\}$ be the set of notes in a diatonic scale with 0 as the root note. From C major scale, let $\langle C, D, E, F, G, A, B\rangle = \langle 0, 1, 2, 3, 4, 5, 6\rangle$. Then by Table 18, we have Table 31:

Table 31: *Musical staves by Quasigroup*

$*$	0	1	2	3	4	5	6
0	C	E	G	B	D	F	A
1	E	G	B	D	F	A	C
2	G	B	D	F	A	C	E
3	B	D	F	A	C	E	G
4	D	F	A	C	E	G	B
5	F	A	C	E	G	B	D
6	A	C	E	G	B	D	F

From Table 31, we note the followings:

(i) The first row from left to right shows the order of musical notes on the lines of the treble staff from the middle C upward to the first ledger line above the staff occupied by A.

(ii) The second row from right to left shows the order of musical notes on the lines of the bass staff from the middle C downward to the first ledger line below the staff occupied by E.

(iii) The second row (third row) from left to right shows the order of notes on the treble (bass) staff starting from the first line of the staff upward.

(iv) The sixth row (seventh row) from left to right shows the order of notes on the spaces of the treble (bass) staff starting from the first space of the staff upward.

(v) To accommodate more extra lower or higher notes, two or more rows (columns) are considered in the same direction provided the next row (column) begins with the note that ended the immediate row (column). For instance, the first and last rows from left to right of this table extend the lines for treble staff from the first row, by adding six ledger lines upward, if the two rows are joined at A.

(vi) From (i), (ii), (iii) and (v) above, this table describes the order of notes on a grand staff.

vii. Motion of two melodies by Quasigroups

In this subsection, some motions obtained by two melodies from some finite quasigroups are considered. We described some motions obtained from a diatonic scale by a quasigroup of order seven. Examples of two melodies with a contrary, parallel, and oblique motions respectively are shown by a quasigroup of order seven. By a quasigroup of order four, an example of a similar motion is given.

Consider the musical notes from F-major scale and let $\langle 0, 1, 2, 3, 4, 5, 6 \rangle = \langle F, G, A, B^\flat, C, D, E \rangle$. Then by Table 20, we have Table 32:

Table 32: *The chart of pitch classes from key F by Example 9.*

$*$	0	1	2	3	4	5	6
0	F	E	D	C	$B^\flat$	A	G
1	G	F	E	D	C	$B^\flat$	A
2	A	G	F	E	D	C	$B^\flat$
3	$B^\flat$	A	G	F	E	D	C
4	C	$B^\flat$	A	G	F	E	D
5	D	C	$B^\flat$	A	G	F	E
6	E	D	C	$B^\flat$	A	G	F

From Table 32, we note the followings:

(i) The first row from the left is in a descending motion and the first column from the top is in an ascending motion. Thus, these melodies departing from index $(1, 1)$ into row and column are in a contrary motion.

(ii) The melodic motion described by first row (column) and last row (column) is a contrary motion.

(iii) Any two melodies departing from index (i,i) into row and column of this table form a contrary motion.

(iv) Any two melodies departing from index (i,i) into the diagonal and row, or, the diagonal and column respectively, form an oblique motion. Clearly, the diagonal is occupied by F while the rows descend from left to right, and columns ascend from the top to the bottom.

Consider the notes of the F-major scale and let $\langle 0,1,2,3,4,5,6\rangle = \langle F,G,A,B^\flat,C,D,E\rangle$. Then by Table 18, we have Table 33:

Table 33: *The chart of pitch classes from key F by Example 8.*

$*$	0	1	2	3	4	5	6
0	F	A	C	E	G	$B^\flat$	D
1	A	C	E	G	$B^\flat$	D	F
2	C	E	G	$B^\flat$	D	F	A
3	E	G	$B^\flat$	D	F	A	C
4	G	$B^\flat$	D	F	A	C	E
5	$B^\flat$	D	F	A	C	E	G
6	D	F	A	C	E	G	$B^\flat$

From Table 33, we note the followings:

(i) The first row (column) contains both the ascending and descending motions. Therefore, both conjunct and disjoint motions occur in the first row (column).

(ii) The first column and row departing from index $(1,1)$ are both ascending with the same interval. Thus, these two melodies form a parallel motion.

(iii) Any two melodies departing from index (i,i) into its row and column of this table respectively form a parallel motion.

Example 14. Let $Q = \{0,1,2,3\}$ be the set of four consecutive notes from a diatonic scale with 0 as the root note. Let $*$ be a binary operation defined on Q as in Table 34.

Table 34: *A musical quasigroup with a right identity element* 0.

$*$	0	1	2	3
0	0	2	3	1
1	1	0	2	3
2	2	3	1	0
3	3	1	0	2

Let $F,G,A,B^\flat$ be the four consecutive notes from F-major scale and let $\langle 0,1,2,3\rangle = \langle F,G,A,B^\flat\rangle$. Then by Table 34, we have the below table:

Table 35: *A musical quasigroup chart from key F*

$*$	0	1	2	3
0	F	A	$B^\flat$	G
1	G	F	A	$B^\flat$
2	A	$B^\flat$	G	F
3	$B^\flat$	G	F	A

From Table 35, we note that at index $(1,1)$, the row and column depart into ascending motions with different intervals, but the first row finished with a descending motion. This implies that, the row and column melodies departed from index $(1,1)$ initially formed a similar motion with each other, but finished with a contrary motion.

viii. Motion between melodies in chords by Quasigroups

In this subsection, some motions in chords obtained by melodies from a finite quasigroup are considered. We considered quasigroup of terads, pentads and hexads with an example for each of them. It is noted that a musical chord of n notes and its inversions can be viewed by a quasigroup of order n. Quasigroup of triads discussed in Section 2.2 shows ascending motion in the first row by a quasigroup with a left identity element and descending motion in the first column by a quasigroup with a right identity element. We note that a melody from any row (column) from a quasigroup of triads is in a disjunct motion, see Section 2.2. These results obtained from quasigroups of triads are also satisfied by some chords in quasigroup of hexads. Here, we give examples of quasigroups of chords whose row (column) contains both conjunct and disjunct motions. By considering some melodic pairs from a row and column of a quasigroup of chords, we obtained examples of contrary, parallel, oblique and similar motions respectively. An example of an oblique motion which one of its melodies is static while the other moves into disjunct and conjunct motions is given.

Motion between melodies by a Quasigroup of Tetrads

A musical chord with four notes and its inversions can be viewed by a quasigroup of order four.

Example 15. Let $Q = \{0,1,2,3\}$ be the set of pitch classes. Define the binary operation $*$ on Q as $a * b = a + 3b (mod\ 4)$ then $(Q,*)$ is a four-tone musical quasigroup (see Table 36).

Table 36: *A musical quasigroup with a right identity element* 0*.*

$*$	0	1	2	3
0	0	3	2	1
1	1	0	3	2
2	2	1	0	3
3	3	2	1	0

Let C, E, G, A be the four consecutive notes from C-Major 6th chord and let $\langle 0,1,2,3\rangle = \langle C, E, G, A\rangle$. Then by Table 36, we have Table 37:

Table 37: *C-Major 6th chord*

$*$	0	1	2	3
0	C	A	G	E
1	E	C	A	G
2	G	E	C	A
3	A	G	E	C

From Table 37, we note the followings:

(i) The first row displays a descending melodic motion when read from left to right; and the first column displays an ascending melodic motion when read from top to bottom. Thus, the two melodies departing from index $(1, 1)$ into the first row and column respectively are in contrary motion.

(ii) The first row and the last row of this table display a similar motion when read from left to right.

(iii) Some rows from left to right of this table display both conjunct and disjunct motions.

(iv) Using the main diagonal and the first row starting from the left, an oblique motion is obtained; such that, one melody is static while the other melody moves into conjunct and disjunct motions.

Motion between melodies by a Quasigroup of Pentads

A musical chord with five notes and its inversions can be viewed by a quasigroup of order five.

Example 16. Let C, E, G, B, D be the five consecutive notes from C-Major 7th with 9 chord and let $\langle 0, 1, 2, 3, 4\rangle = \langle C, E, G, B, D\rangle$. Then by Table 27, we have Table 38:

Table 38: *C-Major 7th with 9 chord*

$*$	C	E	G	B	D
C	C	G	D	E	B
E	G	D	E	B	C
G	D	E	B	C	G
B	E	B	C	G	D
D	B	C	G	D	E

From Table 38, we note the followings:

(i) Parallel motion is obtained, for each pair of melodies departing from index (i, i) into a row and column respectively.

(ii) The rows from left to right of this table display both conjunct and disjunct motions.

Motion between melodies by a Quasigroup of Hexads

A musical chord with six notes and its inversions can be viewed by a quasigroup of order six.

Example 17. Let $Q = \{0,1,2,3,4,5\}$ be the set of pitch classes. Define the binary operation $*$ on Q as $a * b = 5a + b (mod\ 6)$ then $(Q, *)$ is a six-tone musical quasigroup (see Table 39).

Table 39: *Multiplication Table for* $(Q, *)$ *by Example 17.*

$*$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	5	0	1	2	3	4
2	4	5	0	1	2	3
3	3	4	5	0	1	2
4	2	3	4	5	0	1
5	1	2	3	4	5	0

Let $C, E, G, B^\flat, D, F^\sharp$ be the six consecutive notes from C-Augmented 11th chord and let $\langle 0,1,2,3,4,5\rangle = \langle C, E, G, B^\flat, D, F^\sharp \rangle$. Then by Table 39, we have Table 40:

Table 40: *C-Augmented 11th chord.*

$*$	0	1	2	3	4	5
0	C	E	G	$B^\flat$	D	$F^\sharp$
1	$F^\sharp$	C	E	G	$B^\flat$	D
2	D	$F^\sharp$	C	E	G	$B^\flat$
3	$B^\flat$	D	$F^\sharp$	C	E	G
4	G	$B^\flat$	D	$F^\sharp$	C	E
5	E	G	$B^\flat$	D	$F^\sharp$	C

From Table 40, we note the followings:

(i) Each row and column has a disjunct motion.

(ii) Any two melodies of this table departing from index (i, i) into its diagonal and row, or its diagonal and column respectively, form an oblique motion.

ix. Musical Subquasigroup and Normal Subquasigroup

In this subsection, we define a musical subquasigroup and normal subquasigroup. We construct some examples of subquasigroup and normal subquasigroup for pitch classes, and applied them to music. It is shown that a melodic motion which is not descending (nor ascending) may contain a sub-melodic motion which is descending (or ascending). A melodic motion which is disjunct is shown to have a sub-melodic motion which is conjunct. Also, it is shown that a melodic motion may contain a sub-melodic motion whose distance between its consecutive notes are shorter than that of the melodic motion. It is shown that there are paired melodies which are not in contrary motion to each other but have paired sub-melodies which are in contrary (or strict-contrary) motion.

Definition 8. Let $(Q, *)$ be a musical quasigroup. A proper subset W of Q is said to be a musical subquasigroup of Q if W is a musical quasigroup on its own right under the binary operation $*$.

Definition 9. Let n be an even integer and let $\mathbb{Z}_n = \{0, 1, 2, 3, \cdots, n-1\}$ be the set of pitch classes for $n \geq 4$. Define a binary operation $*$ on $\mathbb{Z}_n$ as $a * b = xa + yb(mod\ n)$ where x, y are two distinct elements in $\mathbb{Z}_n \backslash \{0\}$ which are primes such that $(x, y) = 1,\ x + y = n$. An n-tone musical quasigroup $(\mathbb{Z}_n, *)$ is obtained. Let $\mathbb{Z}'_n$ be the set of all even numbers in $\mathbb{Z}_n$, we obtained a musical subquasigroup $(\mathbb{Z}'_n, *)$ of $(\mathbb{Z}_n, *)$.

Example 18. Let $Q = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$ be the set of pitch classes. Define the binary operation $*$ on Q as $a * b = 5a + 7b(mod\ 12)$ then $(Q, *)$ is a twelve-tone musical quasigroup and the set $Q' = \{0, 2, 4, 6, 8, 10\}$ forms a musical subquasigroup under the binary operation $*$. We note that, the subquasigroup $(Q', *)$ is normal (see Tables 41, 42, 43, and 44).

Table 41: *Multiplication Table for $(Q, *)$ by Example 18.*

$*$	0	1	2	3	4	5	6	7	8	9	10	11
0	0	7	2	9	4	11	6	1	8	3	10	5
1	5	0	7	2	9	4	11	6	1	8	3	10
2	10	5	0	7	2	9	4	11	6	1	8	3
3	3	10	5	0	7	2	9	4	11	6	1	8
4	8	3	10	5	0	7	2	9	4	11	6	1
5	1	8	3	10	5	0	7	2	9	4	11	6
6	6	1	8	3	10	5	0	7	2	9	4	11
7	11	6	1	8	3	10	5	0	7	2	9	4
8	4	11	6	1	8	3	10	5	0	7	2	9
9	9	4	11	6	1	8	3	10	5	0	7	2
10	2	9	4	11	6	1	8	3	10	5	0	7
11	7	2	9	4	11	6	1	8	3	10	5	0

Table 42: *The chart of pitch classes from key F by Example 18.*

$*$	0	1	2	3	4	5	6	7	8	9	10	11
0	F	C	G	D	A	E	B	$F^{\sharp}$	$C^{\sharp}$	$G^{\sharp}$	$D^{\sharp}$	$B^{\flat}$
1	$B^{\flat}$	F	C	G	D	A	E	B	$F^{\sharp}$	$C^{\sharp}$	$G^{\sharp}$	$D^{\sharp}$
2	$D^{\sharp}$	$B^{\flat}$	F	C	G	D	A	E	B	$F^{\sharp}$	$C^{\sharp}$	$G^{\sharp}$
3	$G^{\sharp}$	$D^{\sharp}$	$B^{\flat}$	F	C	G	D	A	E	B	$F^{\sharp}$	$C^{\sharp}$
4	$C^{\sharp}$	$G^{\sharp}$	$D^{\sharp}$	$B^{\flat}$	F	C	G	D	A	E	B	$F^{\sharp}$
5	$F^{\sharp}$	$C^{\sharp}$	$G^{\sharp}$	$D^{\sharp}$	$B^{\flat}$	F	C	G	D	A	E	B
6	B	$F^{\sharp}$	$C^{\sharp}$	$G^{\sharp}$	$D^{\sharp}$	$B^{\flat}$	F	C	G	D	A	E
7	E	B	$F^{\sharp}$	$C^{\sharp}$	$G^{\sharp}$	$D^{\sharp}$	$B^{\flat}$	F	C	G	D	A
8	A	E	B	$F^{\sharp}$	$C^{\sharp}$	$G^{\sharp}$	$D^{\sharp}$	$B^{\flat}$	F	C	G	D
9	D	A	E	B	$F^{\sharp}$	$C^{\sharp}$	$G^{\sharp}$	$D^{\sharp}$	$B^{\flat}$	F	C	G
10	G	D	A	E	B	$F^{\sharp}$	$C^{\sharp}$	$G^{\sharp}$	$D^{\sharp}$	$B^{\flat}$	F	C
11	C	G	D	A	E	B	$F^{\sharp}$	$C^{\sharp}$	$G^{\sharp}$	$D^{\sharp}$	$B^{\flat}$	F

Table 43: *A musical subquasigroup by Example 18.*

$*$	0	2	4	6	8	10
0	0	2	4	6	8	10
2	10	0	2	4	6	8
4	8	10	0	2	4	6
6	6	8	10	0	2	4
8	4	6	8	10	0	2
10	2	4	6	8	10	0

Table 44: *A musical subquasigroup from key F by Example 18.*

$*$	0	2	4	6	8	10
0	F	G	A	B	$C^\sharp$	$D^\sharp$
2	$D^\sharp$	F	G	A	B	$C^\sharp$
4	$C^\sharp$	$D^\sharp$	F	G	A	B
6	B	$C^\sharp$	$D^\sharp$	F	G	A
8	A	B	$C^\sharp$	$D^\sharp$	F	G
10	G	A	B	$C^\sharp$	$D^\sharp$	F

Definition 10. Let $(Q,*)$ be a musical quasigroup and let $(W,*)$ be a musical subquasigroup of $(Q,*)$. Then $(W,*)$ is called a musical normal subquasigroup of $(Q,*)$ if:

(i) $nW = Wn$ (ii) $y(xW) = (yx)W$ (iii) $(Wx)y = W(xy)$ for all $n, x, y \in Q$.

Definition 11. Let n be an even integer and let $\mathbb{Z}_n = \{0,1,2,3,\cdots,n-1\}$ be the set of pitch classes for $n \geq 4$. Define a binary operation $*$ on $\mathbb{Z}_n$ as $a * b = xa + yb(mod\ n)$ where x, y are elements in $\mathbb{Z}_n \backslash \{0\}$ and $(x,y) = 1$, $x + y = n$ and $|x - y|$ is a minimum. For a fixed integer n and varying x and y, an n-tone musical quasigroup $(\mathbb{Z}_n,*)$ is obtained. Let $\mathbb{Z}'_n$ be the set of all even numbers in $\mathbb{Z}_n$, we obtain a musical subquasigroup $(\mathbb{Z}'_n,*)$ of $(\mathbb{Z}_n,*)$.

Example 19. Let $Q = \{0,1,2,3,4,5,6,7\}$ be the set of pitch classes. Define the binary operation $*$ on Q as $a * b = 5a + 3b(mod\ 8)$ then $(Q,*)$ is an eight-tone musical quasigroup and the set $Q' = \{0,2,4,6\}$ forms a musical subquasigroup under the binary operation $*$.

Example 10 gives the chart for Q. It is clear that Q' table is given by Table 45 (see also Table 46):

Table 45: *A musical subquasigroup by Example 19*

$*$	0	2	4	6
0	0	6	4	2
2	2	0	6	4
4	4	2	0	6
6	6	4	2	0

Table 46: *A musical subquasigroup chart from key F by Example 19*

$*$	0	2	4	6
0	F	B	A	G
2	G	F	B	A
4	A	G	F	B
6	B	A	G	F

In Example 19, Q' is an example of a normal subquasigroup of $(Q, *)$. Thus, Table 46 is a musical chart of a normal subquasigroup Q' from key F.

Motion of a Single Melody by a Normal Subquasigroup

(A) We consider Tables 42 and 44. Table 44 is a normal subquasigroup table obtained from Table 42.

From Table 42, we note the followings:

(i) Each row and column of this table displays melodic motion of a chromatic scale

(ii) The distance between two notes on this table is either five or seven semitones in each row and column.

(iii) Each row and column of this table displays a disjunct melodic motion.

(iv) The rows (or columns) do not display an ascending or descending motion.

From Table 44, we note the followings:

(i) The first column of this table shows a descending motion when read from the top to the bottom; and shows an ascending motion when read from the bottom to the top.

(ii) The distance between two notes on this table is two semitones in each row and column.

(iii) The motion of each single melody described by each column and row is a conjunct motion.

Comparing the results from Tables 42 and 44, it is clear that:

(i) A melodic motion which is disjunct, may contain a sub-melodic motion which is conjunct.

(ii) A melodic motion which is not ascending (descending), may contain a sub-melodic motion which is ascending (descending).

(iii) The main melodic motion may contain a sub-melodic motion whose distance between its consecutive notes are shorter than that of the main melodic motion.

(B) We consider Tables 23 and 46. Table 46 is a normal subquasigroup table obtained from Table 23.

From Table 23, we note the following:

(i) The table displays some motions of eight consecutive semitones in columns and rows from Key F.

(ii) The distance between two notes on this table is either three or five semitones in each row and column.

(iii) The motion of a single melody described by each row and column is a disjunct motion.

(iv) The columns and rows do not display the ascending or descending motions. Rather, the ascending and descending motions both occur in a single melody, this is shown by the first row.

From Table 46, we note the following:

(i) The first column and row of this table shows an ascending and descending motions respectively.

(ii) The distance between two notes on this table is either two or six semitones in each row and column.

(iii) The motion of a single melody described by each row (column) is either disjunct, or have both conjunct and disjunct motions.

Comparing the results from Tables 23 and 46, it is clear that:

(i) A melodic motion which is not ascending (descending) may contain a sub-melodic motion which is ascending (descending).

(ii) The main melodic motion may have a sub-melodic motion whose distance between its consecutive notes contains more semitones than that of the main melodic motion.

(iii) A melodic motion which is disjunct may contain a sub-melodic motion which is either disjunct, or have both conjunct and disjunct motions.

Motion of Two Melodies by a Normal Subquasigroup

(A) From Table 44, consider the entries with index (i, i):

(i) For $i = 1$, we have the index $(1, 1)$, that is, the position of F on the first row. At the index $(1, 1)$ of Table 44, the column and the row are two different melodies. These two melodies departing from index $(1, 1)$ are in opposite directions. Thus, the two melodies form a strict contrary motion as they depart from the index $(1, 1)$.

(ii) Each of these melodic pairs departing from index (i, i) into rows and columns of this table is seen to be in a strict contrary motion, keeping a distance of two semitones between two notes.

From Table 42, consider the entries with index (i, i). Each of these melodic pairs departing from index (i, i) into rows and columns of this table is not in a contrary motion, and keeps a distance of five or seven semitones between two notes.

Comparing result from Tables 42 and 44, it is clear that, there are some paired melodies which are not in contrary motion to each other but have paired sub-melodies which are in strict contrary motion.

(B) From Table 46, consider the entries with index (i, i):

(i) At the index $(1, 1)$ of Table 46, the column and the row are two different melodies. These two melodies departing from index $(1, 1)$ are in opposite directions. Thus, the two melodies form a contrary motion as they depart from the index $(1, 1)$.

(ii) Each of these melodic pairs departing from index (i, i) into rows and columns of this table is seen to be in a contrary motion.

From Table 23, consider the entries with index (i,i). Each of these melodic pairs departing from index (i,i) into rows and columns of this table is not in a contrary motion, and keeps a distance of three or five semitones between two notes.

Comparing result from Tables 23 and 46, it is clear that, there are some paired melodies which are not in contrary motion to each other but have paired sub-melodies which are in contrary motion.

III. Conclusion

A musical quasigroup is a musical groupoid in which all its left and right translation mappings are permutations. Some examples of quasigroups have been constructed and applied to music, and it is noted that one of the functions of a given quasigroup binary operation on a set of order n is to preserve the permutations of the n symbols defined by the $n \times n$ multiplication. As regarding the main results (Section II), from Sections i, ii, and viii, it has been shown that quasigroup plays important roles in chord progressions and inversions. By Remarks 1, 2 and 4, and Section viii, the function of the left (right, middle) identity element of a quasigroup could be linked to the root of a given chord. The study considered examples from triad, tetrads, pentads, and hexads. In Section iii, examples of musical sequences from triads are given. In Section iv, an example of a twelve-tone matrix is described to be a quasigroup. In Section v, we construct some examples of an n-tone composition chart using quasigroups. By Example 6, a circle of fourths' and fifths' progressions were obtained. In Sections vi and viii, motion of a single melody is described by some quasigroups. It is shown by examples that each row and column of a finite musical quasigroup is a melodic motion on its own right. These examples were used to describe a descending, ascending, disjunct and conjunct melodic motions. Sections vii and viii give examples of musical quasigroups which demonstrate contrary, parallel, oblique and similar motions respectively. Table 37 gives an oblique motion which one of its melodies is static while the other moves into disjunct and conjunct motions. In Section ix, examples of a subquasigroup are constructed and further characterized to be normal subquasigroups. An example of a melodic motion which is disjunct is shown to have a sub-melodic motion which is conjunct. It was obtained that there are paired melodies which are not in contrary motion to each other but have paired sub-melodies which are in contrary (or strict-contrary) motion.

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