Neo-Riemannian Graphs Beyond Triads and Seventh Chords

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Abstract: *This article develops theoretical concepts that allow us to adapt the graphs used for Neo-Riemannian Theory to all classes of trichords and tetrachords, beyond triads and seventh chords. To that end, it will be necessary to determine what the main features of these graphs are and the roles that the members of each set class play in them. Each graph is related to a mode of limited transposition, which contains all the pitch classes occurring in the graph. The musical examples extracted from these graphs reveal passages in which the sets are connected by a consistent voice-leading.*

Keywords*: Neo-Riemannian Theory. Graphs. Voice leading. Contextual inversion.*

I. Introduction

Graphs of Neo-Riemannian Theory proved to be very useful in dealing with triads and seventh chords, covering many possible connections between them. There are several graphs with these sets, including cycles, like HexaCycles, OctaCycles, and EnneaCycles; trees, like Weitzmann Regions and Boretz Regions; and unified models, like Cube Dance, Power Towers, and 4-Cube Trio.

In this paper we will develop the theoretical tools to construct all these types of neo-Riemannian graphs with any trichord or tetrachord beyond the members of sc. (037), (048), (0258), (0358), and (0369). Section II will explore the position and function that the sets have in the graphs and how important is their symmetry in this context. We divided all sets in four types: 1) target sets, 2) pivot sets, 3) bridge sets, and 4) supersets.

Section III will determine labels for the axes of contextual symmetry. The transformations P, L, R, among others, used traditionally in the Neo-Riemannian Theory, do not fit all classes of trichords and tetrachords and, given the purpose of this work to construct graphs with all the sets of these cardinality, it was necessary to introduce different terminology. Taking advantage of the

*This work on the graph-theoretic approach of Neo-Riemannian Theory has begun in 2017 when I was at CUNY under the guidance of Distinguished Professor, Joseph Straus, to whom I am immensely grateful for his many insightful ideas, which were invaluable to the writing of this article. I would also like to thank Prof. Dr. Paulo de Tarso Salles, who was my doctoral supervisor at ECA/USP and who first introduced me to Neo-Riemannian Theory and its graphs.

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relation between the transformations and the contextual inversions, this terminology was defined according to the axes of these inversions in relation to the normal form of the sets.

Section IV will explore cycles of same set class members. In the earlier part of this section the main features of the Hexatonic Cycles, components of the graph known as HexaCycles, will be listed and then, based on these features, new cycles that include all sets of trichords and tetrachords related by contextual inversion will be constructed. All the new cycles will share three of four main features of Hexatonic Cycles. The musical examples for each cycle will show passages where the sets are connected by a consistent voice-leading. This consistency results from the fact that all the sets are limited to two only sum classes. This section will also explore the cycles with symmetric sets that are not related by contextual inversion, but by transposition. All the sets in these cycles are limited to a single sum class and they are connected by pure contrary motion.

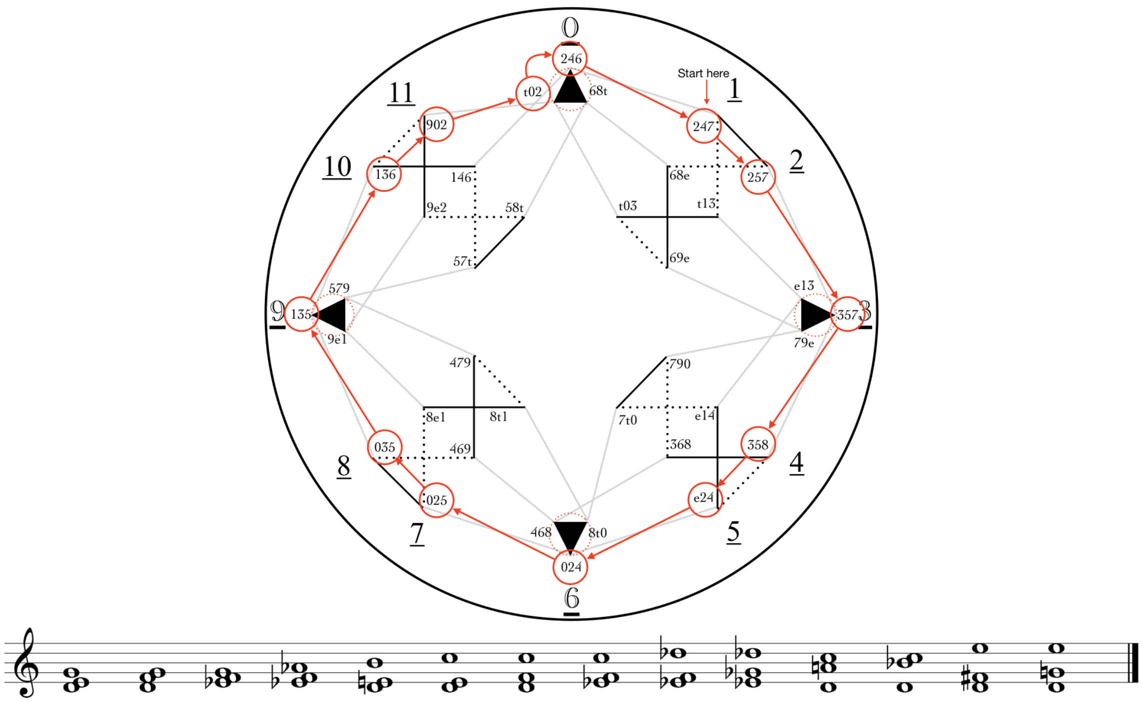
The graphs built in Section V include members of two sum classes and Weitzmann graphs, and Boretz Spiders will be used as models for that task. As with cycles, the new graphs will also share three of the four main features of their models, and the musical examples for them will also show passages with a consistent voice-leading, where the sets are limited to three sum classes.

The same strategy will be used to build the unified models in Section VI. The graphs known as Cube Dance and Power Towers will be used as models to the new graphs, which share three of the their four main features. All the cycles and graphs built in Sections IV and V will be subgraphs in these new Cube Dances and Power Towers, and the musical examples will show passages where the sets are connected by a consistent voice-leading, even if its members are distributed in all sum classes. One can see an example of this consistent voice-leading, which will be the focus of this article, in Figure 1 that shows in its upper part a Cube Dance which is built with members of sc. (024) and (025). The path drawn over the graph indicates a passage with the sets that are noted in the lower part of Figure 1. Note how all the sets are linked by a kind of voice-leading, parsimonious or not, that keeps the sets in two adjacent sum classes, the only exception being between the antepenultimate and penultimate set of the passage, in which the sets remain in the same class of sum because of the pure contrary motion. All the supplementary material for this article are available as at `https://axesofcontextualinversion.wordpress.com/` (see Figure 1).

II. Types of Sets in the Graphs

All neo-Riemannian graphs, even those that include more than one set class, have a main set. We will refer to them as target sets. All the cycles used in Neo-Riemannian Theory have just a single set class that is the target set of the cycle, thus in the HexaCycles ([3, p. 243, Fig. 3]) and in the OctaCycles ([3, p. 247, Fig. 5]) consonant triads, sc. (037), are the target sets in each component, and in EnneaCycles ([3, p. 247, Fig. 6]) the target sets are members of sc. (0258), half-diminished and dominant seventh chords.

There is another type of graph, called tree, whose components include members of two different set classes limited to three sum class. Weitzmann Graph ([1, p. 94, Ex. 6]), the OctaTowers ([3, p. 246, Fig. 4]) and the Boretz regions ([3, p. 153, Tab. 7.2]) are examples of trees. Consonant triads are also the target sets in each component of the Weitzmann graph. They are all connected to a single augmented triad, sc (048). Members of sc. (0258) are the target sets in each component of the OctaTowers and of the Boretz regions. In the former, four pairs, made by one half-diminished chord and one dominant seventh chord, are connected to four minor seventh chords that are members of sc. (0358). In the latter, all members of sc. (0258) are connected to a single diminished seventh chord, sc. (0369). We will refer to the augmented triads in Weitzmann region and the diminished seventh chord in the Boretz regions as pivot sets, due to their quality in connecting with all the other members of the graph, and we will refer to the minor seventh chords in the



Sets	[2,4,7]	[2,5,7]	[3,5,7]	[3,5,8]	[e,2,4]	[0,2,4]	[0,2,5]	[0,3,5]	[1,3,5]	[1,3,6]	[9,0,2]	[t,0,2]	[2,4,6]	[2,4,7]
Pitch Space	G (7)	→ G (7)	→ G (7)	$\xrightarrow{+1}$ A♭ (8)	$\xrightarrow{+3}$ B (11)	$\xrightarrow{+1}$ C (12)	→ C (12)	→ C (12)	$\xrightarrow{+1}$ D♭ (13)	→ D♭ (13)	$\xrightarrow{-1}$ C (12)	→ C (12)	$\xrightarrow{+4}$ E (16)	→ E (16)
	E (4)	$\xrightarrow{+1}$ F (5)	→ F (5)	→ F (5)	$\xrightarrow{-1}$ E (4)	→ E (4)	$\xrightarrow{+1}$ F (5)	→ F (5)	→ F (5)	$\xrightarrow{+1}$ G♭ (6)	$\xrightarrow{+3}$ A (9)	$\xrightarrow{+1}$ B♭ (10)	$\xrightarrow{-4}$ F♯ (6)	$\xrightarrow{+1}$ G (7)
	D (2)	→ D (2)	$\xrightarrow{+1}$ E♭ (3)	→ E♭ (3)	$\xrightarrow{-1}$ D (2)	→ D (2)	→ D (2)	$\xrightarrow{+1}$ E♭ (3)	→ E♭ (3)	→ E♭ (3)	$\xrightarrow{-1}$ D (2)	→ D (2)	→ D (2)	→ D (2)
P-int Sum		+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	0	+1
PC Space	G (7)	→ G (7)	→ G (7)	$\xrightarrow{1}$ A♭ (8)	$\xrightarrow{3}$ B (11)	$\xrightarrow{1}$ C (0)	→ C (0)	→ C (0)	$\xrightarrow{1}$ D♭ (1)	→ D♭ (1)	$\xrightarrow{11}$ C (0)	→ C (0)	$\xrightarrow{4}$ E (4)	→ E (4)
	E (4)	$\xrightarrow{1}$ F (5)	→ F (5)	→ F (5)	$\xrightarrow{11}$ E (4)	→ E (4)	$\xrightarrow{1}$ F (5)	→ F (5)	→ F (5)	$\xrightarrow{1}$ G♭ (6)	$\xrightarrow{3}$ A (9)	$\xrightarrow{1}$ B♭ (10)	$\xrightarrow{8}$ F♯ (6)	$\xrightarrow{1}$ G (7)
	D (2)	→ D (2)	$\xrightarrow{1}$ E♭ (3)	→ E♭ (3)	$\xrightarrow{11}$ D (2)	→ D (2)	→ D (2)	$\xrightarrow{1}$ E♭ (3)	→ E♭ (3)	→ E♭ (3)	$\xrightarrow{11}$ D (2)	→ D (2)	→ D (2)	→ D (2)
PC-int sum (Mod 12)		1	1	1	1	1	1	1	1	1	1	1	0	1
Sum Class	1	2	3	4	5	6	7	8	9	10	11	0	0	1

Figure 1: *Example of a passage with members of sc. (024) and (025) arranged in a Cube Dance.*

OctaTowers as bridge sets, due to its quality in connecting with two target sets of the graph.

The graph known as Cube Dance ([2, p. 104, Fig. 5.24]) is a "unified model of triadic voice-leading space" ([2, p. 83]) because it "includes the four hexatonic cycles and the four Weitzmann regions as contiguous subgraphs" ([2, p. 83]). The target sets are all members of sc. (037) that are placed in the voice-leading zones 1, 2, 4, 5, 7, 8, 10, and 11 and the members of sc. (048), placed in the voice-leading zones 0, 3, 6, and 9, are the pivot sets, as they connect to all sets in its two adjacent voice-leading zones. In the same way, we can consider the Power Towers ([3, p. 256, Fig. 10]) as a unified model of seventh chords voice-leading space, since it includes the three octatonic towers and the three Boretz regions as contiguous subgraphs. In this graph, all the members of sc.

(0258), placed in the odd voice-leading zones, are the target sets, while the members of sc. (0369), diminished seventh chords placed in voice-leading zones 2, 6, and 10, are the pivot sets, and the members of sc. (0358), minor seventh chords placed in voice-leading zones 0, 4, and 8, are the bridges sets.

Douthett and Steinbach have shown how the components of these graphs are embedded in one set which is associated to one of modes of limited transposition ([3, pp. 245–247]). We will refer to these sets as supersets. The hexatonic collection, sc. (014589), is the superset of each component of the HexaCycles and of each cube of Cube Dance; the octatonic collection, sc. (0134679t), is the superset of each component of the OctaCycles, each component of OctaTowers, and also of each circuit constituted by all sets between two diminished seventh chords in the Power Towers[1]. Nonatonic collection, sc. (01245689t), is the superset of each component of the Weitzmann graph and each component of the EnneaCycles.

Therefore, there are four types of sets in neo-Riemannian graphs: target sets, pivot sets, bridge sets, and supersets, each one of them playing a different role in the graphs. Since our goal in this work is to build graphs for sets other than triads and seventh chords, we shall summarize and generalize these roles.

- **Target sets** are the main sets of a graph; they are necessarily members of a set class with neither inversion or transpositional symmetry, which therefore have 24 sets; these 24 sets connect with themselves and with members of other set classes in the graph. For members of a set class to be target sets in a graph of trichords, they must be distributed in the sum classes[2] 1, 2, 4, 5, 7, 8, 10, and 11[3], they are sc. (013), (014), (016), (025), (026), (027) and (037). In tetrachord graphs, all members of the target sets must be distributed in odd sum classes. They are sc. (0124), (0126), (0135), (0137), (0146), (0148), (0157), (0236), (0247) and (0258).
- **Pivot sets** make connections with every target set placed in their adjacent voice-leading zones; they are necessarily member of a set class with inversional and transpositional symmetry, with its pitches dividing equally the octave by a single interval. Pivot sets therefore belong to set classes that have a maximum of 6 members[4]. In trichord graphs, all the members of the pivot set must be distributed in the sum classes 0, 3, 6, and 9, and in tetrachord graphs, all members of the pivot sets must be distributed in sum classes 2, 6, and 10. Sc. (048) is the only pivot set among the trichords, while sc. (0369) is the only pivot set among the tetrachords.

[1]Douthett and Steinbach have changed the original Power Tower including the French-sixth chord in voice-leading zones 0, 4, and 8 along with minor seventh chords ([3, p. 262]). In this way, the circuit constituted by all sets between two diminished seventh chords in the Power Towers becomes a tesseract. They named this version of graph as 4-Cube Trio, since it connects three tesseracts in the same way that Cube Dance connects four cubes. The octatonic collection is the superset of each tesseract of the 4-Cube Trio.

[2]According to Joseph Straus: "Two pitch sets are equivalent as members of the same sum class if their pitch integers have the same sum" ([9, p. 2]). This concept can be associated to the concept of voice-leading zones, since two sets in the same sum class are necessarily in the same voice-leading zone. In this work, we will use both concepts, as we find it more appropriate.

[3]In his article named Sum Class ([9]), Straus provides a table with all trichords that shows that they are divided into two groups, one group whose members are distributed in sum classes 1, 2, 4, 5, 7, 8, 10, and 11 - trichords that can be target sets in a graph - and other group whose members are distributed in sum classes 0, 3, 6, and 9 - trichords that can be pivot or bridges sets in a graph - ([9, p. 22]). He also provides a table with all tetrachords that shows that they are divided into three groups, one group whose members are distributed in odd sum classes - tetrachords that can be target sets in a graph - a second group whose members are distributed in sum classes 2, 4, and 6 - tetrachords that can be pivot or bridge sets in a graph - and a third group whose members are distributed in sum classes 0, 4, and 8 - tetrachords that can be bridge sets in a graph - ([9, pp. 41–44]).

[4]I exclude the aggregate of the 12 pitches because it is the only one of cardinality 12 and therefore it has no way to be the pivot in a graph.

- **Bridge sets** make connection with some, but not all target sets placed in their adjacent voice-leading zones; they can be members of a set class with inversional or transpositional symmetry, which therefore have a maximum of 12 sets, but they also can be member of a non-symmetric set class. In trichord graphs, all the members of the bridge sets must be distributed in the sum classes 0, 3, 6, and 9. They are sc. (012), (015), (024), (027) and (036). In tetrachord graphs, all members of the pivot sets must be distributed in sum classes 2, 6, and 10 or in sum classes 0, 4, and 8. They are sc. (0123), (0125), (0127), (0134), (0136), (0145), (0147), (0156), (0158), (0167), (0235), (0237), (0246), (0248), (0257), (0268), (0347), and (0358).
- **Supersets** are collections that embody all the pitches of the sets of a graph or of one component of a graph; they are necessarily member of a set class with inversional and transpositional symmetry, which therefore have a maximum of 6 sets. Here we will relate these collection to Messiaen's modes of limited transposition, thus the label WT (the whole-tone collection) will be used for his first mode, sc. (02468t), which interval cycle is a **<2>** chain; OCT (the octatonic collection) for his second mode, sc. (0134679t), which interval cycle is a **<1,2>** chain; NON (the nonatonic collection) will be used for his third mode, sc. (01245689t), which interval cycle is a **<1,1,2>** chain; MM4 will be used for his fourth mode, sc.(01236789), which interval cycle is a **<1,1,3>** chain; MM5 will be used for his fifth mode, sc. (012678), which interval cycle is a **<1,1,4>** chain; MM6 will be used for his sixth mode, sc. (0124678t), which interval cycle is a **<1,1,2,2>** chain; and MM7 will be used for his seventh mode, sc. (0123466789t), which interval cycle is a **<1,1,1,2>** chain. In addition to these collections, we will use the label HEX for the hexatonic collection, sc. (014589), which is not one of the Messiaen's modes and its cycle is a **<1,3>**im chain and AGG for the aggregated of the 12 pitches. Table 1 summarizes the main features of these sets.

III. Axes of Contextual Inversions

Neo-Riemannian theorists use transformations labels as **P**, **L**, **R**, **N**, **S**, among others, to determine how sets connect each other. These labels are based on voice-leading work that privileges common pitches and parsimonious movements between sets. In this article we will not use these traditional labels for two reasons: 1) although some authors such as Morris ([5, pp. 186–193]) and Straus ([7, pp. 53–67]) have adapted these transformations to sets other than triads and seventh chords, those adaptations can bring some subtle difficulties, as one of these labels may be associated to more than one connection, or a single connection may be associated to more than one label for some sets[5]; 2) for constructing graphs representing the voice-leading space for all trichords and tetrachords we need to describe a greater number of connections between sets than those available with the traditional labels of Neo-Riemannian Theory, and, even if we create new labels based on the same voice-leading work principles, they would also be subject to same difficulty mentioned above.

The approach we present here to label the connections between sets is based on contextual inversions, so the labels will indicate the position of the axis on which the two sets invert themselves. Contextual inversions only occur between target sets, and this type of set class always contains 24 members divided into two different types of OPC equivalent sets: 12 members represented by normal form A and 12 members represented by normal form B[6]. We will determine

[5] For example, following Straus ([7, p. 56, Tab. 1]) **L** transforms [0,2,4] in both [t,0,2] and [2,4,6] because these connections retain i2. However, these same two connections can be labeled **R**, because it also retains i2 for this set.

[6] Following Solomon's set class table available at `http://solomonsmusic.net/pcsets.htm`, we will call normal form A those that are most packed to the left, and normal form B those that are most packed to the right.

Table 1: *List of the collections that can be supersets in a graph.*

Meessiaen's Mode	PCs	Set	Collection	T_n	Modes	Interval Cycle
Mode 1 (MM1)	6	(02468t)	Whole-Tone	2	WT_0 [0,2,4,6,8,t]; WT_1 [1,3,5,7,9,e]	<2>
Mode 2 (MM2)	8	(0134679t)	Octatonic	2	$OCT_{0,1}$ [0,1,3,4,6,7,9,t]; $OCT_{1,2}$ [1,2,4,5,7,8,t,e]; $OCT_{2,3}$ [2,3,5,6,8,9,t,0]	<1,2>
Mode 3 (MM3)	9	(01245689t)	Nonatonic	4	$NON_{0,1,2}$ [0,1,2,4,5,6,8,9,t] $NON_{1,2,3}$ [1,2,3,5,6,7,9,t,e] $NON_{2,3,4}$ [2,3,4,6,7,8,t,e,0] $NON_{3,4,5}$ [3,4,5,7,8,9,e,0,1]	<1,1,2>
Mode 4 (MM4)	8	(01236789)	-	6	$MM4_{0,1,2,3}$ [0,1,2,3,6,7,8,9] $MM4_{1,2,3,4}$ [1,2,3,4,7,8,9,t] $MM4_{2,3,4,5}$ [2,3,4,5,8,9,t,e] $MM4_{3,4,5,6}$ [3,4,5,6,9,t,e,0] $MM4_{4,5,6,7}$ [4,5,6,7,t,e,0,1] $MM4_{5,6,7,8}$ [5,6,7,8,e,0,1,2]	<1,1,3>
Mode 5 (MM5)	6	(012678)	-	6	$MM5_{0,1,2}$ [0,1,2,6,7,8] $MM4_{1,2,3}$ [1,2,3,7,8,9] $MM5_{2,3,4}$ [2,3,4,8,9,t] $MM5_{3,4,5}$ [3,4,5,9,t,e] $MM5_{4,5,6}$ [4,5,6,t,e,0] $MM5_{5,6,7}$ [5,6,7,e,0,1]	<1,1,4>
Mode 6 (MM6)	8	(0124678t)	-	6	$MM6_{0,1,2}$ [0,1,2,4,6,7,8,t] $MM6_{1,2,3}$ [1,2,3,5,7,8,9,e] $MM6_{2,3,4}$ [2,3,4,6,8,9,t,0] $MM6_{3,4,5}$ [3,4,5,7,9,t,e,1] $MM6_{4,5,6}$ [4,5,6,8,t,e,0,2] $MM6_{5,6,7}$ [5,6,7,9,e,0,1,3]	<1,1,4>
Mode 7 (MM7)	8	(012346789t)	-	6	$MM7_{0,1,2,3,4}$ [0,1,2,3,4,6,7,8,9,t] $MM7_{1,2,3,4,5}$ [1,2,3,4,5,7,8,9,t,e] $MM7_{2,3,4,5,6}$ [2,3,4,5,6,8,9,t,e,0] $MM7_{3,4,5,6,7}$ [3,4,5,6,7,9,t,e,0,1] $MM7_{4,5,6,7,8}$ [4,5,6,7,8,t,e,0,1,2] $MM7_{5,6,7,8,9}$ [5,6,7,8,9,e,0,1,2,3]	<1,1,2,2>
Not a Messiaen's Mode (HEX)	6	(014589)	Hexatonic	4	$HEX_{0,1}$ [0,1,4,5,8,9] $HEX_{1,2}$ [1,2,5,6,9,t] $HEX_{2,3}$ [2,3,6,7,t,e] $HEX_{3,4}$ [3,4,7,8,e,0]	<1,3>

the label by the two opposite positions of the axis[7] relative to the first pitch of the normal form A and the last pitch of the normal form B that will always be mirrored. The axis rotates every half semitone and therefore it has 12 different positions that will be labelled with letters from **A** to **L**. We will use **A** as the label if the position of the axis is over the first pitch of normal form A and six semitones above it, or if it is over the last pitch of the normal form B and six semitones below it. **B** will be the label for the first rotation of the axis, **C** will be the label for the second rotation, and so

[7]Since we are working in a pitch class space, that traditionally is represented by the clock face, the axis always passes over two opposite points separated by 6 semitones.

on. Figure 2 shows how the 12 positions of the axis relate to the set [0,1,3] with all its possible inversions.

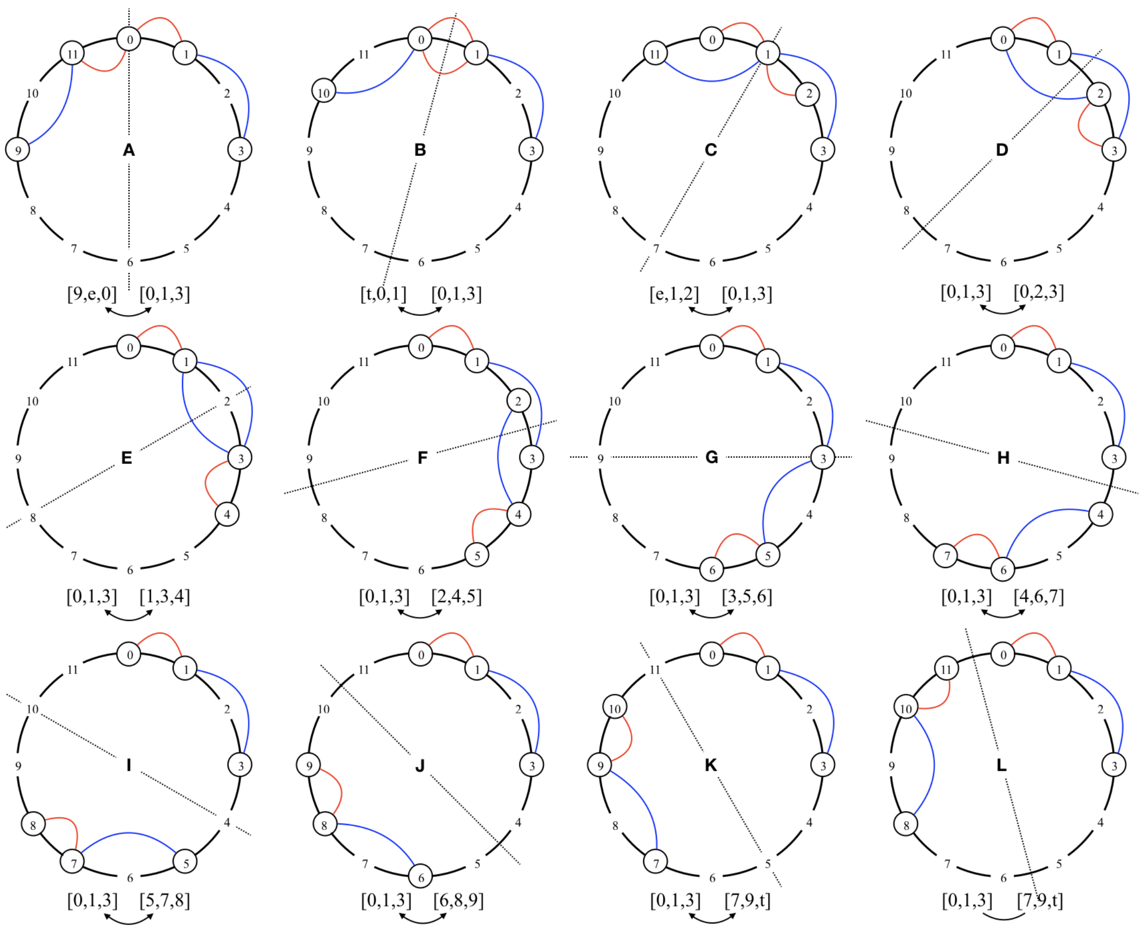


Figure 2: *The position of the axes for the 12 contextual inversion of set [0,1,3], all sets with pitches connected by the inside of the circle are of the normal form B.*

The difference between the axes of contextual inversion shown in Figure 2 and those used for the traditional inversion operation presented by Straus ([8, p. 61, Ex. 2.28]) is that the contextual inversion axes are not fixed in the clock face and move according to the first pitch of the normal form A or to the last pitch of the normal form B of a set[8]. Since there are 12 position of axis relating 12 pairs of sets, there will be 144 connections for each set class[9].

[8]That's why the axis **A** connects set [0,1,3] to the set [9,e,0] that is related by I_0, and also connects set [3,4,6] to the set [0,2,3] that is related by I_3.

[9]For more information about axis of contextual inversion, see [11], and for tables showing the axis that connect all the members of all set classes see [12, Vol. II, pp. 3–232].

IV. Cycles

A cycle graph is a connected graph[10] that is regular of degree 2[11] [13, p. 17]. Cycles are a very common type of neo-Riemannian graphs: they may be a component of a larger graph which is a union of several similar cycles, such as hexatonic cycles are components of HexaCycles ([3, p. 245, Fig. 3]) or eneatonic cycles are components of the EnneaCycles ([3, p. 247, Fig. 5]), or may be a subgraph of a unified model, such as Cube Dance ([3, p. 254, Fig. 9]) or Power Towers ([3, p. 256, Fig. 10]).

All the neo-Riemannian cycles are chains of transformations using target set of the same type. Each one of the four Hexatonic cycles, for example, is a **<PL>** chain of sc. (037), but if we use the contextual inversion axis labels presented in the previous section, these cycles become <HD> chains, as Figure 3 shows.

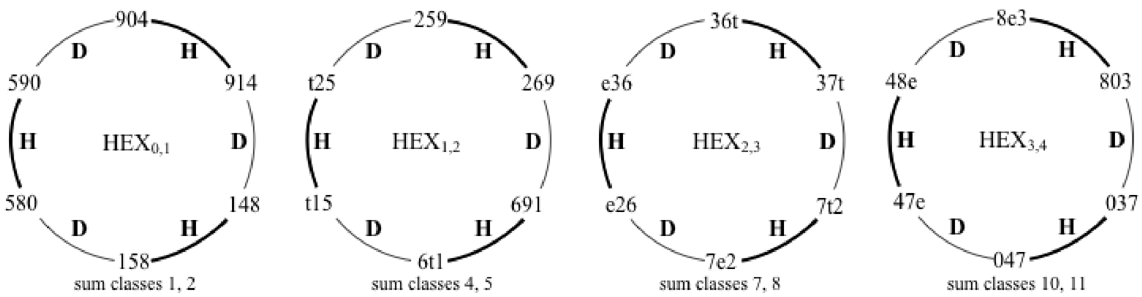


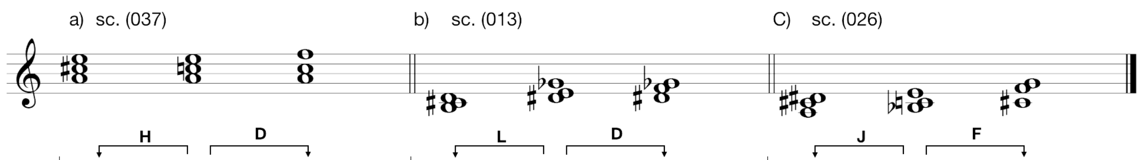
Figure 3: *The graph known as HexaCycles, with the four Hexatonic cycles as* ***<HD>*** *chain.*

Hexatonic cycles are subgraphs for both HexaCycles and Cube Dance graphs. This is because they have several important features that make them special cases among the trichord cycles used by the Neo-Riemannian Theory. Here are of some of these features:

- **1) All the sets belong to the same set class.** This is a common feature to all cycles used by the Neo-Riemannian Theory, because they all relate sets by a chain of contextual inversions[12].
- **2) All notes of each cycle are embedded in a symmetric superset.** The superset of each hexatonic cycle is the hexatonic collection that is listed inside them. All cycles used in the Neo-Riemannian Theory are embedded in one of the symmetric collections shown in Table 1, or in the aggregate of all PCs.
- **3) All sets of each cycle belong to two adjacent sum classes.** Sets in the cycle embedded in $HEX_{0,1}$ belong only to sum class 1 or 2; sets in the cycle embedded in HEX1,2 belong only to sum class 4 or 5; sets in the cycle embedded in $HEX_{2,3}$ belong only to sum class 7 or 8; and sets in the cycle embedded in $HEX_{3,4}$ belong only to sum class 10 or 11.
- **4) Parsimonious voice-leading[13].** Since **H** and **D** are the only position of the axis that hold fixed two notes between sets with the remaining note moving by a single semitone, it is possible to connect by parsimonious voice-leading between all the members of sc. (037) positioned at adjacent vertices in the hexatonic cycles.

[10]Connected graphs are those in which each vertex is connect to any other vertex by a path of edges.
[11]Degree is the number of connections of a vertex. If all vertices have a same degree, the graph is regular.
[12]However, it would certainly be possible to create cycles with members of two or more set classes if the criterion to relating these sets were neither transposition nor inversion.
[13]In this paper, I will consider that parsimonious voice-leading occurs only between two sets that can be transformed by moving one of its pitches by one semitone while the others remain fixed.

Only the hexatonic cycles have all these four features because just the sc. (037), among all sets of cardinality 3, can connect one of its members to two others by parsimonious voice-leading. Since parsimonious voice-leading has been a very important feature for the development of Neo-Riemannian Theory, all the trichords graphs have been limited to exclusively using members of sc. (037) as target sets. In order to build graphs with target sets other than sc. (037), it is necessary give up the voice-leading parsimony in all connections between its members. It should be noted that voice-leading parsimony is a particular case of connection between members of two adjacent sum classes, and if this kind of connection is only possible between members of sc. (037), there are other kinds of voice-leading that can connect members of trichords (013), (014), (016), (025), (026), and (027) in two adjacent sum classes. Figure 4 shows some examples of connections that keep sets in two adjacent sum classes.



	a)	b)	c)
Sets	[9,14] [9,0,4] [5,0,9]	[e,1,2] [3,4,6] [3,5,6]	[9,1,3] [t,0,4] [1,5,7]
Pitch Space	E (16) ← E (16) +1 → F (17) C♯ (13) ← +1 C (12) → C (12) A (9) ← A (9) → A (9)	D (2) ← -4 G♭ (6) → G♭ (6) C♯ (1) ← -3 E (4) +1 → F (5) B (-1) ← -4 D♯ (3) → D♯ (3)	D♯ (3) ← -1 E (4) +3 → G (7) C♯ (1) ← +1 C (0) +5 → F (5) A (-3) ← -1 B♭ (-2) +3 → C♯ (1)
P-int sum	+1 +1	-11 +1	-1 +11
Pitch-Class Space	E (4) ← E (4) 1 → F (5) C♯ (1) ← 1 C (0) → C (0) A (9) ← A (9) → A (9)	D (2) ← 8 G♭ (6) → G♭ (6) C♯ (1) ← 9 E (4) 1 → F (5) B (-1) ← 8 D♯ (3) → D♯ (3)	D♯ (3) ← 11 E (4) 3 → G (7) C♯ (1) ← 1 C (0) 5 → F (5) A (9) ← 11 B♭ (10) 3 → C♯ (1)
PC-int sum (mod 12)	1 1	1 1	11 11
Sum Class	2 1 2	2 1 2	1 2 1

Figure 4: *Connections between members of sc. (037), (013) and (026) that keep sets in two adjacent sum classes.*

Figure 4 shows examples of connections between members of sc. (037), (013), and (026) that keep the sets in sum classes 1 and 2. The upper part of Figure 4 shows the music notation for the sets, while the table below shows their representation on both pitch and pitch-class space with arrows showing the voice-leading (with solid lines connecting two common pitches, and dashed lines representing the movement between two different pitches).

Straus shows that two sets of the same cardinality are in the same sum class if the sum of the directed pitch intervals (pitch space class) or direct pitch-class intervals (pitch-class space) is 0 ([9, p. 1]). In the same way, we can conclude that two sets of the same cardinality are in two adjacent sum classes if the sum of the directed pitch intervals or the direct pitch-class intervals is either 1 or 11 (mod 12). One can see in Figure 5a) how position **H** of the axis connects set [9,0,4] to set [9,1,4], and position **D** connects it to set [5,0,9]. All members of sc. (037) belonging to hexatonic cycle embedded in $HEX_{0,1}$. The parsimonious voice-leading in those connections comes from the fact that the sum of either the directed pitch intervals or the direct pitch-class intervals, is equal to 1, and hence the three sets are in sum classes 1 or 2. We can see similar features in Figure 5b), where the set [3,4,6] connects by **L** with [e,1,2] and by **D** with [3,5,6]. All the three sets are also in sum classes 1 and 2 because the sum of either the directed pitch intervals or the direct pitch-class intervals is equal to 1, but in these example only [3,4,6] and [3,5,6] are connect by the parsimony voice-leading. Figure 5c) shows that the set [t,0,4] connects by **J** with [9,1,3], and by **F** with [1,5,7] and, even neither of these connections are by parsimonious voice-leading, all the three sets are in sum classes 1 or 2 again, since the sum of either the directed pitch intervals or the direct pitch-class intervals is equal to 11. With these kind of connections it is possible to make cycles with members of trichords (013), (014), (016), (025), (026), and (027) which, as in hexatonic cycles, are divided into two adjacent sum classes. Figure 5 shows all cycles for target sets with cardinality 3. One can see that within each hexatonic cycle (Figure 5f)) is indicated the hexatonic collection which is its superset. The superset for all the remaining cycles is the aggregated of the 12 pitches.

Figure 6 shows some examples of voice-leading in cycles with sc. (014) (Figure 5b)), with sc. (026) (Figure 5e)) and with sc. (016) (Figure 5c)). Note that no matter which pitch arrangement is used for each set, the PC intervals sum of the voice-leading will always be 1 or 11, which keeps the sets in two adjacent sum classes.

It is not possible to build similar cycles with tetrachords because none of them have members in adjacent sum classes. In order to build cycles with the target sets of cardinality 4, their members must be in two adjacente odd sum classes. This will generate two types of graphs for each set class: those in which the components keep the sets in the sum classes 11/1, 3/5, and 7/9, and those in which the sets are in the sum classes 1/3, 5/7, and 9/11. Some of those graphs are divided in three components (cycles) and others are divided in six, as shown in Figure 7[14]. The supersets, except the aggregate of 12 pitches, are listed within each cycle according to Table 1.

Figure 8 shows some examples of voice-leading in tetrachords cycles with sc. (0148) (Figure 7f)), with sc. (0157) (Figure 7g)) and with sc. (0236) (Figure 7h)), in the same way that Figure 6 shows with trichords. Note that no matter which pitch arrangement is used for each set, the PC intervals sum of the voice-leading will always be 2 or 10, which keeps the sets in two adjacent even or odd sum classes.

Back to the set of cardinality 3, all the remaining trichords that are not included in the cycles of Figure 6 are either bridge or pivot, sets and have their members in sum classes 0, 3, 6, and 9, and therefore there is no way to build cycles with sets in adjacent sum classes as done before. For those set classes we are going to build cycles with members in the same sum class, using the labels for the position of contextual inversion axis for sc. (015) cycle, and transposition (T4 and T8) for sc. (012), (024), (027), and (036)[15]. In both cases the sets are connected by "pure contrary voice-leadings", in which, according to Tymoczko, "the amount of ascending motion exactly balances the amount of descending motion" ([13, p. 89]). Figure 9 shows examples of this kind of connection between members of sc. (015) and (027).

Figure 10 shows cycles for all bridge sets in which all members are connected by pure contrary voice leadings, and therefore are in the same sum class. Unlike the cycles shown in Figure 6, which the sets are related by contextual inversion, the sets in almost all of these cycles are related by transposition of 4 semitones, the exception being the cycle with sc. (015), as this is the only trichord without inverse symmetry in which its sets are distributed in the sum classes 0, 3, 6, and 9.

Figure 11 shows some examples of voice-leading in cycles with sc. (015) (Figure 10b) and with sc. (027) (Figure 10d). Note that any pitch arrangement that is used for sets results in a connection with pure contrary voice-leading, which keeps the sets in the same sum class.

[14]Some set classes have more than one chain that keep its members in two adjacent odd sum classes, like sc. (0236) and (0247).

[15]It is not possible to build this kind of cycle with members of sc. (048) since they map onto themselves both by inversion and by transposition.

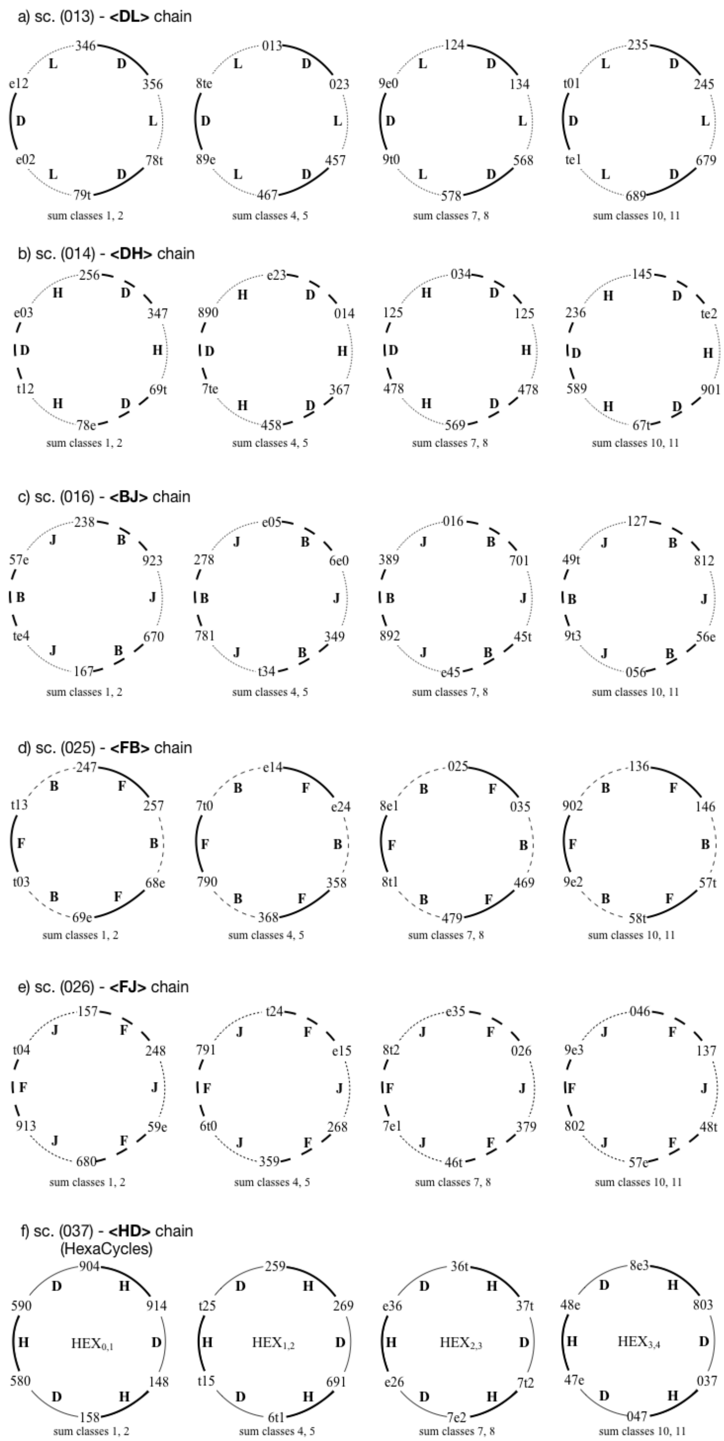


Figure 5: *Cycles for all target sets with cardinality 3 (solid lines represent parsimonious connections while dashed and dotted lines represent non-parsimonious connections).*

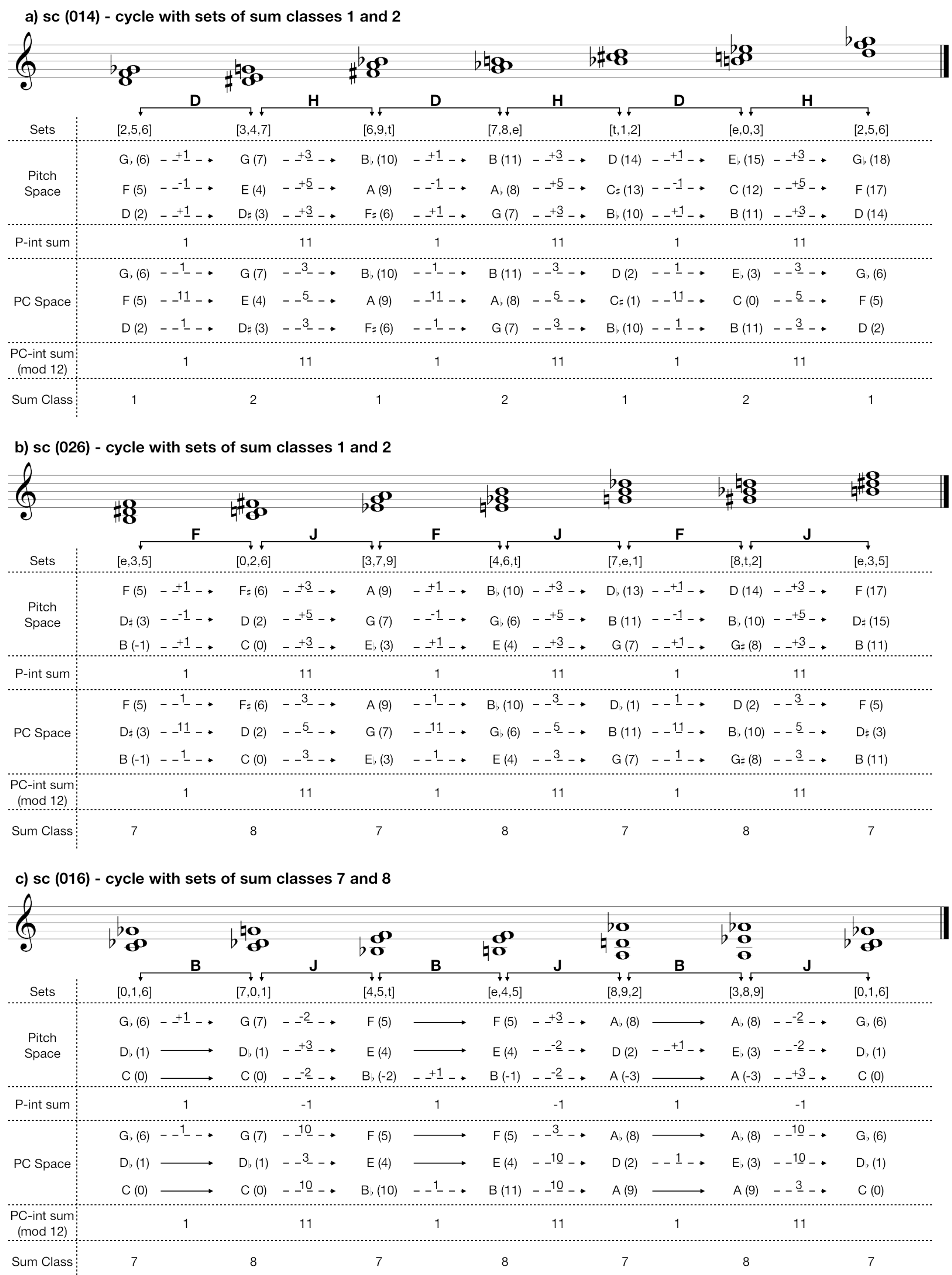
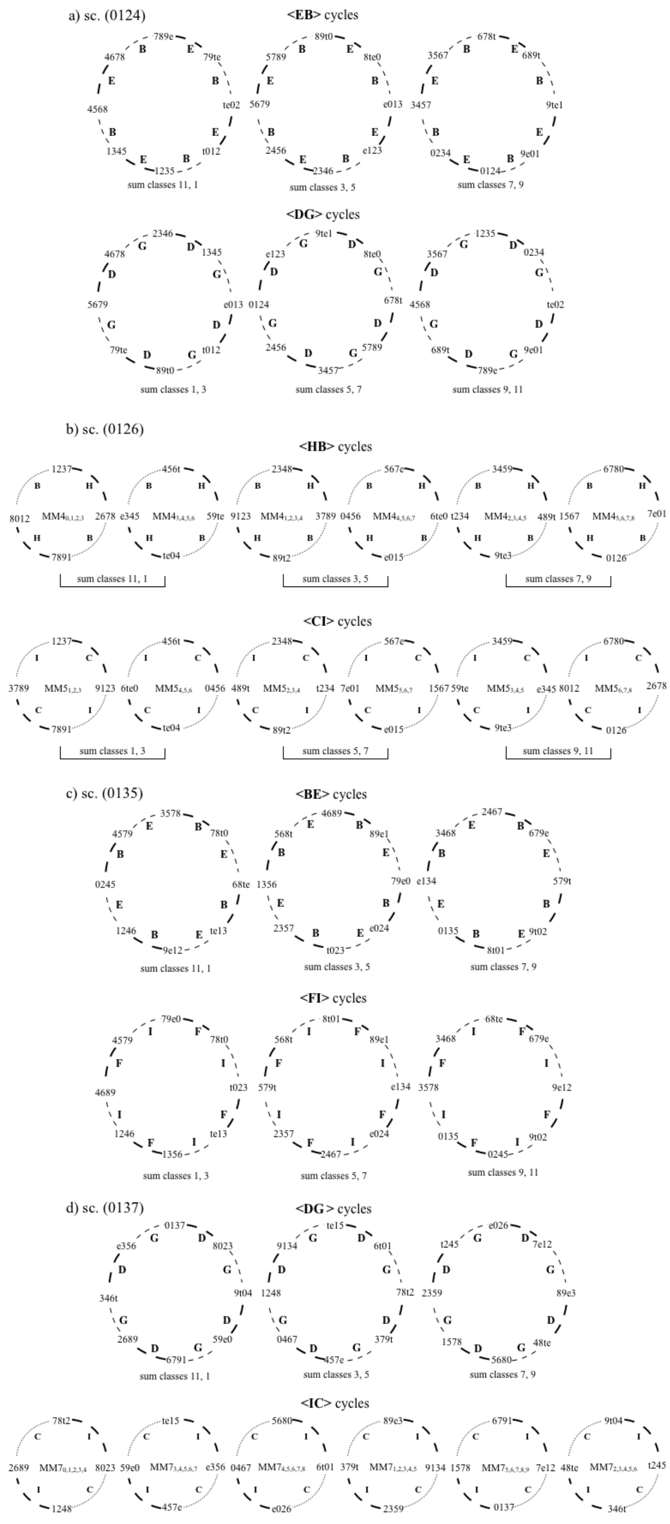
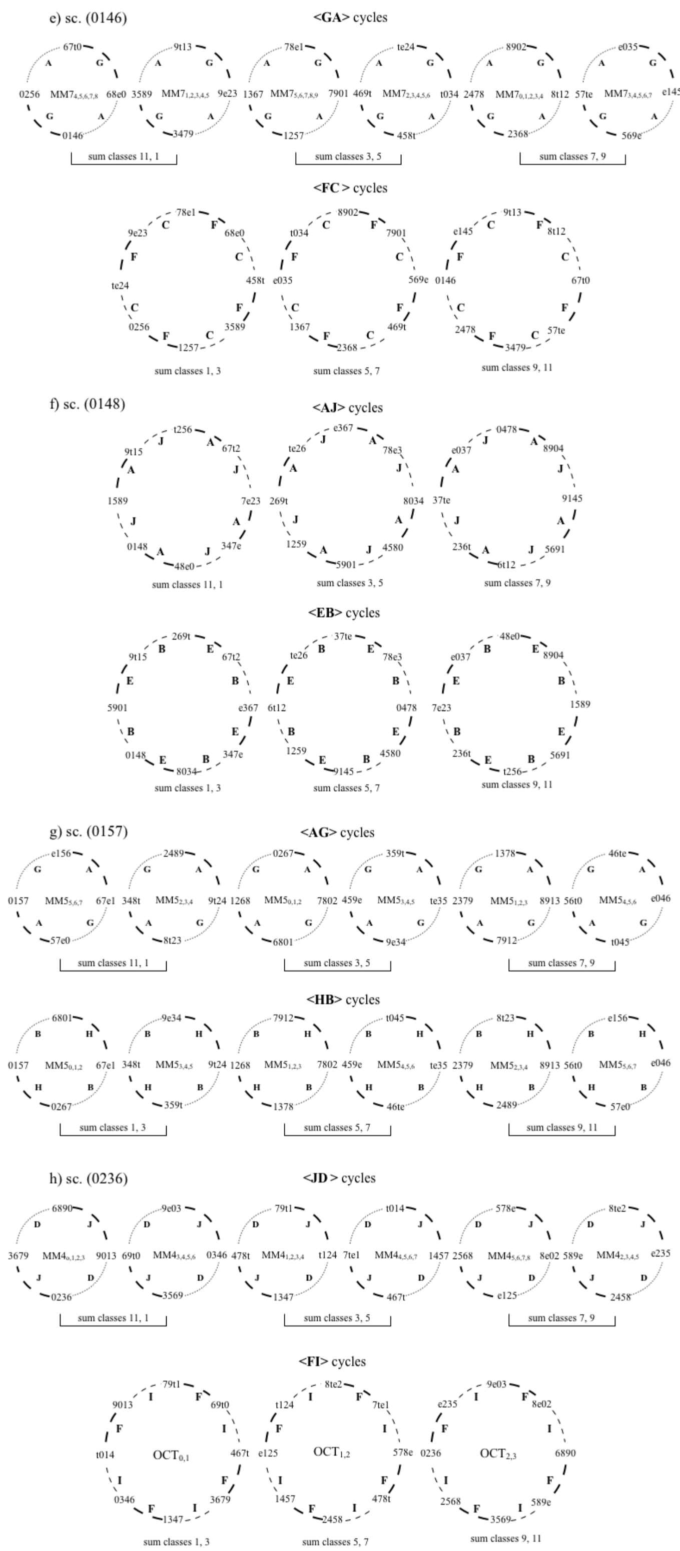


Figure 6: *Examples of voice-leading in cycles with sc. (014), sc. (026) and sc. (016).*


a) sc. (0124)
<EB> cycles
sum classes 11, 1
sum classes 3, 5
sum classes 7, 9
<DG> cycles
sum classes 1, 3
sum classes 5, 7
sum classes 9, 11
b) sc. (0126)
<HB> cycles
sum classes 11, 1
sum classes 3, 5
sum classes 7, 9
<CI> cycles
sum classes 1, 3
sum classes 5, 7
sum classes 9, 11
c) sc. (0135)
<BE> cycles
sum classes 11, 1
sum classes 3, 5
sum classes 7, 9
<FI> cycles
sum classes 1, 3
sum classes 5, 7
sum classes 9, 11
d) sc. (0137)
<DG> cycles
sum classes 11, 1
sum classes 3, 5
sum classes 7, 9
<IC> cycles
sum classes 1, 3
sum classes 5, 7
sum classes 9, 11


e) sc. (0146)
<GA> cycles
sum classes 11, 1
sum classes 3, 5
sum classes 7, 9
<FC> cycles
sum classes 1, 3
sum classes 5, 7
sum classes 9, 11
f) sc. (0148)
<AJ> cycles
sum classes 11, 1
sum classes 3, 5
sum classes 7, 9
<EB> cycles
sum classes 1, 3
sum classes 5, 7
sum classes 9, 11
g) sc. (0157)
<AG> cycles
sum classes 11, 1
sum classes 3, 5
sum classes 7, 9
<HB> cycles
sum classes 1, 3
sum classes 5, 7
sum classes 9, 11
h) sc. (0236)
<JD> cycles
sum classes 11, 1
sum classes 3, 5
sum classes 7, 9
<FI> cycles
sum classes 1, 3
sum classes 5, 7
sum classes 9, 11

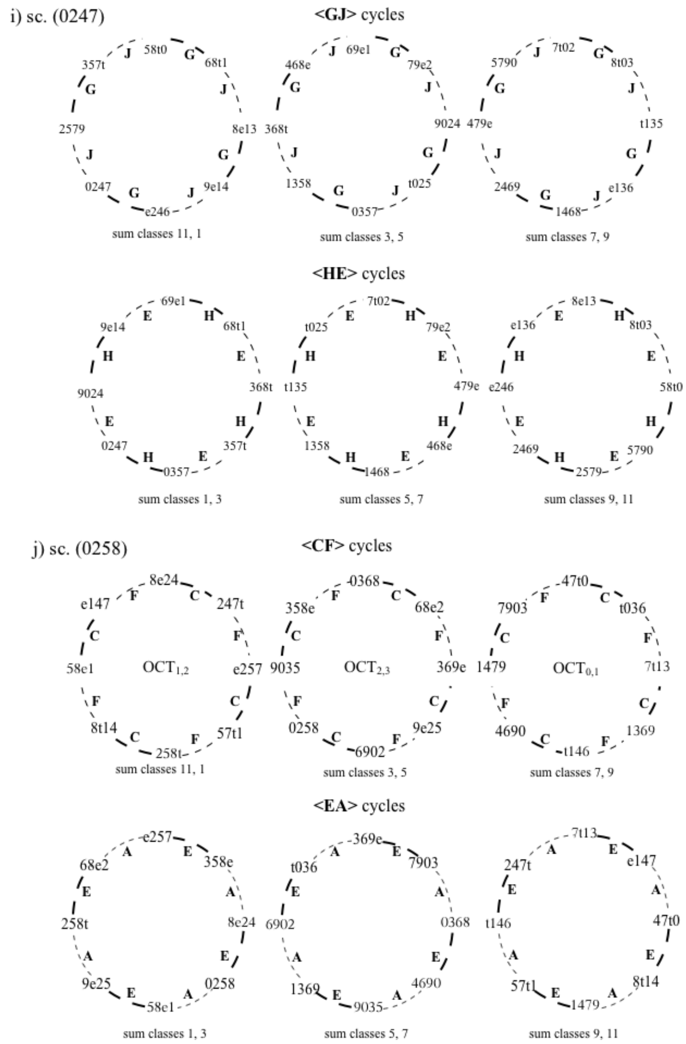


Figure 7: *Cycles for all target sets with cardinality 4.*

The bridge sets with cardinality 4 have its members in even sum classes and are divided in two groups: bridge sets I, which members are in sum classes 0, 4, and 8; bridge sets II, which members are in sum classes 2, 6, and 10. It is possible to build tetrachords cycles similar to those with trichords shown in Figure 10, in which all members of a target set are of the same class of sum. The sets of these cycles can be related by a chain of contextual inversions or by a chain of transpositions of 3 semitones. Table 2 provides the labels of the relation that connects bridge sets I and II with supersets of each cycle.

V. Trees and other graphs with two different set classes

In addition to the cycles, *trees*, which are connected graphs containing no cycles ([4, p. 18]), are another type of graph used by the Neo-Riemannian Theory. The two most well known neo-Riemannian tree graphs are the Weitzmann graph and the Boretz Spiders. Cohn have designed these graphs using the chord symbols but, in order to keep the same criteria used in our previous

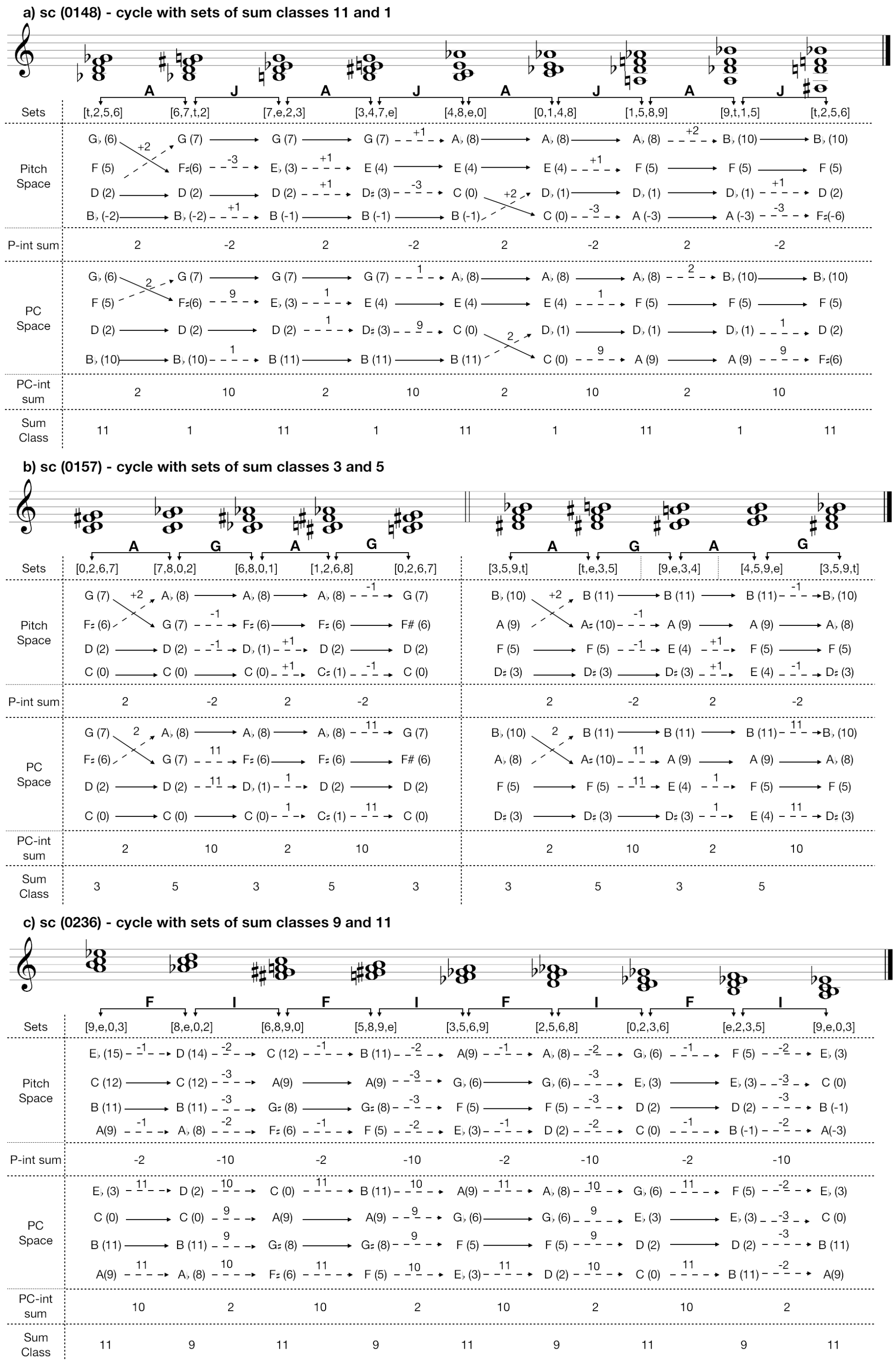
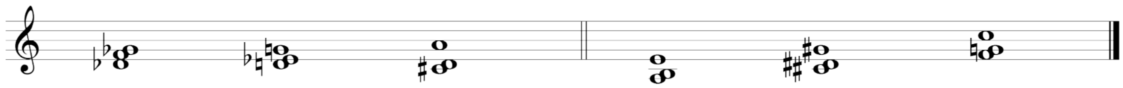


Figure 8: *Examples of voice-leading in cycles with sc. (0148), sc. (0157) and sc. (0236).*

a) sc. (015) b) sc. (027)



		A		E			T_8		T_4	
Sets	[1,5,6]		[2,3,7]		[9,1,2]	[9,e,4]		[1,3,8]		[5,7,9]
Pitch Space	G♭ (6)	← -1	G (7)	+2 →	A (9)	E (4)	← -4	G♯ (8)	+4 →	A (9)
	F (5)	← +2	E♭ (3)	-1 →	D (2)	B (-1)	← -4	D♯ (3)	+4 →	G (7)
	D♭ (1)	← -1	D (2)	-1 →	C♯ (1)	A (-3)	← -4	C♯ (1)	+4 →	F (5)
P-int sum		0		0			-12		12	
Pitch-Class Space	G♭ (6)	← 11	G (7)	2 →	A (9)	E (4)	← 8	G♯ (8)	4 →	A (9)
	F (5)	← 2	E♭ (3)	11 →	D (2)	B (11)	← 8	D♯ (3)	4 →	G (7)
	D♭ (1)	← 11	D (2)	11 →	C♯ (1)	A (9)	← 8	C♯ (1)	4 →	F (5)
PC-int sum (mod 12)		0		0			0		0	
Sum Class	0		0		0	0		0		0

Figure 9: *Connections between members of sc. (015) and (027) that keep sets in the same sum class.*

graphs, they are reproduced in Figure 12 with all sets notated in the normal form. The nodes in the center of each graph is related to the pivot sets. They are connect with other sets by a gray solid line. The solid line, as before, represents a parsimonious connection between two graphs, and the gray color indicates that this connection is between members of different set classes.

Since these two graphs are subgraphs of both Cube Dance and Power Towers, and may therefore serve as a model to build other graphs, we will highlight some of their main features as we did previously with HexaCycles.

- **1) All the sets belong to two different set classes.** Both graphs have four components with a root (a node that is connect with all others) that is related to a pivot set (sc. (048) in Weitzmann graph and sc. (0369) in Boretz Spiders). All remaining nodes are related to a target sets (sc. (037) in Weitzmann graph and sc. (0258) in Boretz Spiders);
- **2) Each component is embedded in a symmetric collection.** The superset of each component in Weitzmann graph is the nonatonic collection listed below it, and the superset of each Boretz spider is the aggregated of 12 pitches;
- **3) All sets of each component belong to three adjacent sum classes.** All components of both graphs are limited to only three sum classes, with the pivot set of each component being in the sum class between the two sum classes of the target sets.
- **4) Parsimonious voice-leading between the pivot and the target sets.** Since the target sets do not connect to each other in either graph, all connections are between members of different sets of classes, which is a remarkable difference between tree graphs and cycles previously discussed. However, the target and the pivot sets in each graph were chosen because they connect parsimoniously.

We can use Weitzmann graphs and Boretz Spiders as models in order to build new graphs using different target sets, but since there are only one pivot set for cardinality 3 and one for

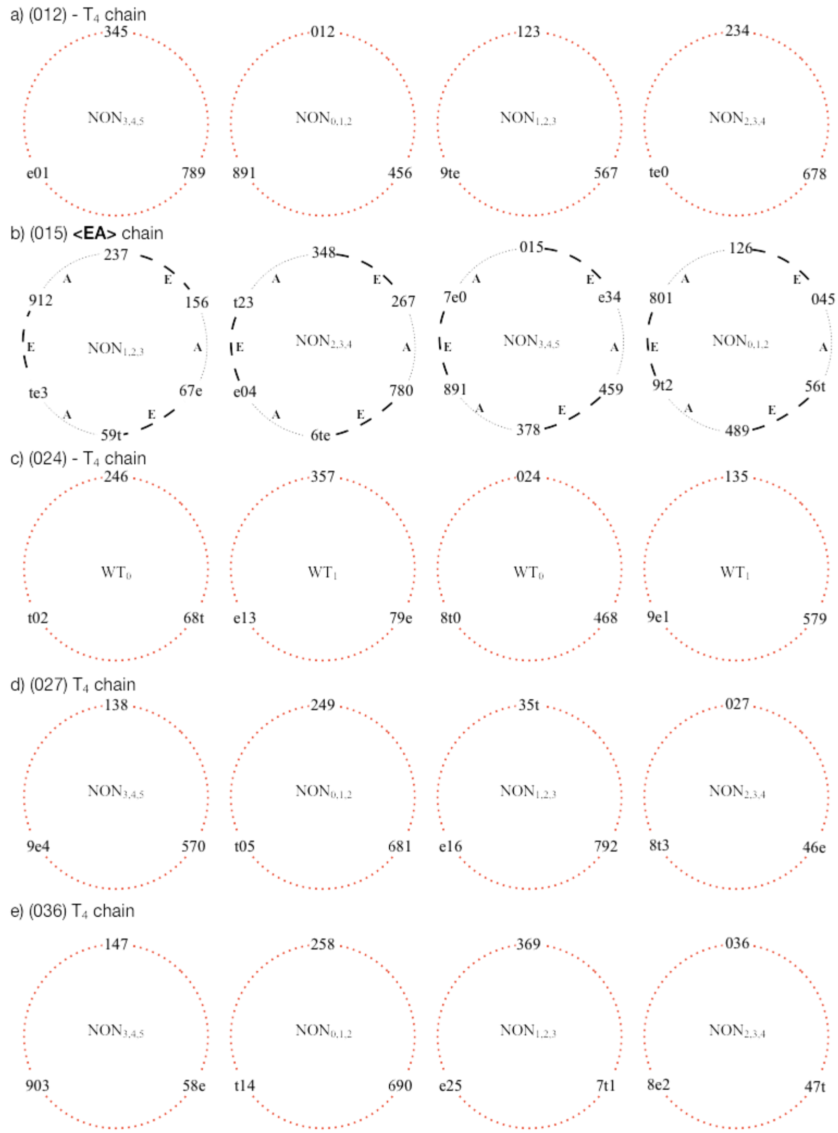


Figure 10: *Cycles for all pivot sets with cardinality 3 (solid black lines represent parsimonious connections, dashed and dotted black lines represent non-parsimonious connections, and red dotted lines represent T4 connections).*

cardinality 4 (sc. (048) and sc. (0369), respectively), we will have to replace them by bridge sets that can connect to these target sets by parsimonious voice-leading. As previously mentioned, the difference between the pivot and bridge sets is that the former make connections with every target set placed in their adjacent voice-leading zones, while the latter make connections with some, but not all target sets placed in their adjacent voice-leading zones. Because of this, we will replace in both trichord and tetrachord graphs the nodes of the pivot sets by cycles that we have previously built, and which connect bridge sets in the same sum class by transposition. In this way, each bridge set in the cycle will connect at least to two target sets (see the models for those graphs in Figure 13). Note that although we have used the graphs of Figure 14 as models, these cannot be considered as tree graphs because they include a cycle of bridge sets. In this way we can build

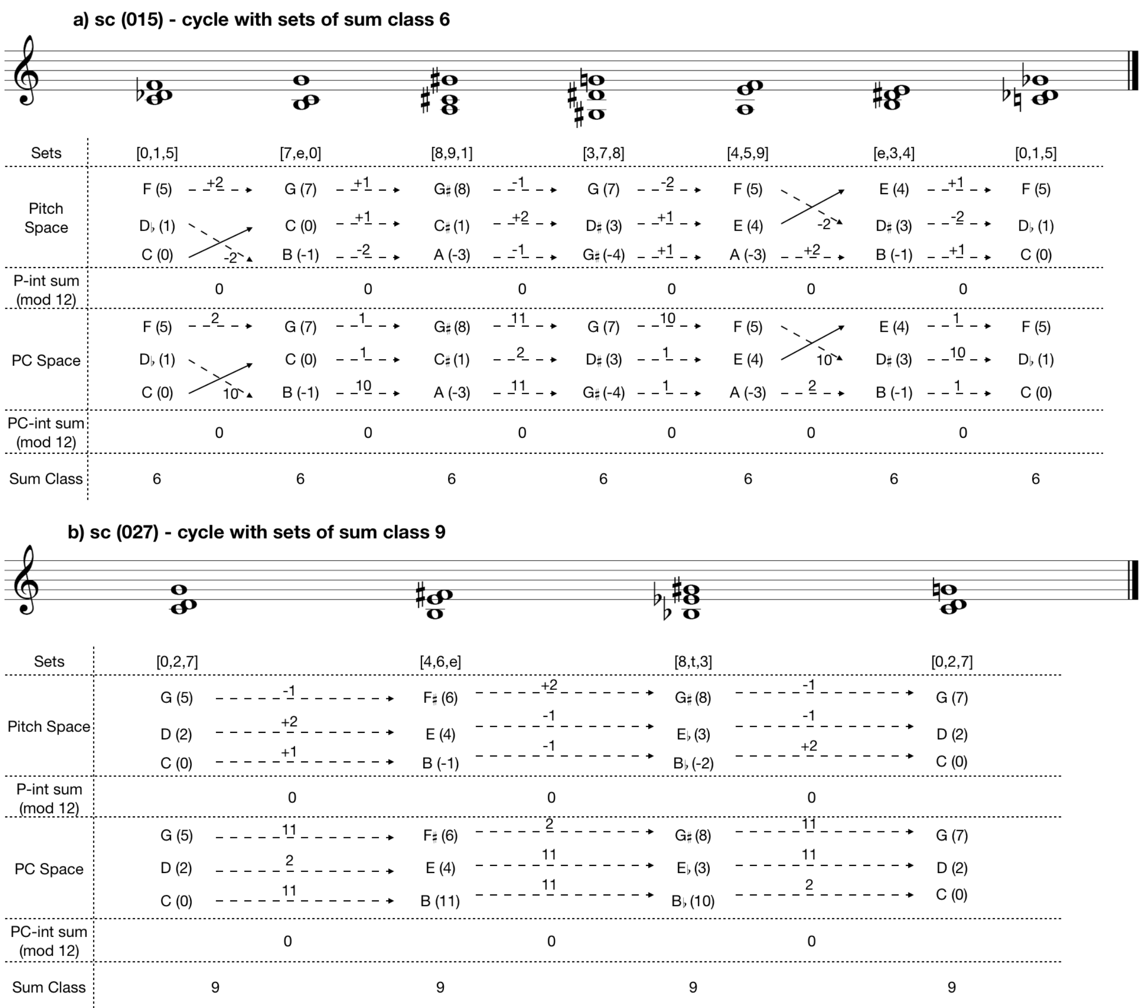


Figure 11: *Examples of voice-leading in cycles with sc. (015) and sc. (027).*

12 graphs that connect two sets of trichords by parsimony voice-leading. Those graphs include all set classes of cardinality 3. Each graph has four components and the sets of each one are in three adjacent sum classes listed below them. The supersets of those components are listed in the middle of the cycle, and if not, the superset is the aggregate of 12 pitches. The arrangement of the sets in all trichords graphs which sc. (015) is the bridge set is different from those with the symmetrical sc. (012), (024), (027), and (036), and therefore there are two different types of designs for the components of these graphs shown in Figure 13. Components of type 1 have a central cycle in which members of a inversional symmetric set class are connected by T4. Each bridge set is also connected to two target sets, so this type of component has 9 members. Components of type 2 have a central cycle in which members of sc. (015) are connected by contextual inversion. Each bridge set is also connected to one target set, so this type of component has 12 members.

Figure 14 shows all 12 graphs that connect two different sets of trichords. There are 8 with components of type 1 and 4 with components of type 2. I will represent the graphs using the prime form of both sets separated by a slash, the first is always the pivot or the bridge set while the second is the target set.

Table 2: *The table on the left provides information about tetrachord cycles with bridge sets I and, the table on the right provides information about tetrachord cycles with bridge sets II.*

Bridge Sets I Sum Classes 0, 4 and 8				Bridge Sets II Sum Classes 2, 6 and 10		
Set Class	Cycle (all sets in the same sum class)	Superset		Set Class	Cycle (all sets in the same sum class)	Superset
A (0125) B (0345)	six cycles **<HB>** chain	MM4		(0123)	three cycles T_3 related	AGG
(0134)	three cycles T_3 related	MM2		(0127)	three cycles T_3 related	AGG
A (0147) B (0367)	three cycles **<AG>** chain	MM7		A (0136) B (0356)	**<CF>** and **<FI>** chains three cycles each	AGG
(0156)	three cycles **<AG>** chain	AGG		(0145)	three cycles T_3 related	AGG
A (0237) B (0457)	six cycles **<JD>** chain	MM4		(0158)	three cycles T_3 related	AGG
(0246)	three cycles T_3 related	AGG		(0167)	three cycles T_3 related	MM2
(0268)	three cycles T_3 related	MM2		(0235)	three cycles T_3 related	MM2
(0358)	three cycles T_3 related	MM2		(0248)	three cycles T_3 related	AGG
				(0257)	three cycles T_3 related	AGG
				(0347) T_3 related	three cycles	MM2

Next we can describe the voice-leading between sets in two components of two different graphs. Figure 15a) is an example of voice-leading between sets of the third component (left to right) of graph 024/014 (shown in Figure 14b)), while Figure 15b) is an example of voice-leading between sets of the third component (left to right) of graph 015/026 (shown in Figure 14l)). One can see in the left side of Figure 15 the patch I choose to connect the sets on the graph and in the right side one of the possible voice-leadings for these sets. Note that I have used parsimony voice-leading for every connection between members of two different sets (red arrows in the graphs), while the members of the same set were connected by pure contrary motion (blue arrows).

There are 57 graphs that connect two different tetrachords sets by parsimony voice-leading, and each one has three components with its members in three adjacent sum classes listed below them. The components of those graphs are divided in 5 different types: components of type 1 have a central cycle whose members of a inversional and transpositional symmetric set classes are connected by T_6. Each one of the bridge sets is also connected to four target sets, so this type of component has 10 members. Components of type 2 have a central cycle whose members of a inversional symmetric set classes are connected by T_3. Each one of the bridge set is also connected to two target sets, so this type of component has 12 members. Components of type 3 have a central cycle whose members are connected by contextual inversion. Each one of the bridge set is also connected to one target set, so this type of component has 16 members arranged in two subtypes

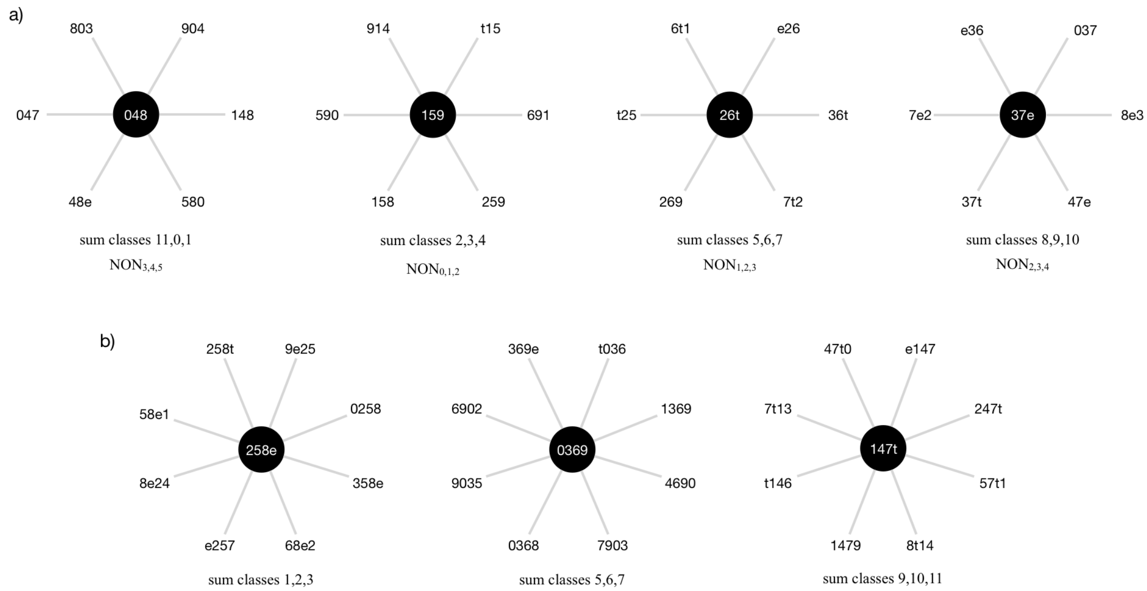


Figure 12: *a) Weitzmann graph; b) Boretz Spiders.*

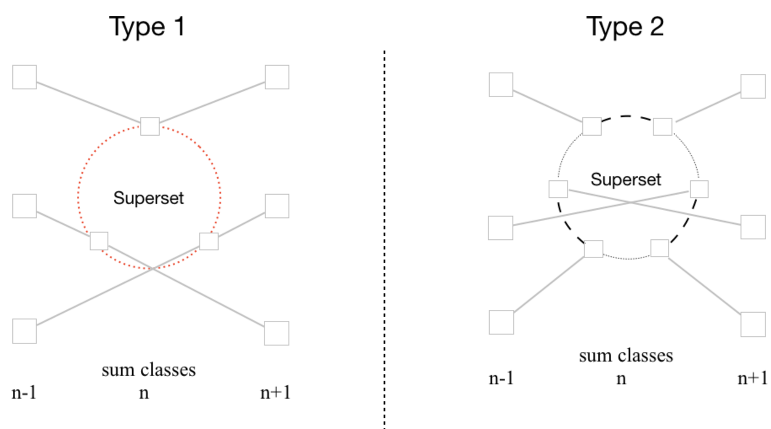
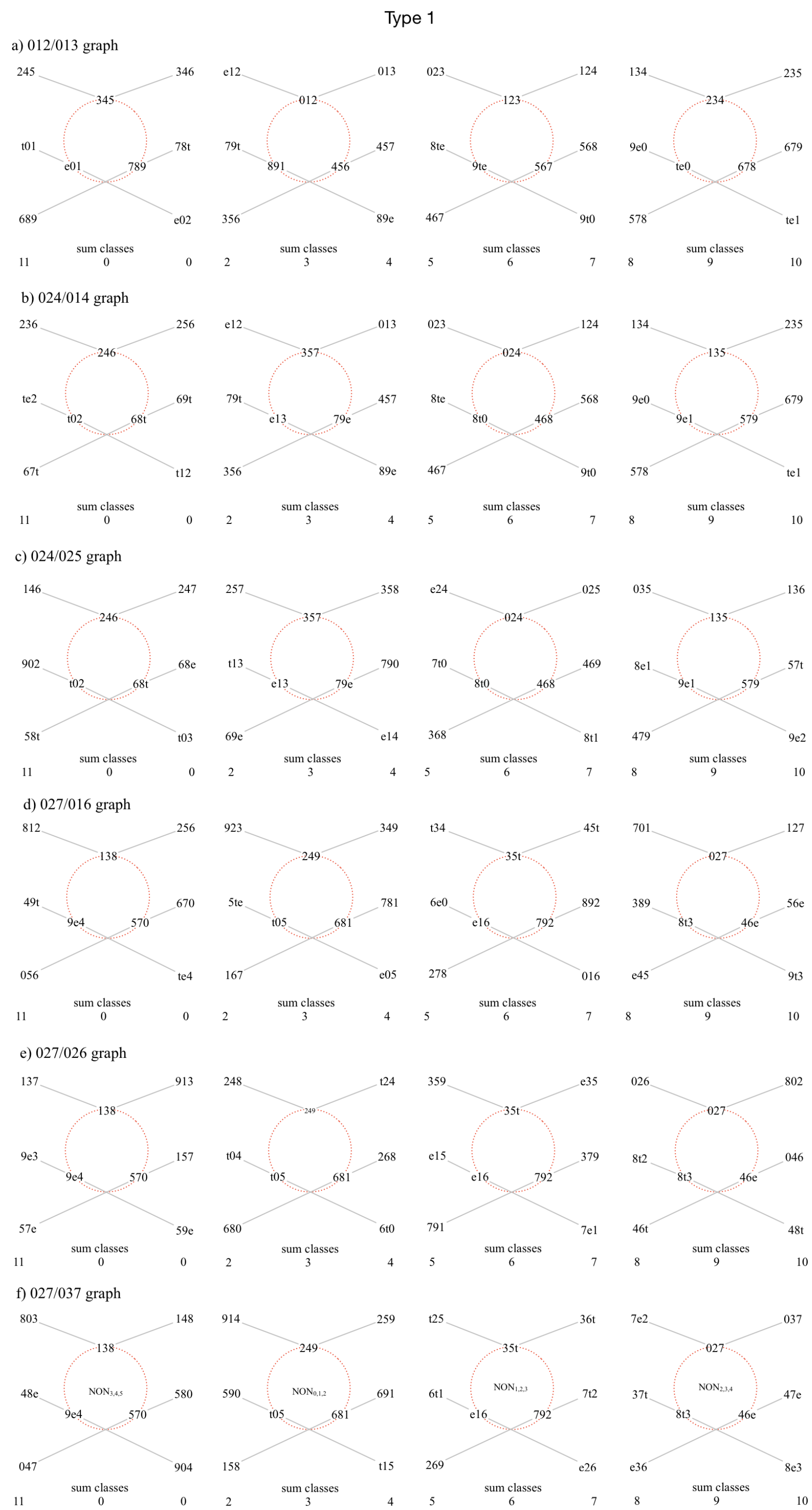


Figure 13: *Two types of components in graphs that connect two sets of trichords.*

of components, 3a and 3b. Each component of type 4 is divided in two cycles in which 4 members are connect by contextual inversion. Each one of the bridge set is also connected to one target set, so this type of component has 16 members arranged in two subtypes of components, 4a and 4b. There is only one graph with components of type 5, and we will analyze it separately below. Figure 16 shows the design of the components of type 1, 2, 3, and 4.

Figure 17 shows one example of graphs that connect two different tetrachords with each type and subtype of component. The graph of Figure 17a) shows three components of type 1 with members of sc. (0268) as bridge sets and members of sc. (0157) as target sets; Figure 17b) shows a graph divided in components of type 2 with members of sc. (0127) as bridge sets and members of sc. (0126) as target sets; Figure 17c) shows a graph divided in components of type 3a with members of sc. (0136) as bridge sets and members of sc. (0126) as target sets; Figure 17d) shows a graph divided in components of type 3b with members of sc. (0136) as bridge sets and members


Type 1
a) 012/013 graph
245 346 345 t01 78t e01 789 689 e02
sum classes 11 0 0
e12 013 012 79t 457 891 456 356 89e
sum classes 2 3 4
023 124 123 8te 568 9te 567 467 9t0
sum classes 5 6 7
134 235 234 9e0 679 te0 678 578 te1
sum classes 8 9 10
b) 024/014 graph
236 256 246 te2 69t t02 68t 67t t12
sum classes 11 0 0
e12 013 357 79t 457 e13 79e 356 89e
sum classes 2 3 4
023 124 024 8te 568 8t0 468 467 9t0
sum classes 5 6 7
134 235 135 9e0 679 9e1 579 578 te1
sum classes 8 9 10
c) 024/025 graph
146 247 246 902 68e t02 68t 58t t03
sum classes 11 0 0
257 358 357 t13 790 e13 79e 69e e14
sum classes 2 3 4
e24 025 024 7t0 469 8t0 468 368 8t1
sum classes 5 6 7
035 136 135 8e1 57t 9e1 579 479 9e2
sum classes 8 9 10
d) 027/016 graph
812 256 138 49t 670 9e4 570 056 te4
sum classes 11 0 0
923 349 249 5te 781 t05 681 167 e05
sum classes 2 3 4
t34 45t 35t 6e0 892 e16 792 278 016
sum classes 5 6 7
701 127 027 389 56e 8t3 46e e45 9t3
sum classes 8 9 10
e) 027/026 graph
137 913 138 9e3 157 9e4 570 57e 59e
sum classes 11 0 0
248 t24 249 t04 268 t05 681 680 6t0
sum classes 2 3 4
359 e35 35t e15 379 e16 792 791 7e1
sum classes 5 6 7
026 802 027 8t2 046 8t3 46e 46t 48t
sum classes 8 9 10
f) 027/037 graph
803 148 138 NON3,4,5 48e 580 9e4 570 047 904
sum classes 11 0 0
914 259 249 NON0,1,2 590 691 t05 681 158 t15
sum classes 2 3 4
t25 36t 35t NON1,2,3 6t1 7t2 e16 792 269 e26
sum classes 5 6 7
7e2 037 027 NON2,3,4 37t 47e 8t3 46e e36 8e3
sum classes 8 9 10

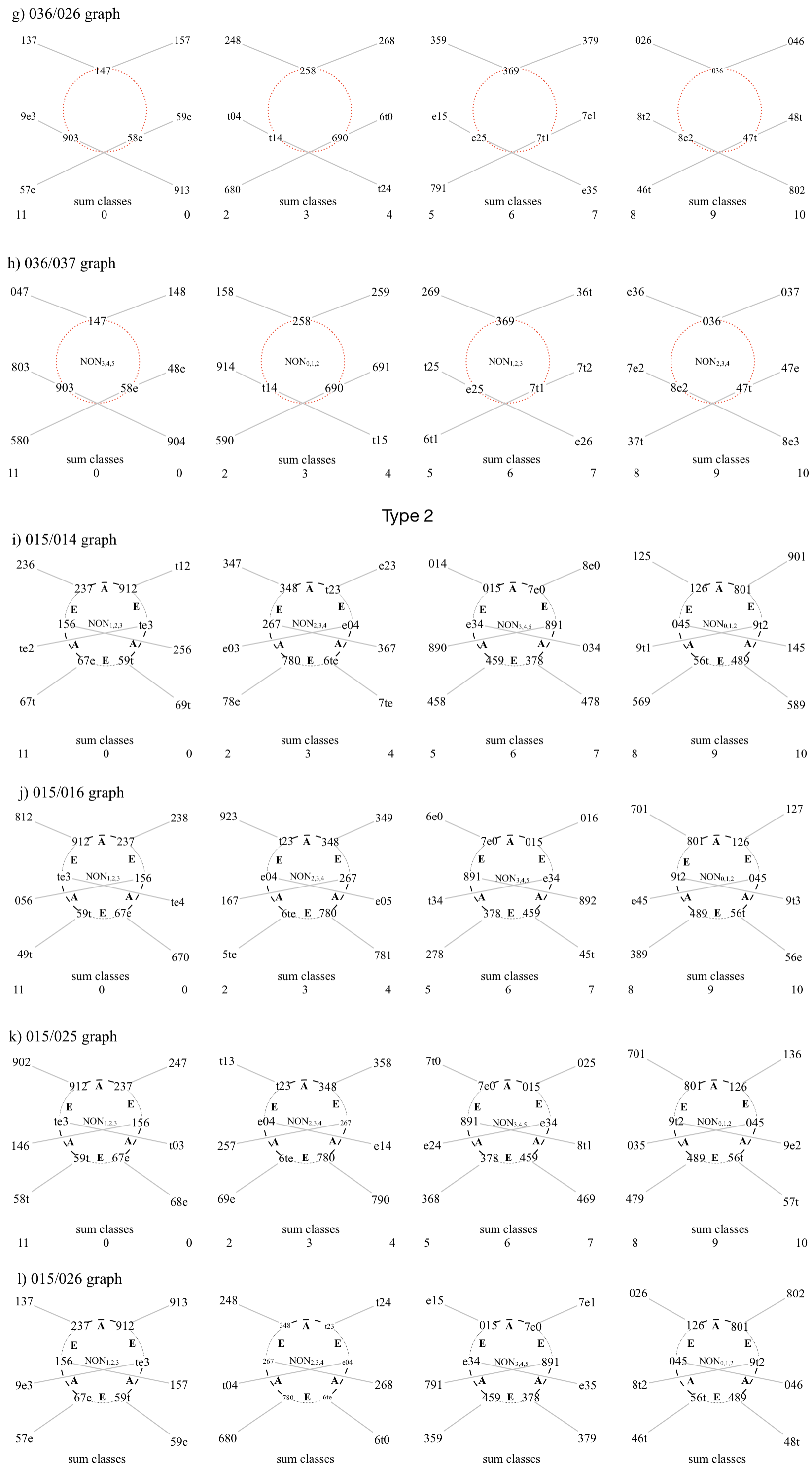


Figure 14: *Graphs that connect two sets of trichords by parsimony voice-leading.*

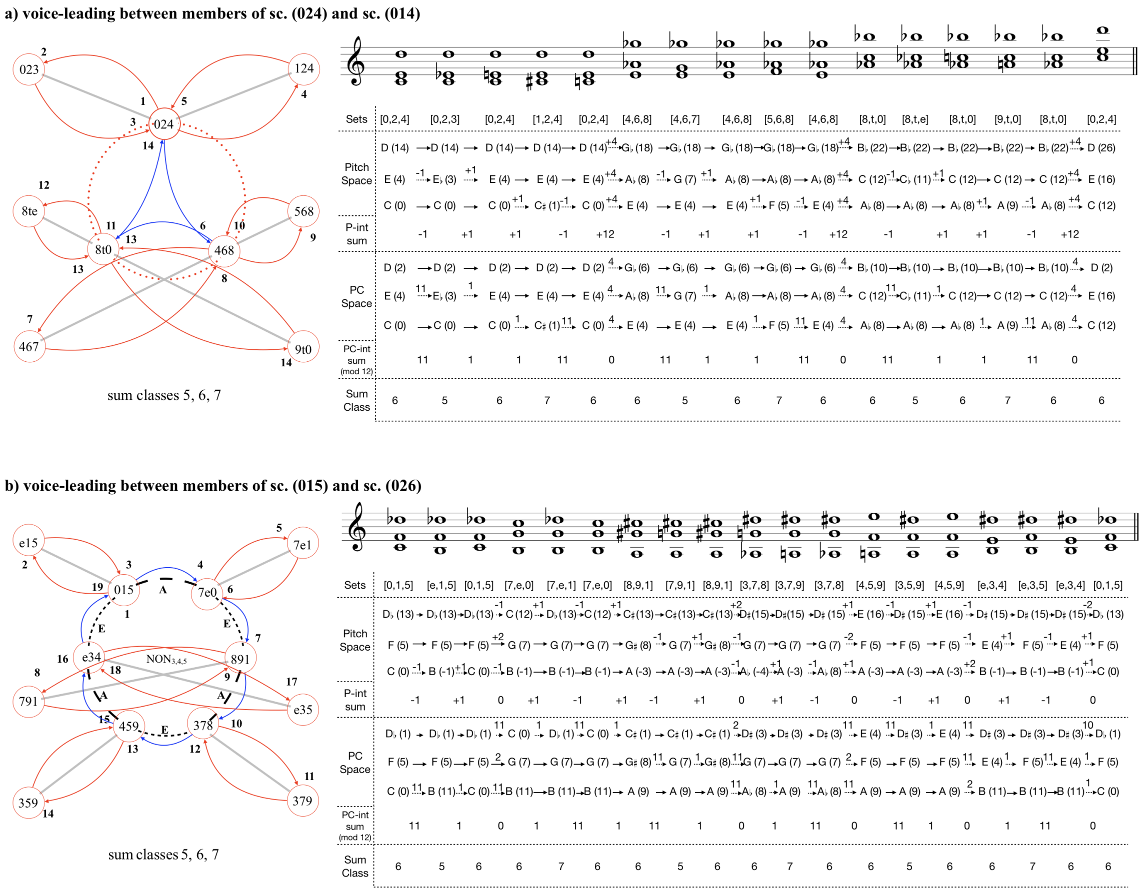


Figure 15: *Examples of voice-leading for sets of the 024/014 and 015/026 graphs.*

of sc. (0236) as target sets; Figure 17e) shows a graph divided in components of type 4a with members of sc. (0237) as bridge sets and members of sc. (0236) as target sets; Figure 17f) shows a graph divided in components of type 4b with members of sc. (0147) as bridge sets and members of sc. (0146) as target sets.

In Table 3 of Appendix are listed all 57 graphs that connect two different tetrachords. In it all the graphs are divided by the type of its components, which are listed in the left column. In the middle column, the set classes of the graphs are listed. The first is the bridge set, and the second is the target set on the chart. It can be seen in the right column, in which sum classes the members of the three components of each graph are distributed.

Graph 0358/0258 is the only one with components of type 5. This graph is similar to the Douthett/Steinbach's OctaTowers ([3, p. 246, Fig. 4]), but Figure 18 shows it with an alternative design and with the normal forms for each set instead the chord symbols in order to keep same appearance in all graphs. Therefore, this graph has three components with a central cycle whose members of sc. (0358) are connected by T_3. Each member of this set class is also connected to four members of sc. (0258), so all the components have 12 members.

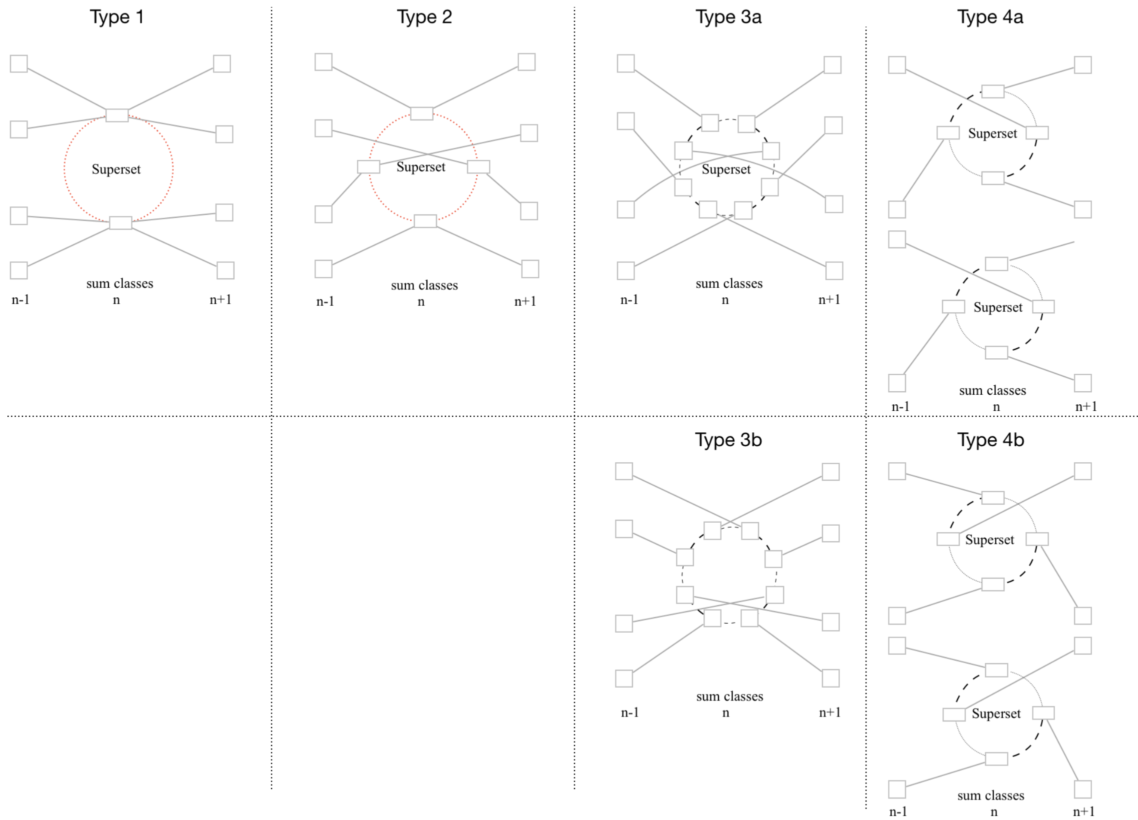


Figure 16: *Components of type 1,2,3, and 4 for graphs that connect two sets of tetrachords.*

VI. Unified Models of Voice Leading Space

Richard Cohn refers to the graph known as Cube Dance ([2, p. 83, Fig. 5.24]) as a "unified model of triadic voice-leading space" ([2, p. 83]) since it "includes the four hexatonic cycles and the four Weitzmann regions as contiguous subgraphs" ([2, p. 85, Fig. 5.24]). I will refer to this graph as 048/037 Cube Dance, since members of sc. (048) are the pivot sets and members of sc. (037) are the target sets in it. Figure 19 shows 048/037 Cube Dance with some changes to the original graph: the sets are represented by its normal forms instead of the chord symbols; the black lines represent connections between members of the same set class and the gray lines represent connections between members of different set classes; thickest black lines represent connections using the contextual inversion axis **H**, while the thinner black lines represent connections using the contextual inversion axis **D**.

Next we list the main features of the original Cube Dance, so that these can be used to build other graphs as done above with the cycles and with tree graphs.

- **1) All the sets in the graph belong to two different set classes.** As observed by Cohn, Cube Dance has the hexatonic cycles and the Weitzmann regions as contiguous subgraphs. These subgraphs are arranged in order to make four cubes with members of sc. (048) in the intersections between them, and members of sc. (037) in the remaining vertices.
- **2) Sets are placed in voice-leading zones.** The entire graph is placed inside a clock face. The sum of pitches modulo 12 of each set is equal to the number next to it. These numbers

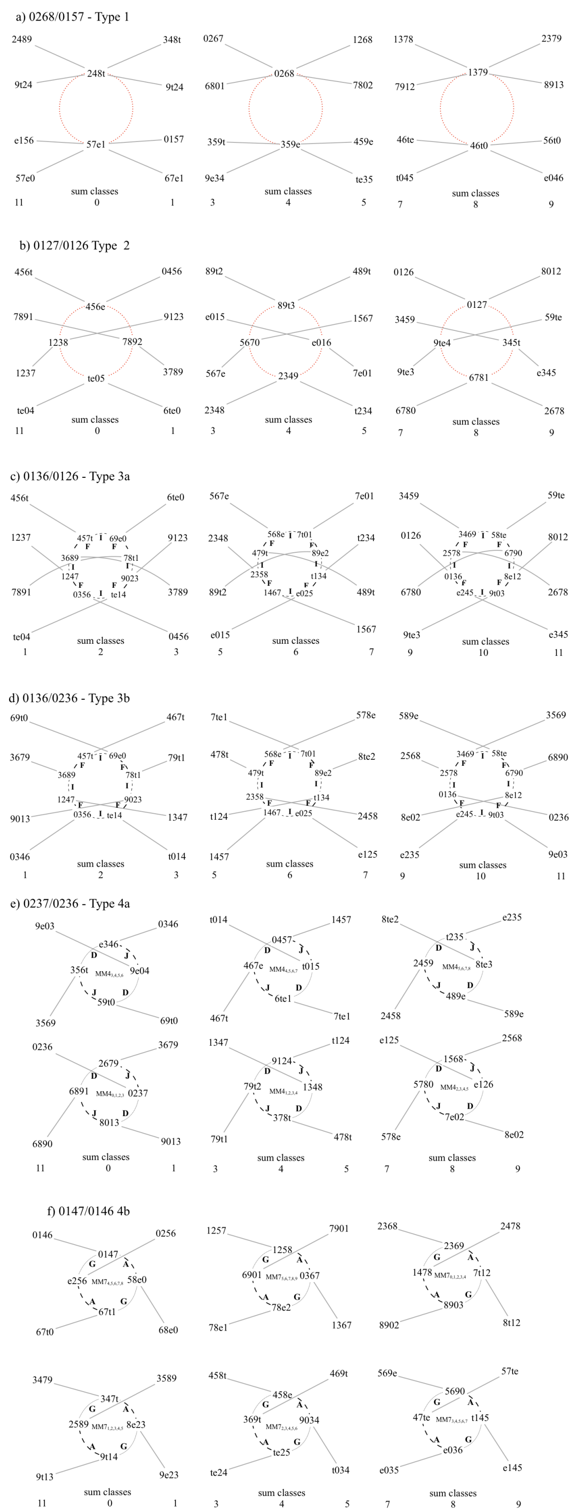


Figure 17: *Examples of graphs that connect two sets of tetrachords.*

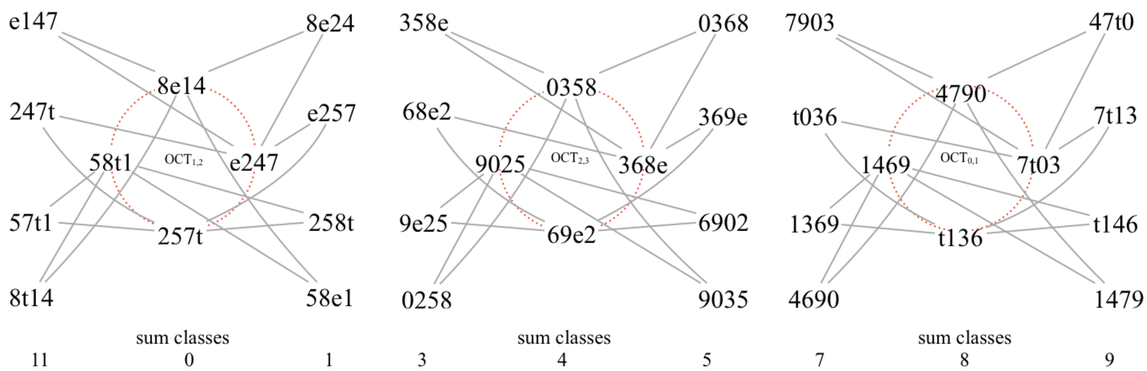


Figure 18: *The 0358/0258 graph is similar to Douthett/Steinbach's OctaTowers.*

are the voice-leading zones. In order words, the sets in Cube Dance are organized in adjacent sum classes.

- **3) Each cube has a symmetrical superset.** All the sets in each cube are embedded in a particular hexatonic collection. The sets in northeast cube are in $HEX_{0,1}$; sets in southeast cube are $HEX_{1,2}$; sets in southwest cube are $HEX_{2,3}$; and sets in northwest cube are $HEX_{3,4}$.
- **4) Parsimonious voice-leading between all connected sets.** Any connection between two sets in Cube Dance is parsimonious, regardless of whether the sets belong to the same set class or not. So it is possible to make a path that connects several chords with a voice-leading work that uses only displacements of one semitone.

We can build Cube Dances using trichords other than members of sc. (048) and (037). They will share the first three features of the original Cube Dance listed before[16]. As seen in cycles of trichords shown in Figure 5, the only set class in which members can relate parsimoniously by contextual inversion to two other members is sc. (037). So in order to build Cube Dances with different trichords one must admit that the voice-leading between zones 1-2, 4-5, 7-8, and 10-11 may not necessarily be parsimonious, although it connects members of adjacent sum classes.

Figure 20 shows the 024/014 Cube Dance, a graph that connects all members of these two set classes with similar patterns of voice-leading throughout their vertices. In this new Cube Dance all the vertices where were the pivot sets (in zones 0, 3, 6, and 9) in the original Cube Dance were replaced by the four cycles with sc. (024) shown in Figure 10c), so the members of this set class are the bridge sets in the graph. In the zones 1-2, 4-5, 7-8, and 10-11, the hexatonic cycle were replaced by the four cycles with sc. (014) shown in Figure 5b), and the members of this set class are the target set in the graph. In other words, the 024/014 Cube Dance includes the 024/014 graph shown in Figure 14b) and the cycle of **<DH>** chain with sc. (014) as contiguous subgraphs. The red and the black lines in the graph represent connections between members of same set class, red lines connect them by transposition and black lines by contextual inversion. Gray lines connect members of different set classes. Solid lines represent connection with parsimonious voice-leading, while dashed and dotted lines represent connections using the contextual inversions axes **D** and **H**, respectively. The graph of Figure 20 gives an example of connection with this sets represented by the path with red arrows and circles. The voice leading between these sets are shown in Figure 21. Certainly there are countless other possible ways to connect these sets that can be traced in this Cube Dance.

Figure 22 shows the 027/016 Cube Dance. It includes the 027/016 graph, shown in Figure 14d),

[16] About the third feature of the list: the supersets of each cube in all graphs is the aggregate of the 12 pitches.

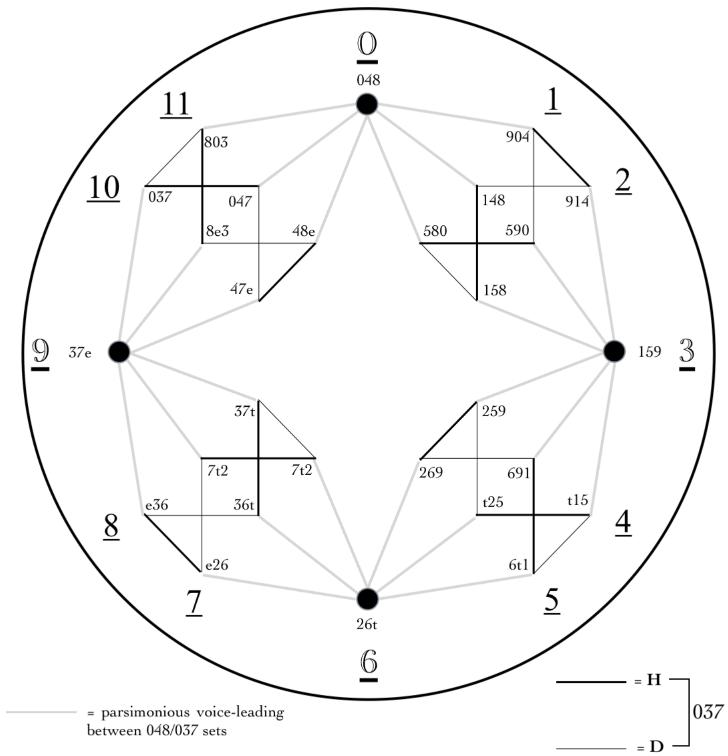


Figure 19: *048/037 Cube Dance.*

and the cycle of **<BJ>** chain with sc. (016) as contiguous subgraphs. The pattern of the connection lines is the same used in Figure 20, but the black solid and dotted lines represent connections using the contextual inversions axes **B** and **J**, respectively. The path drawn over this graph is an example of a connection that explores the parsimonious voice leading in this graph, as shows Figure 23.

Figure 24 shows the 015/025 Cube Dance. This graph has a different design from the previous ones, since members of sc. (015) are the bridge sets and, as previously observed, this set class is the only one in which its members are distributed in the voice-leading zones 0, 3, 6, and 9 and is not symmetrical. Thus, the vertices in the intersections between the four cubes in the original graph were replaced by the cycles of **<DH>** chain with sc. (015) shown in Figure 11b), and this is what changes the model of the graph, since cycles with 6 instead of 3 sets, are placed in those positions. That is, the Cube Dances of Figures 20 and 22 included graphs of type 1 as subgraphs between the voice-leading zones 11-0-1, 2-3-4, 5-6-7, and 8-9-10, while the 015/025 Cube Dance includes graphs of type 2 as subgraphs in these same positions (the types of graphs that connect two sets of trichords are shown in Figure 13). Another important difference between this graph and the previous ones is that in this Cube Dance there are no T_4 direct connections, and all the members of the same set class are connected by contextual inversion. It should be noted that members of sc. (015) are connect with only one target set in adjacent voice-leading zone, so to cross between target sets in two different cubes[17] it is necessary to connect with two members of

[17]By replacing the intersecting vertices of the original Cube Dance with cycles, the graph is deformed and the solids between the voice-leading zones 0-1-2-3, 3-4-5-6, 6-7-8-9, and 9-10-11-0 are no longer exactly cubes. However, I shall still use the term cube, considering that the intersection between these solids is a unit, even if it is represented by a cycle. In the same spirit, I shall still refer to these graphs as Cube Dance.

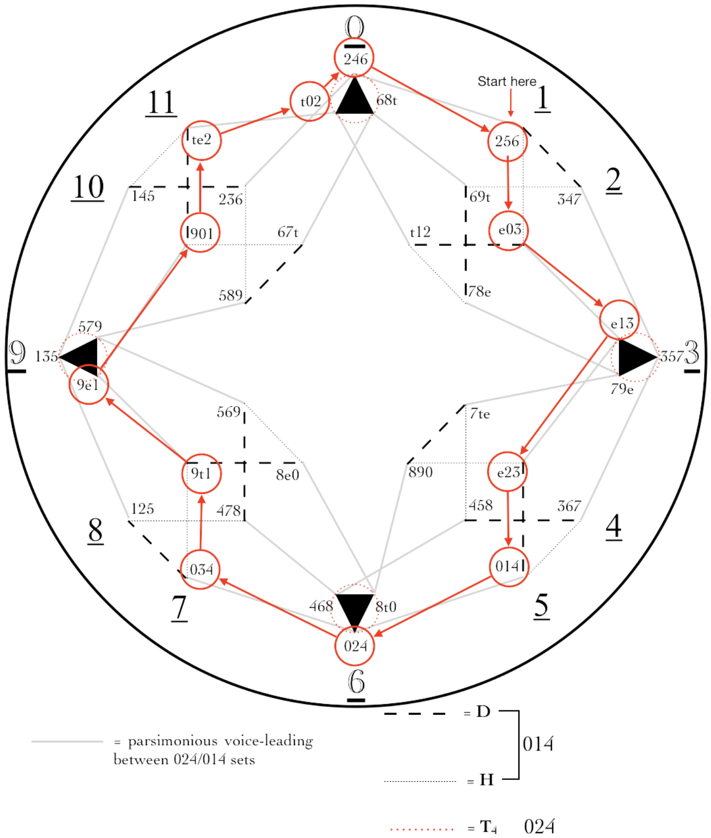
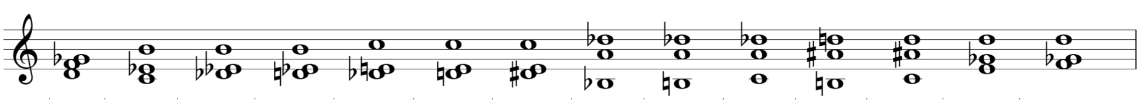


Figure 20: *024/014 Cube Dance.*



Sets	[2,5,6]	[e,0,3]	[e,1,3]	[e,2,3]	[0,1,4]	[0,2,4]	[0,3,4]	[9,t,1]	[9,e,1]	[9,0,1]	[t,e,2]	[t,0,2]	[2,4,6]	[2,5,6]
Pitch Space	G♭ (6)	+5 ⇢ B (11)	→ B (11)	→ B (11)	+1 ⇢ C (12)	→ C (12)	→ C (12)	+1 ⇢ D♭ (13)	→ D♭ (13)	→ D♭ (13)	+1 ⇢ D (14)	→ D (14)	→ D (14)	→ D (14)
	F (5)	-2 ⇢ E♭ (3)	→ E♭ (3)	→ E♭ (3)	+1 ⇢ E (4)	→ E (4)	→ E (4)	+5 ⇢ A (9)	→ A (9)	→ A (9)	+1 ⇢ A♯ (10)	→ A♯ (10)	-4 ⇢ G♭ (6)	→ G♭ (6)
	D (2)	-2 ⇢ C (0)	+1 ⇢ D♭ (1)	+1 ⇢ D (2)	-1 ⇢ D♭ (1)	+1 ⇢ D (2)	+1 ⇢ D♯ (3)	-5 ⇢ B♭ (-2)	+1 ⇢ B (-1)	+1 ⇢ C (0)	-1 ⇢ B (-1)	+1 ⇢ C (0)	+4 ⇢ E (4)	+1 ⇢ F (5)
P-int Sum		+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	0	+1
PC Space	G♭ (6)	5 ⇢ B (11)	→ B (11)	→ B (11)	1 ⇢ C (0)	→ C (0)	→ C (0)	1 ⇢ D♭ (1)	→ D♭ (1)	→ D♭ (1)	1 ⇢ D (2)	→ D (2)	→ D (2)	→ D (2)
	F (5)	10 ⇢ E♭ (3)	→ E♭ (3)	→ E♭ (3)	1 ⇢ E (4)	→ E (4)	→ E (4)	5 ⇢ A (9)	→ A (9)	→ A (9)	1 ⇢ A♯ (10)	→ A♯ (10)	8 ⇢ G♭ (6)	→ G♭ (6)
	D (2)	10 ⇢ C (0)	1 ⇢ D♭ (1)	1 ⇢ D (2)	11 ⇢ D♭ (1)	1 ⇢ D (2)	1 ⇢ D♯ (3)	7 ⇢ B♭ (10)	1 ⇢ B (11)	1 ⇢ C (0)	11 ⇢ B (11)	1 ⇢ C (0)	4 ⇢ E (4)	1 ⇢ F (5)
PC-int sum (Mod 12)		1	1	1	1	1	1	1	1	1	1	1	0	1
Sum Class	1	2	3	4	5	6	7	8	9	10	11	0	0	1

Figure 21: *An example of voice-leading for the path shown over the 024/014 Cube Dance in Figure 20.*

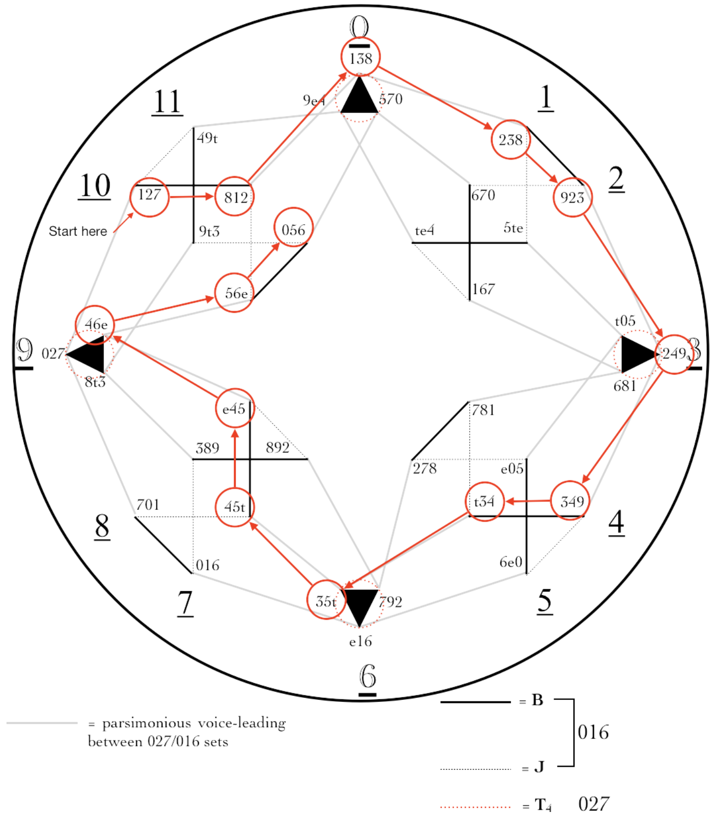
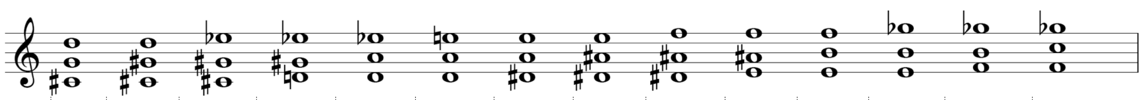


Figure 22: *027/016 Cube Dance.*



Sets	[1,2,7]	[8,1,2]	[1,3,8]	[2,3,8]	[9,2,3]	[2,4,9]	[3,4,9]	[t,3,4]	[3,5,t]	[4,5,t]	[e,4,5]	[4,6,e]	[5,6,e]	[0,5,6]
Pitch Space	D (14) →	D (14) ⇢+1	E♭ (15) →	E♭ (15) →	E♭ (15) ⇢+1	E (16) →	E (16) →	E (16) ⇢+1	F (17) →	F (17) →	F (17) ⇢+1	G♭ (18) →	G♭ (18) →	G♭ (18)
	G (7) ⇢+1	G♯ (8) →	G♯ (8) →	G♯ (8) ⇢+1	A (9) →	A (9) →	A (9) ⇢+1	A♯ (10) →	A♯ (10) →	A♯ (10) ⇢+1	B (11) →	B (11) →	B (11) ⇢+1	C (12)
	C♯ (1) →	C♯ (1) →	C♯ (1) ⇢+1	D (2) →	D (2) →	D (2) ⇢+1	D♯ (3) →	D♯ (3) →	D♯ (3) ⇢+1	E (4) →	E (4) →	E (4) ⇢+1	F (5) →	F (5)
P-int Sum	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	
PC Space	D (2) →	D (2) ⇢1	E♭ (3) →	E♭ (3) →	E♭ (3) ⇢1	E (4) →	E (4) →	E (4) ⇢1	F (5) →	F (5) →	F (5) ⇢1	G♭ (6) →	G♭ (6) →	G♭ (6)
	G (7) ⇢1	G♯ (8) →	G♯ (8) →	G♯ (8) ⇢1	A (9) →	A (9) →	A (9) ⇢1	A♯ (10) →	A♯ (10) →	A♯ (10) ⇢1	B (11) →	B (11) →	B (11) ⇢1	C (0)
	C♯ (1) →	C♯ (1) →	C♯ (1) ⇢1	D (2) →	D (2) →	D (2) ⇢1	D♯ (3) →	D♯ (3) →	D♯ (3) ⇢1	E (4) →	E (4) →	E (4) ⇢1	F (5) →	F (5)
PC-int sum (Mod 12)	1	1	1	1	1	1	1	1	1	1	1	1	1	
Sum Class	10	11	0	1	2	3	4	5	6	7	8	9	10	11

Figure 23: *An example of voice-leading for the path shown over the 027/016 Cube Dance in Figure 22.*

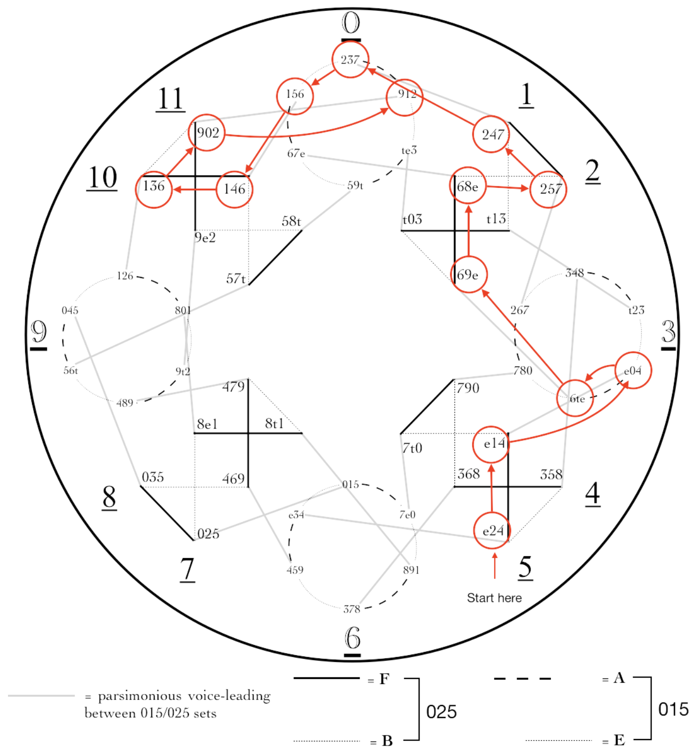
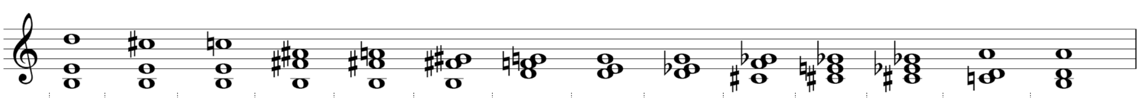


Figure 24: *015/025 Cube Dance.*

sc. (015). The path drawn over this Cube Dance gives an example of how the voice-leading can be with these sets. Figure 25 shows the details of this example.



Sets	[e,2,4]		[e,1,4]		[e,0,4]		[6,t,e]		[6,9,e]		[6,8,e]		[2,5,7]		[2,4,7]		[2,3,7]		[1,5,6]		[1,4,6]		[1,3,6]		[9,0,2]		[9,e,2]
Pitch Space	D (14)	-1	C♯ (11)	-1	C (12)	-2	A♯ (10)	-1	A (9)	-1	G♯ (8)	-1	G (7)	→	G (7)	→	G (7)	+1	G♭ (6)	→	G♭ (6)	→	G♭ (6)	+3	A (9)	→	A (9)
	E (4)	→	E (4)	→	E (4)	+2	F♯ (6)	→	F♯ (6)	→	F♯ (6)	-1	F (5)	-1	E (4)	+1	E♭ (3)	⋯→	F (5)	-1	E (4)	-1	E♭ (3)	-1	D (2)	→	D (2)
	B (-1)	→	B (-1)	→	B (-1)	→	B (-1)	→	B (-1)	→	B (-1)	+3	D (2)	→	D (2)	→	D (2)	-1	C♯ (1)	→	C♯ (1)	→	C♯ (1)	-1	C (0)	-1	B (-1)
P-int Sum		-1		-1		0		-1		-1		+1		-1		+1		0		-1		-1		+1		-1	
PC Space	D (2)	11	C♯ (1)	11	C (0)	10	A♯ (10)	11	A (9)	11	G♯ (8)	11	G (7)	→	G (7)	→	G (7)	11	G♭ (6)	→	G♭ (6)	→	G♭ (6)	3	A (9)	→	A (9)
	E (4)	→	E (4)	→	E (4)	2	F♯ (6)	→	F♯ (6)	→	F♯ (6)	11	F (5)	11	E (4)	11	E♭ (3)	2	F (5)	11	E (4)	11	E♭ (3)	11	D (2)	→	D (2)
	B (11)	→	B (11)	→	B (11)	→	B (11)	→	B (11)	→	B (11)	3	D (2)	→	D (2)	→	D (2)	11	C♯ (1)	→	C♯ (1)	→	C♯ (1)	11	C (0)	1	B (11)
PC-int sum (Mod 12)		11		11		0		11		11		1		11		11		0		11		11		1		11	
Sum Class	5		4		3		3		2		1		2		1		0		0		11		11		10		11

Figure 25: *An example of voice-leading for the path shown over the 015/025 Cube Dance in Figure 24.*

It is possible to build fourteen Cube Dances combining trichord cycles shown in Figure 5 and graphs that connect two trichords shown Figure 14[18]. In Table 4 of the Appendix, a list of all these Cube Dances with its subgraphs and supersets is given.

We can call the Douthett/Steinbach's Power Towers ([3, p. 256, Fig. 10]) a unified model too, since it includes the Boretz Regions and the OctaTowers as subgraphs, and the authors themselves state that "Power Towers is the seventh chord analog to Cube Dance" ([3, p. 255]). I will refer to it as 0369/0358/0258 Power Towers, since members of sc. (0369) in voice-leading zones 2, 6, and 10 are the pivot sets, members of sc. (0358) in voice-leading zones 0, 4, and 8 are the bridge sets, and members of sc. (0258) in the odd voice-leading zones are the target set in this graph. Figure 26 shows the 0369/0358/0258 Power Towers[19] with some changes to the original graph: as done before with the Cube Dances, the sets are represented by their normal forms instead the chord symbols; the OctaTowers are represented as the graph shown in Figure 18; the dotted red lines represent connection between members of same set class related by T_3; the solid gray lines represent connections between members of different set class by parsimony voice-leading (there are no black lines because members of same set class are not connect by contextual inversion in this graph).

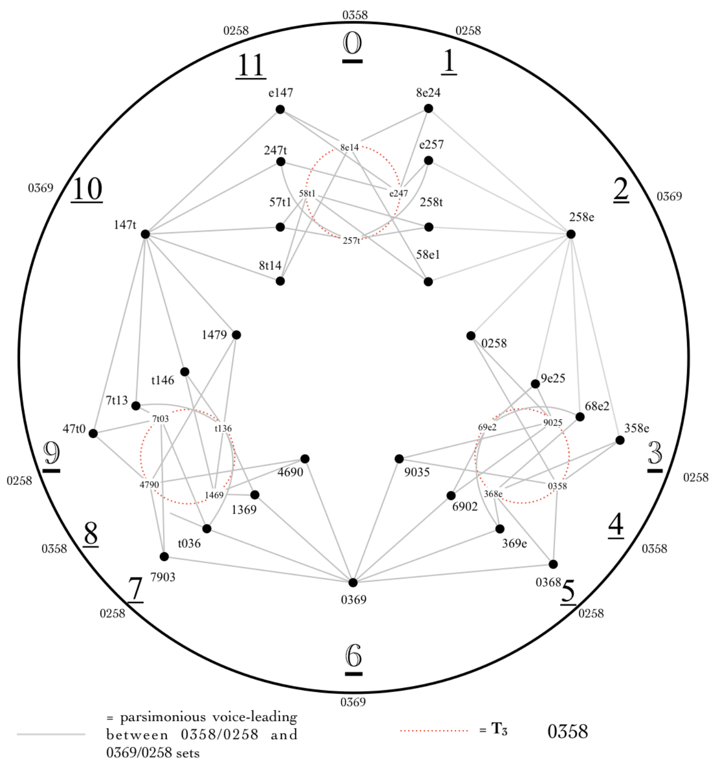


Figure 26: *The 0369/0358/0258 Power Towers.*

We now list the main features of the original Power Dance, as done for the original Cube Dance

[18] In all fourteen Cube Dances we keep the parsimony voice-leading between the bridges and the target sets. However, it would be possible to build Cube Dances without this kind of the connection, which would considerably increase the number of possible graphs.

[19] In Cohn's Book there is a version of this graph that also includes the members of sc. (0268) in the same voice-leading zones of members of sc. (0358) which he calls "4-Cube Trio" ([2, p. 158, Fig. 7.16]). However, for our purpose in this article it will better to limit the graph to three different set classes of the Power Towers.

before:

- **1) All the sets in the graph belong to three different set classes.** Power Tower includes set (0258) as target set in all odd voice-leading zones. Since there is no member of this set class in even voice-leading zones, it is necessary to include members of two different sets of sets to fill these gaps. Because of this, members of sc. (0369) are placed in voice-leading zones 2, 6, and 10 as pivot sets and members of sc. (0358) are placed in voice-leading zones 0, 4, and 8 as pivot sets.
- **2) Sets are placed in voice-leading zones.** As in Cube Dance, all the sets are organized in adjacent sum classes.
- **3) Symmetrical superset.** All sets in voice-leading zone 10 to 2 are embedded in $OCT_{1,2}$, all sets in voice-leading zone 2 to 6 are embedded in $OCT_{2,3}$ and all sets in voice-leading zones 6 to 10 are embedded in $OCT_{0,1}$.
- **4) Parsimonious voice-leading between all connected sets.** As in Cube Dance, any connection between two sets is parsimonious.

Next we can build Power Towers with different sets in the same way we did with the Cube Dances, and these new graphs will also share the first three features of previous list. Figure 27 shows the 0167/0147/0157 Power Towers. It includes graphs 0147/0157 (voice-leading zones 11-0-1, 3-4-5, and 7-8-9) and 0167/0157 (voice-leading zones 1-2-3, 5-6-7, and 9-10-11) as contiguous subgraphs. In this graph the red lines connect two members of sc. (0167) related by T_6 and the black dashed and dotted lines represent connections between members of sc. (0147) using the contextual inversions axes **A** and **G**, respectively. The path drawn over this Power Towers shows an example of a pattern of voice-leading that combine parsimony and pure contrary motion with these three sets of trichords. The details of the voice-leading of this path is shown in Figure 28.

Figure 29 shows the 0136/0246/0135 Power Towers which includes graphs 0246/0135 (voice-leading zones 11-0-1, 3-4-5, and 7-8-9) and 0136/0135 (voice-leading zones 1-2-3, 5-6-7, and 9-10-11) as contiguous subgraphs. Red lines connect two members of sc. (0246) related by T_3 and the black dashed and dotted lines represent connections between members of sc. (0136) using the contextual inversions axes **D** and **F**, respectively. The path drawn over this Power Towers shows another example of a pattern of voice-leading that combine parsimony and pure contrary motion, and the details of this voice-leading is shown in Figure 30.

It is possible to build 87 Power Towers combining the tetrachord graphs of Table 3 in the Appendix, where we also give a list of all these Power Towers with its subgraphs and supersets in Table 5.

VII. Conclusion

The visual advantages that are offered by neo-Riemannian graphs are of unquestionable importance both to determine how sets that are not embedded in a scale or collection connect in a given passage, and to highlight certain types of voice-leading. Graphs have performed these two tasks efficiently and have been widely used, especially in the analyzes, in the last decades. However, the fact that most neo-Riemannian graphs include only triads or seventh chords limited the scope of these analyses to a specific type of repertoire primarily comprised of works composed in the 19th century, chord progressions that Cohn call "pantriadic" ([2, p. 34]).

In this paper we have proposed ways to construct several types of graphs that include any set class of trichord and tetrachord. If on the one hand this approach had to renounce the requirement that all the connections between the sets included in the graphs are parsimonious, on the other

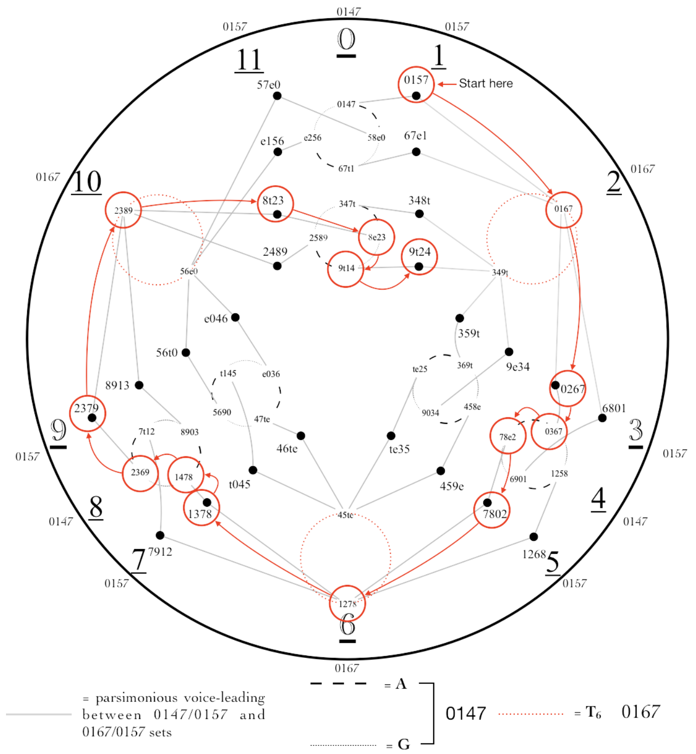


Figure 27: *0167/0147/0157 Power Towers.*

Sets	[0,1,5,7]	[0,1,6,7]	[0,2,6,7]	[0,3,6,7]	[7,8,e,2]	[7,8,0,2]	[1,2,7,8]	[1,3,7,8]	[1,4,7,8]	[2,3,6,9]	[2,3,7,9]	[2,3,8,9]	[8.t.2.3]	[8.e.2.3]	[9,t,1,4]	[9,t,2,4]
Pitch Space	D♭ (13)	→ D♭ (13)	+1 → D (14)	+1 → E♭ (15)	-1 → D (14)	→ D (14)	→ D (14)	+1 → E♭ (15)	+1 → E (14)	-1 → E♭ (15)	→ E♭ (15)	→ E♭ (15)	→ E♭ (15)	→ E♭ (15)	+1 → E (16)	→ E (16)
	G (7)	→ G (7)	→ G (7)	→ G (7)	+1 → A♭ (8)	→ A♭ (8)	→ A♭ (8)	→ A♭ (8)	→ A♭ (8)	+1 → A (9)	→ A (9)	→ A (9)	+1 → B♭ (10)	+1 → B (11)	-1 → B♭ (10)	→ B♭ (10)
	F (5)	+1 → F♯ (6)	→ F♯ (6)	→ F♯ (6)	+1 → G (7)	→ G (7)	→ G (7)	→ G (7)	→ G (7)	-1 → F♯ (6)	+1 → G (7)	+1 → G♯ (8)	→ G♯ (8)	→ G♯ (8)	+1 → A (9)	→ A♭ (8)
	C (0)	→ C (0)	→ C (0)	→ C (0)	-1 → B (-1)	+1 → C (0)	+1 → C♯ (1)	→ C♯ (1)	→ C♯ (1)	+1 → D (2)	→ D (2)	→ D (2)	→ D (2)	→ D (2)	-1 → C♯ (1)	+1 → D (2)
P-int Sum		+1	+1	+1	0	+1	+1	+1	+1	0	+1	+1	+1	+1	0	+1
PC Space	D♭ (13)	→ D♭ (13)	1 → D (14)	1 → E♭ (15)	11 → D (14)	→ D (14)	→ D (14)	1 → E♭ (15)	1 → E (14)	11 → E♭ (15)	→ E♭ (15)	→ E♭ (15)	→ E♭ (15)	→ E♭ (15)	1 → E (16)	→ E (16)
	G (7)	→ G (7)	→ G (7)	→ G (7)	1 → A♭ (8)	→ A♭ (8)	→ A♭ (8)	→ A♭ (8)	→ A♭ (8)	1 → A (9)	→ A (9)	→ A (9)	1 → B♭ (10)	1 → B (11)	11 → B♭ (10)	→ B♭ (10)
	F (5)	1 → F♯ (6)	→ F♯ (6)	→ F♯ (6)	2 → G (7)	→ G (7)	→ G (7)	→ G (7)	→ G (7)	11 → F♯ (6)	1 → G (7)	1 → G♯ (8)	→ G♯ (8)	→ G♯ (8)	1 → A (9)	→ A♭ (8)
	C (0)	→ C (0)	→ C (0)	→ C (0)	11 → B (-1)	1 → C (0)	1 → C♯ (1)	→ C♯ (1)	→ C♯ (1)	1 → D (2)	→ D (2)	→ D (2)	→ D (2)	→ D (2)	11 → C♯ (1)	+1 → D (2)
PC-int sum (Mod 12)		1	1	1	0	1	1	1	1	0	1	1	1	1	0	1
Sum Class	1	2	3	4	4	5	6	7	8	8	9	10	11	0	0	1

Figure 28: *An example of voice-leading for the path shown over the 0167/0147/0157 Power Towers in Figure 27.*

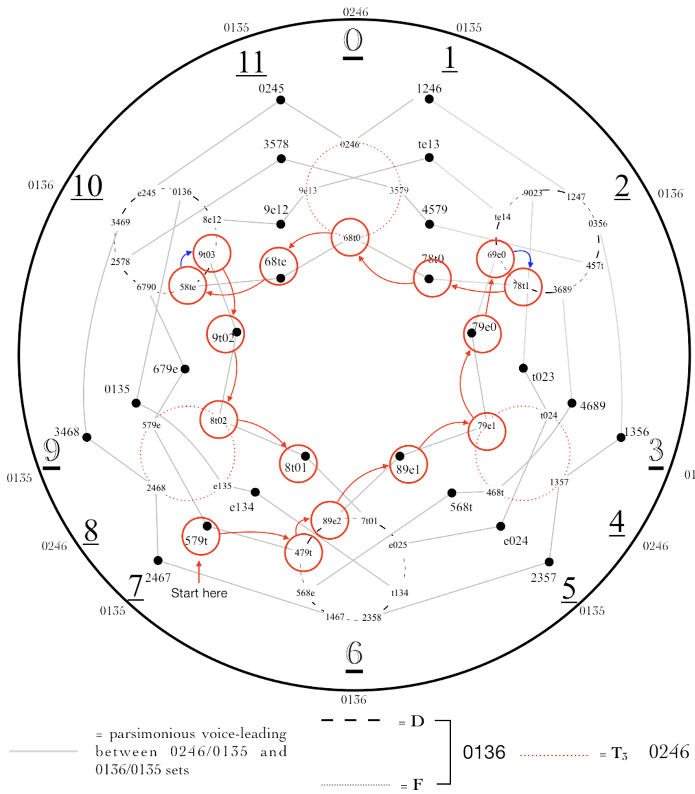


Figure 29: *The 0136/0246/0135 Power Towers.*

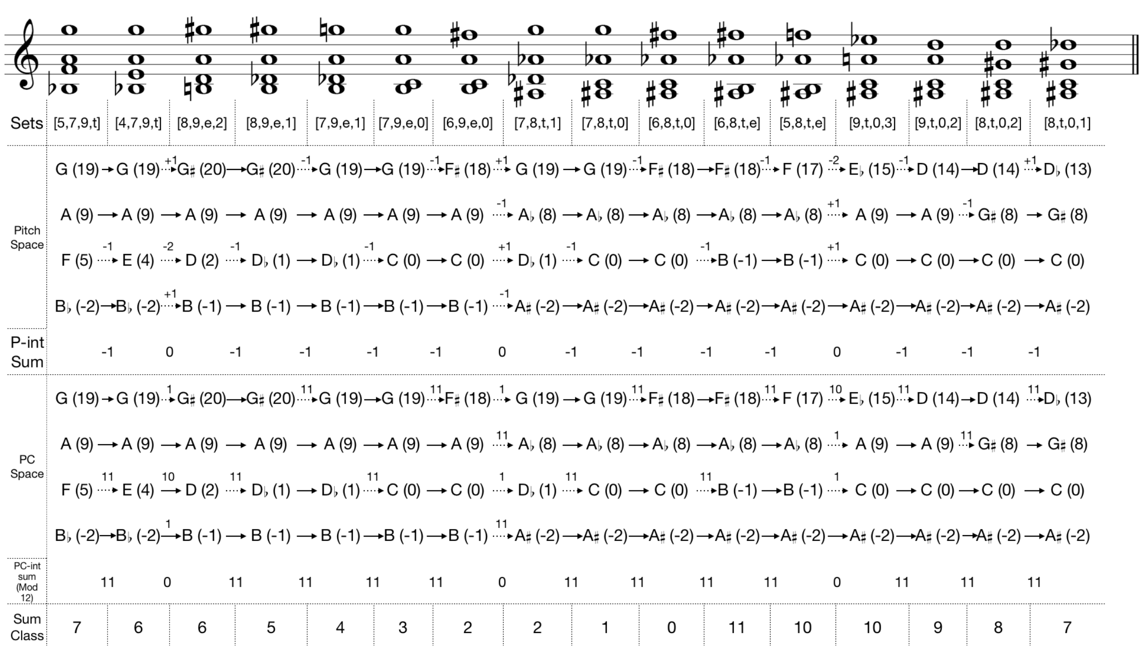


Figure 30: *An example of voice-leading for the path shown over the 0136/0246/0135 Power Towers in Figure 29.*

hand in the new graphs the sets are connected in adjacent voice-leading zones in the same way as in traditional graphs, and this maintains the leading voice consistency in the passages of the musical examples created with them. It is hoped that these new graphs can contribute both to analyses of music composed after the 19th century and to pre-compositional work of new music.

This approach can be easily expanded in many different ways, such as for graphs that would include sets of other cardinalities (dyads, pentachords, hexachords, etc.), graphs in which the pivot/bridge and target sets do not connect parsimoniously, graphs that include more than three set classes in their vertices, graphs in mod. 7, mod. 8, etc., among others.

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A. Appendix

Table 3: *List of all graphs that connect two sets of tetrachords.*

Type of Components	Bridges/Targets	Sum Classes	Superset
TYPE 1	0167/0157	1,2,3-5,6,7-9,10,11	AGG
	0268/0157	11,0,1-3,4,5-7,8,9	AGG
	0268/0258	11,0,1-3,4,5-7,8,9	MM2
TYPE 2	0123/0124	1,2,3-5,6,7-9,10,11	AGG
	0127/0126	11,0,1-3,4,5-7,8,9	AGG
	0127/0137	1,2,3-5,6,7-9,10,11	AGG
	0134/0124	11,0,1-3,4,5-7,8,9	AGG
	0134/0135	11,0,1-3,4,5-7,8,9	AGG
	0145/0135	1,2,3-5,6,7-9,10,11	AGG
	0145/0146	1,2,3-5,6,7-9,10,11	AGG
	0156/0146	11,0,1-3,4,5-7,8,9	AGG
	0156/0157	11,0,1-3,4,5-7,8,9	AGG
	0158/0148	1,2,3-5,6,7-9,10,11	AGG
	0158/0157	1,2,3-5,6,7-9,10,11	AGG
	0158/0258	1,2,3-5,6,7-9,10,11	AGG
	0235/0124	1,2,3-5,6,7-9,10,11	AGG
	0235/0135	1,2,3-5,6,7-9,10,11	AGG
	0235/0236	1,2,3-5,6,7-9,10,11	MM2
	0246/0135	11,0,1-3,4,5-7,8,9	AGG
	0246/0146	11,0,1-3,4,5-7,8,9	AGG
	0246/0236	11,0,1-3,4,5-7,8,9	AGG
	0246/0247	11,0,1-3,4,5-7,8,9	AGG
	0248/0137	1,2,3-5,6,7-9,10,11	AGG
	0248/0148	1,2,3-5,6,7-9,10,11	AGG
	0248/0247	1,2,3-5,6,7-9,10,11	AGG
	0248/0258	1,2,3-5,6,7-9,10,11	AGG
	0257/0146	1,2,3-5,6,7-9,10,11	AGG
	0257/0157	1,2,3-5,6,7-9,10,11	AGG
	0257/0247	1,2,3-5,6,7-9,10,11	AGG
	0257/0258	1,2,3-5,6,7-9,10,11	AGG
	0347/0148	1,2,3-5,6,7-9,10,11	AGG
	0347/0236	1,2,3-5,6,7-9,10,11	MM2
	0347/0247	1,2,3-5,6,7-9,10,11	AGG
	0358/0247	11,0,1-3,4,5-7,8,9	AGG
	0358/0148	11,0,1-3,4,5-7,8,9	AGG

Table 3: Continuation			
Type of Components	Bridges/Targets	Sum Classes	Superset
TYPE 3a	0136/0126	1,2,3-5,6,7-9,10,11	AGG
	0136/0135	1,2,3-5,6,7-9,10,11	AGG
	0136/0247	1,2,3-5,6,7-9,10,11	AGG
TYPE 3b	0136/0137	1,2,3-5,6,7-9,10,11	AGG
	0136/0146	1,2,3-5,6,7-9,10,11	AGG
	0136/0236	1,2,3-5,6,7-9,10,11	AGG
TYPE 4a	0125/0124	11,0,1-3,4,5-7,8,9	AGG
	0125/0236	11,0,1-3,4,5-7,8,9	AGG
	0147/0148	11,0,1-3,4,5-7,8,9	MM7
	0147/0157	11,0,1-3,4,5-7,8,9	MM7
	0147/0247	11,0,1-3,4,5-7,8,9	MM7
	0237/0236	11,0,1-3,4,5-7,8,9	MM4
	0237/0247	11,0,1-3,4,5-7,8,9	AGG
TYPE 4b	0125/0126	11,0,1-3,4,5-7,8,9	AGG
	0125/0135	11,0,1-3,4,5-7,8,9	AGG
	0147/0137	11,0,1-3,4,5-7,8,9	MM4
	0147/0146	11,0,1-3,4,5-7,8,9	MM7
	0147/0258	11,0,1-3,4,5-7,8,9	MM7
	0237/0126	11,0,1-3,4,5-7,8,9	AGG
	0237/0137	11,0,1-3,4,5-7,8,9	MM4
	0237/0148	11,0,1-3,4,5-7,8,9	AGG
Type 5	0358/0258	11,0,1-3,4,5-7,8,9	mm2

Table 4: *List of fourteen Cube Dances.*

Cube Dance (Intersection/target)	Subgraph 1 (Sum 1/2, 4/5, 7/8 and 10/11)	Superset	Subgraph 2 (Sum 11/0/1, 2/3/4, 5/6/7 and 8/9/10)	Superset
012/013	four **<DL>** chains of 013 sets	AGG	four graphs combining sets 012 and 013 (type 1)	AGG
015/014	four **<DH>** chains of 014 sets	AGG	four graphs combining sets 015 and 014 (type 1)	AGG
024/014	four **<DH>** chains of 024 sets	AGG	four graphs combining sets 014 and 014 (type 1)	AGG
015/016	four **<BJ>** chains of 016 sets	AGG	four graphs combining sets 015 and 016 (type 2)	AGG
027/016	four **<BJ>** chains of 016 sets	AGG	four graphs combining sets 027 and 016 (type 1)	AGG
015/025	four **<FB>** chains of 025 sets	AGG	four graphs combining sets 015 and 025 (type 2)	AGG
024/025	four **<FB>** chains of 025 sets	AGG	four graphs combining sets 024 and 025 (type 1)	AGG
036/026	four **<FJ>** chains of 026 sets	AGG	four graphs combining sets 036 and 026 (type 1)	AGG
015/026	four **<FJ>** chains of 026 sets	AGG	four graphs combining sets 015 and 026 (type 2)	AGG
027/026	four **<FJ>** chains of 026 sets	AGG	four graphs combining sets 027 and 026 (type 1)	AGG
036/026	four **<FJ>** chains of 026 sets	AGG	four graphs combining sets 036 and 026 (type 1)	AGG
027/037	four **<HD>** chains of 037 sets	HEX	four graphs combining sets 027 and 037 (type 1)	MM3
036/037	four **<HD>** chains of 037 sets	HEX	four graphs combining sets 036 and 037 (type 1)	MM3
048/037	four **<HD>** chains of 037	HEX	four Weitzmann Regions	MM3

Table 5: *List of 87 Power Towers.*

Power Towers	Subgraph 1 (Sum 1/2/3, 5/6/7, and 9/10/11)	Superset	Subgraph 2 (Sum 11/0/1, 3/4/5, and 7/8/9/)	Superset
0123-0125-0124	graph 0123/0124	AGG	graph 0125/0124	AGG
0235-0125-0124	graph 0235/0124	AGG	graph 0125/0124	AGG
0123-0134-0124	graph 0123/0124	AGG	graph 0134/0124	AGG
0235-0134-0124	graph 0235/0124	AGG	graph 0134/0124	AGG
0127-0125-0126	graph 0127/0126	AGG	graph 0125/0126	AGG
0136-0125-0126	graph 0136/0126	AGG	graph 0125/0126	AGG
0237-0127-0126	graph 0127/0126	AGG	graph 0237/0126	AGG
0136-0237-0126	graph 0136/0126	AGG	graph 0237/0126	AGG
0136-0125-0135	graph 0136/0135	AGG	graph 0125/0135	AGG
0145-0125-0135	graph 0145/0135	AGG	graph 0125/0135	AGG
0235-0125-0135	graph 0235/0135	AGG	graph 0125/0135	AGG
0136-0134-0135	graph 0136/0135	AGG	graph 0134/0135	AGG
0145-0134-0135	graph 0145/0135	AGG	graph 0134/0135	AGG
0235-0134-0135	graph 0235/0135	AGG	graph 0134/0135	AGG
0136-0246-0135	graph 0136/0135	AGG	graph 0246/0135	AGG
0145-0246-0135	graph 0145/0135	AGG	graph 0246/0135	AGG
0235-0246-0135	graph 0235/0135	AGG	graph 0246/0135	AGG
0127-0147-0137	graph 0127/0137	AGG	graph 0147/0137	MM4
0136-0147-0137	graph 0136/0137	AGG	graph 0147/0137	MM4
0248-0147-0137	graph 0248/0137	AGG	graph 0147/0137	MM4
0127-0237-0137	graph 0127/0137	AGG	graph 0237/0137	MM4
0136-0237-0137	graph 0136/0137	AGG	graph 0237/0137	MM4
0248-0237-0137	graph 0248/0137	AGG	graph 0237/0137	MM4
0136-0147-0146	graph 0136/0146	AGG	graph 0147/0146	MM7
0145-0147-0146	graph 0145/0146	AGG	graph 0147/0146	MM7
0257-0147-0146	graph 0257/0146	AGG	graph 0147/0146	MM7
0136-0156-0146	graph 0136/0146	AGG	graph 0156/0146	AGG
0145-0156-0146	graph 0145/0146	AGG	graph 0156/0146	AGG
0257-0156-0146	graph 0257/0146	AGG	graph 0156/0146	AGG
0136-0246-0146	graph 0136/0146	AGG	graph 0246/0146	AGG
0145-0246-0146	graph 0145/0146	AGG	graph 0246/0146	AGG
0257-0246-0146	graph 0257/0146	AGG	graph 0246/0146	AGG
0158-0147-0148	graph 0158/0148	AGG	graph 0147/0148	MM7
0248-0147-0148	graph 0248/0148	AGG	graph 0147/0148	MM7
0158-0147-0157	graph 0158/0157	AGG	graph 0147/0157	MM7
0167-0147-0157	graph 0167/0157	AGG	graph 0147/0157	MM7
0257-0147-0157	graph 0257/0157	AGG	graph 0147/0157	AGG
0158-0156-0157	graph 0158/0157	AGG	graph 0156/0157	AGG
0167-0156-0157	graph 0167/0157	AGG	graph 0156/0157	AGG
0257-0156-0157	graph 0257/0157	AGG	graph 0156/0157	AGG
0158-0268-0157	graph 0158/0157	AGG	graph 0268/0157	AGG
0167-0268-0157	graph 0167/0157	AGG	graph 0268/0157	AGG
0257-0268-0157	graph 0257/0157	AGG	graph 0268/0157	AGG
0136-0125-0236	graph 0136/0236	AGG	graph 0125/0236	AGG

Table 5: Continuation

Power Towers	Subgraph 1 (Sum 1/2/3, 5/6/7, and 9/10/11)	Superset	Subgraph 2 (Sum 11/0/1, 3/4/5, and 7/8/9/)	Superset
0235-0125-0236	graph 0235/0236	MM2	graph 0125/0236	AGG
0347-0125-0236	graph 0347/0236	AGG	graph 0125/0236	AGG
0136-0237-0236	graph 0136/0236	AGG	graph 0237/0236	MM4
0235-0237-0236	graph 0235/0236	MM2	graph 0237/0236	MM4
0347-0237-0236	graph 0347/0236	AGG	graph 0237/0236	MM4
0136-0246-0236	graph 0136/0236	AGG	graph 0246/0236	AGG
0235-0246-0236	graph 0235/0236	MM2	graph 0246/0236	AGG
0347-0246-0236	graph 0347/0236	AGG	graph 0246/0236	AGG
0235-0237-0236	graph 0235/0236	MM2	graph 0237/0236	MM4
0347-0237-0236	graph 0347/0236	AGG	graph 0237/0236	MM4
0136-0246-0236	graph 0136/0236	AGG	graph 0246/0236	AGG
0235-0246-0236	graph 0235/0236	MM2	graph 0246/0236	AGG
0347-0246-0236	graph 0347/0236	AGG	graph 0246/0236	AGG
0136-0147-0247	graph 0136/0247	AGG	graph 0147/0247	MM7
0248-0147-0247	graph 0248/0247	AGG	graph 0147/0247	MM7
0347-0147-0247	graph 0347/0247	AGG	graph 0147/0247	MM7
0136-0237-0247	graph 0136/0247	AGG	graph 0237/0247	AGG
0248-0237-0247	graph 0248/0247	AGG	graph 0237/0247	AGG
0347-0237-0247	graph 0347/0247	AGG	graph 0237/0247	AGG
0136-0246-0247	graph 0136/0247	AGG	graph 0246/0247	AGG
0248-0246-0247	graph 0248/0247	AGG	graph 0246/0247	AGG
0347-0246-0247	graph 0347/0247	AGG	graph 0246/0247	AGG
0158-0147-0258	graph 0158/0258	AGG	graph 0147/0258	AGG
0248-0147-0258	graph 0248/0258	AGG	graph 0147/0258	MM7
0257-0147-0258	graph 0257/0258	AGG	graph 0147/0258	MM7
0369-0147-0258	graph 0369/0258	AGG	graph 0147/0258	MM7
0158-0268-0258	graph 0158/0258	AGG	graph 0268/0258	MM2
0248-0268-0258	graph 0248/0258	AGG	graph 0268/0258	MM2
0257-0268-0258	graph 0257/0258	AGG	graph 0268/0258	MM2
0369-0268-0258	graph 0369/0258	AGG	graph 0268/0258	MM2
0158-0358-0258	graph 0158/0258	AGG	graph 0358/0258	MM2
0248-0358-0258	graph 0248/0258	AGG	graph 0358/0258	MM2