# Measuring the Amount of Freedom for Compositional Choices in a Textural Perspective

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DOI: 10.46926/musmat.2021v5n1.126-156

***Abstract**: In this paper I discuss the relation between the number of available compositional choices and the complexity in dealing with them in the scope of musical texture. First, I discuss the paradigm of compositional choice in light of the number of variables for a given situation. Then, I introduce the concept of compositional entropy–-a proposal for measuring the amount of freedom that is implied in each compositional choice when selecting a given musical object. This computation depends on the number of available variables provided by the chosen musical object so that the higher the compositional entropy, the more complex is the choosing process as it provides a high number of possibilities to be chosen. This formulation enables the discussion of compositional choices in a view of probability and combinatorial permutations. In the second part of the article, I apply this concept in the textural domain. To do so, I introduce a series of concepts and formulations regarding musical texture to enable such a discussion. Finally, I demonstrate how to measure the compositional entropy of textures, considering both the number of possible textural configurations a composer may manage for a given number of sounding components (exhaustive taxonomy of textures) and how many different ways a given configuration can be realized as music in the score, considering only textural terms (exhaustive taxonomy of realizations).*

***Keywords**: Compositional entropy. Musical texture. Textural layout. Exhaustive taxonomy. Probability and combinatorial permutations.*

## I. Introduction

As Marisa Rezende ([27, p. 77]) states: "composing means, among other things, to make choices." [1] This means that *choosing* is an intrinsic aspect of the compositional process as the endemic characteristic of a piece is determined by the composer's idiosyncratic choices of materials (pitches, pitch classes, rhythmic structures, dynamics, timbre, textures, etc.),

*I would like to thank Prof. Dr. Hugo Carvalho and Prof. Dr. Carlos Almada for their contribution concerning the mathematical issues presented in this article.

**Received**: May 31st, 2021
**Approved**: June 23th, 2021

[1] *Compor significa, entre outras coisas, fazer escolhas.*

aesthetic orientation, compositional techniques, sonic means, and the like. Each choice within the compositional process may contribute to unfolding the musical form so that being aware of the creative implications of it is a crucial expertise for any composer. In this paper I am concerned with mapping the number of compositional choices that are available for composers during the compositional process in a textural perspective. To do so, I will first discuss the relation between the number of choices and the complexity in dealing with them, which can be measured by what I call *compositional entropy*. Then, I will present the theoretical framework of musical texture that underlies this paper so that the set of available choices a given texture may hold, that is, their exhaustive taxonomy to be implemented as music, can be introduced.

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## II. Compositional Choices

In general, I call *compositional choice* a decision made by the composer that contributes to advance the construction of a piece of music. It may involve from the simple selection of materials to the very definition of the compositional strategy to be used. Composers deal with compositional choices all over the creative process in a more or less controlled (or conscious) way. From a temporal perspective, compositional choices can occur in either in *real-time* or *out of time*. The difference between both applications is in the order of bottom-up/top-down compositional approach. Real-time compositional choices involve definitions in a linear perspective, i.e., the composer defines over time how to advance a piece of music considering the possibilities that are suitable to his/her aesthetic orientation. Therefore, each choice is conditioned in part by the context it is made considering its contribution to unfold the musical form. Out-of-time compositional choices, on the other hand, consists of defining a priori how a piece of music shall be constructed. In other words, it involves prior decisions to define materials, structures, processes, or any other instance that may lead the construction of a piece. In any case, compositional choices can be related to creativity given that the more possibilities of choosing composers are aware of, the more creative and imaginative solutions they may explore in their music. For each musical parameter, there is a possible compositional choice to be made so that a piece of music may be understood as the sum (or combination) of all compositional choices made during the creative process (Figure 1).[2]

In this paper, I will focus on out-of-time compositional choices, discussing their creative implications and potentialities within the compositional process. Thus, I shall henceforth refer to them as just compositional choices to avoid wordy expressions.

Most often, compositional choices involve the construction of a compositional design (or plan) to organize the various elements of the piece beforehand. This stage is prior to the very process of writing the chosen elements in the musical score. In this context, compositional choices consist of the process of defining an abstract musical object in the scope of specific musical parameters (or attribute[3]) from its set of possibilities.

Assume, for example, that $H$ is a set of given musical parameter that hold three discrete objects ($H = \{a, b, c\}$). If $H$ is a sequence of notes, then compositional choices concerns choosing any number of notes from the set $H$ from one (a unitary object) up to three (the whole set or the *universe*

[2]Note that the musical parameters in the figure are not ordered hierarchically, that is, the top-down disposition does not necessarily define order each choice was made. Moreover, in a wider compositional spectre, these compositional choices could be interpreted as the actual piece without the urge of being combined with the others.

[3]In the present paper, musical texture is not understood as a musical parameter, as it emerges from the combination of them. Rather, it is a musical attribute that concerns organization. See section IV

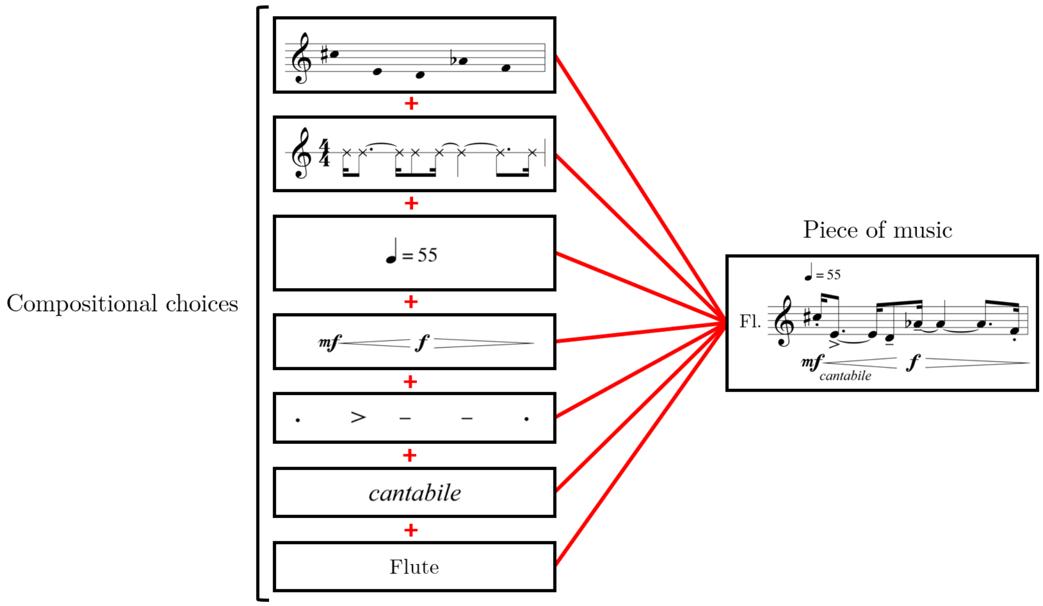


**Figure 1:** *The combination of various compositional choices applied to different musical parameters resulting in a piece of music.*

*set*[4]). If the compositional choice involves choosing a single object, then there are three possible choices available for the composer: $\{a\}$, $\{b\}$, or $\{c\}$. A compositional choice of two objects, in turn, may also provide three possibilities of choice, considering the pairwise combination: $\{a,b\}$, $\{a,c\}$, and $\{b,c\}$.[5] Finally, a compositional choice of three objects in $H$ comprises a unique possibility of choice that is the set $H$ itself.

The calculation of the number of possibilities available in a compositional choice is given by the combination formula $C(n,r)$ (read as "$n$ choose $r$"), where $n$ is the number of variables of the set and $r$ is the number of components to be chosen so that $r$ must be equal to or less than $n$ (Equation 1):

$$C(n,r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}, \text{ for } \ r \leq n. \tag{1}$$

Table 1 shows the application of this formula in the scope of pitch classes, with the number of compositional choices ($r$) ranging from one to twelve.[6] Besides the number of possibilities for $r = 6$, all possibilities have a pair in a different number of compositional choices. This happens because they are the complement to one another, that is, for each possibility $K$ of a given $r$

[4]In Set Theory, the universe set, noted as $\mathbb{U}$, stands for a collection that contains all possible variables of a given element to be considered in a specific situation or in a given purpose. Any set of elements can be understood as $\mathbb{U}$ so that it is taken as a reference to its possible subsets. Note that the definition of a $\mathbb{U}$ is not applicable in any contexts whatsoever. Within the scope of Zermelo-Fraenkel. It is possible to define the more general object so-called *category*, but it is outside of the scope of this paper to deal with such objects. Furthermore, when $\mathbb{U}$ contains all possible variables of a given element, then it is indeed the exhaustive taxonomy of that element.

[5]Note that each one of them is a subset of $H$.

[6]Note that this number of possibilities does not consider class principles defined in pitch-class set theory formulation as number of prime forms of pitch-class set is considerably lower. See [5].

**Table 1:** *The relation between the number of compositional choices (r) and the number of possibilities for the set of twelve pitch classes (n) given by $C(n,r)$.*

| Number of choices ($r$) | Number of possibilities |
|---|---|
| 1 | 12 |
| 2 | 66 |
| 3 | 220 |
| 4 | 495 |
| 5 | 792 |
| 6 | 924 |
| 7 | 792 |
| 8 | 495 |
| 9 | 220 |
| 10 | 66 |
| 11 | 12 |
| 12 | 1 |

there is a possibility $J$ so that their union contains all twelve pith classes ($\mathbb{U}$).[7] As could not be otherwise, the compositional choice with the lower number of variables to choose corresponds to the whole set of pitch classes. One may note therefore that exhaustive taxonomies are of greatest interest for composers within compositional choices since the wider is the $\mathbb{U}$ of variables, the more compositional choices can be made from that.

Compositional choices may be cumulative within the same musical parameter. After choosing a pitch class, for example, the next compositional choice may involve defining its position in the register (pitch), its duration, its timbre, the way it shall be articulated and, so on. Each stage of compositional choice leads toward the realization of the piece as music.[8] A distinction shall be made among the various stages of compositional choices. In the present work, the process of defining a musical object is referred to as a compositional choice of the first instance so that any choice made afterward is of the second instance. Thus, the second instance concerns decisions to be made considering the available options provided by objects defined in the first instance. This implies that a composer may choose a musical object instead of other in the first instance based on the number of possible realizations it provides for the second instance. In other words, the number of possible ways of realizing a given musical object as music may be the most decisive criteria for choosing it beforehand.

Consider, for example, the information provided by Table 1. One may say that the compositional choice involving the selection of twelve pitch classes leaves no room for compositional choices since the unique option to be chosen is the whole itself. Nevertheless, in the second instance several other compositional choices may coordinate it in a myriad of ways. For example, by simply including an ordering factor to organize it sequentially in time, a composer can create up to 479,001,600 of different twelve-tone rows (12!). Furthermore, its realization also involve cumulative compositional choices concerning various musical aspects, such as pitches, their absolute duration, their dynamic, articulation, timbre, and so on. As a consequence, the number of possibilities within

[7]Note that the complement for $r = 12$ is $r = 0$.

[8]In this paper, the idea of *realizing as music* expresses process thereby a musical object (or abstraction) is written in the score, considering their temporal organization, as well as the inclusion of a series of other variables that are not implied by them. Therefore, music that is not supported by musical notation is out of the scope of the present paper. Additionally, I shall consider the realization only in the terms of traditional notation, which excludes graphical indications, open musical forms, interpretative indeterminacy, and the like.

each compositional choice to achieve its realization is astronomic, given the multiple variables and their possible combinations. In this paper, for the sake of simplicity, the discussion of the possible realizations of musical objects will consist of only an intermediary stage between the compositional design, where the objects are to be chosen, and their actual realization in the music score. Thus, when I refer to a possible realization of a given musical object, I am, in fact, speaking about its available possibilities in the second instance in terms of this intermediary stage. A intermediary stage may be as simple as the example above regarding the inclusion of order for the set of pitch classes.

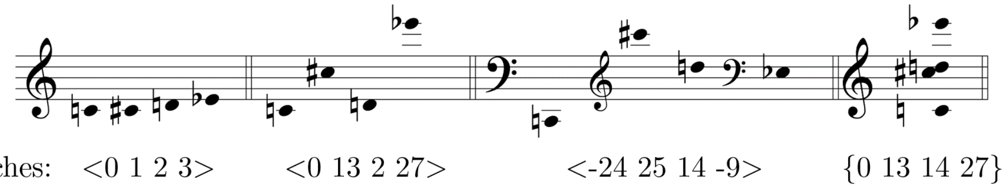


**Figure 2:** *Four realizations of the ordered set of pitch classes <0123> considering pitches.*

Based on the information provided by Table 1, a composer may choose four pitch classes among 495 possible compositional alternatives. A compositional choice of the second instance may involve the simple inclusion of order. The number of compositional choices in this second instance is defined by the possible permutation of notes, which is equal to 24 (4!). Another compositional choice would involve the articulation of this sequence as pitches—a note in a specific register—so that the number of available choices is defined by the number of possible realizations of these pitch classes as pitches. This computation is not as simple as in compositional choices of the first instance or as that one regarding order as several factors shall be considered. For example, it is necessary to consider the range of the instrumental mean to calculate how many different pitches are available for each pitch class of the sequence. Within an octave, the number of available compositional choices for their realization as pitches in a sequential order is equal to the number of choices involved in the order (24). In these terms, the sequence of pitch classes may have different possibilities depending on the chosen instrument to be realized. Therefore, the very choice of the instrumental mean is a crucial factor for computing the number of possibilities in the instance of realization. If the composer defines the sequence will be written in a flute, for example, the number of possibilities would be considerably smaller than if he/she chooses a piano solo given their difference in the range of available pitches. This number may also increase by including the possibility of pitches coinciding in time.

Figure 2 illustrates some of the possible realizations of the set of pitch classes [0123], considering both order and pitch.[9] In all realizations, the order of pitch classes is preserved. In the first measure, the notation of pitch classes and pitches is equivalent; pitch classes are arranged from low to high in both time and register from the middle-C.[10] In measures two and three, pitch classes are ordered only in time as pitches are placed in several registral spans. Finally, in the last measure, all pitches are articulated at once in a single temporal span, which means the order is defined only in the register from low to high.

[9] The notation of pitch classes consists of ascribing an integer for each pitch class so that 0 refers to C-natural, 1 stands for C-sharp or D-flat, 2 corresponds to D-natural, and so on up to 11 that refers to B-natural.

[10] In the notation of pitches, each note is written according to its intervallic relation with middle-C, that is labeled as 0. Then, the D-flat (or C-sharp) one semitone above is 1, the adjacent D is 2, and so on. The B directly below middle-C is defined as –1, the B-flat (or A-sharp) a semitone below is –2, and so on. See [22].

In the context of other musical parameters, computing the number of possible realizations in the second instance would involve different factors suitable to their specific natures. If the musical object is a duration, for example, its variables could imply, for example, the definition of its metric position in the temporal span. Some compositional aspects are naturally non-practical in terms of realizations of the second instance; the very definition in the first instance provides its realization. Consider, for example, the definition in the first instance of a given tempo marking (e.g. quarter note equals to 100bpm). This choice already defines its unique realization that is the tempo marking itself. So there are no variables to be chosen in the second instance.

It should be clear now, from the discussion provided so far, how compositional choices operate in the creative process to define the elements and their possible realizations. On the face of it, a composer may argue that the wider the set of musical objects to be chosen and/or their possible realizations in the second instance, the better for compositional choices as it provides more options to be chosen. Although this might be true, an extensive number of possibilities may also intricate the compositional process.

Concerning the way people deal with the choosing process, Sheena S. Iyengar and Mark R. Lepper [9, pp. 999-1000], based on various researchers, say that in front of an extension of choices some people:

> may actually feel more committed to the choice-making process; that is, that they may feel more responsible for the choices they make because of the multitude of options available. These enhanced feelings of responsibility, in turn, may inhibit choosers from exercising their choices, out of fear of later regret. In other words, choice-makers in extensive-choice contexts might feel more responsible for their choices given the potential opportunity of finding the very best option, but their inability to invest the requisite time and effort in seeking the so-called best option may heighten their experience of regret with the options they have chosen. If so, choosers in extensive-choice contexts should perceive the choice-making process to be more enjoyable given all the possibilities available. They should at the same time, however, find it more difficult and frustrating given the potentially overwhelming and confusing amount of information to be considered.[9, p. 1000]

This situation is perfectly suitable to the compositional context, in which the choice overload may lead to a creative block—a state in which the composer is unable to decide a path to follow in the face of the universe of possibilities. The so-called "dilemma of the blank page", where the composer has no idea on how to start his piece or what kind of music he/she wants to compose, is a trivial example of this creative block. This situation can be explained by the complexity of dealing with a wide set of variables at once. This can be a critical issue for composers to manage both objects and their realizations.

Liduino Pitombeira proposes the idea of an *organized complexity*, "a category in which is located the vast majority of human problems, including music analysis and composition." ([25, pp. 39-40]). Based on this idea, Pitombeira defines a continuum of complexity in such a way that organized complexity is located between *organized simplicity*, which involves simple deterministic problems holding up to four variables, that "can be handled, for example, by calculus and differential equations", and *disorganized complexity*, "which involves the use of probability and statistics to deal with an astronomical number of variables." (Figure 3).[11]

In the compositional process, organized complexity may be understood as the realm in which composers create their music in a more or less restricted way. Disorganized complexity is therefore

[11] This proposal is based on the work "Science and Complexity" by Warren Weaver.

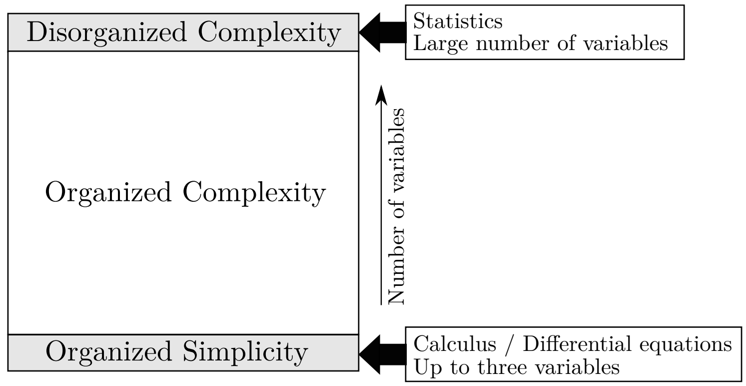


**Figure 3:** *Continuum of complexity from organized simplicity to disorganized complexity ([25, pp. 40]).*

the most chaotic compositional situation, where the number of available choices tends to infinite—the blank page. Organized simplicity, in turn, may be understood as the most strict situation, in which there is only one possible compositional path to be taken. In terms of compositional choices, the continuum of complexity impacts the number of available possibilities in both instances. This means the higher the number of variables, the more possibilities the composer may choose, and, consequently, more complex will be the very choosing process. This complexity, of course, impacts both instances.[12] Let me illustrate the creative implications of the number of variables in this continuum of complexity by discussing the simple task of creating a list of things.

If a task requires to randomly list ten different things whatsoever, without specifying a category, a topic, or rules, one could be taken too long to even start. Moreover, the choice overload could even inhibit the starting point as a creative block. After all, choosing anything from an infinitude of possibility is not an elementary task. Some people would start listing in a pragmatic way, choosing at once something that is in their sight or the very first haphazard idea that pops up in their mind. Sheena S. Iyengar and Mark R. Lepper relate that this kind of response would be a way to "simply strive to end the choice-making ordeal by finding a choice that is merely satisfactory, rather than optimal". ([9, p. 999]) In any case, the decision for the first item in the list would probably mitigate the choosing process. Presumably, the other elements of the list would be related somehow to the first chosen item, because it provides a guiding principle to be followed. This means the infinitude of possibilities would become now finite and manageable, being confined to guidelines established from the first choice. This explains why a task requiring to list ten different fruits or animals would be probably easier to most anyone as it involves the definition of a specific category, reducing the number of variables in the choosing process.

The idea of reducing the number of variables to be chosen may imply a sense of control. On this, Igor Stravinsky advocates that "the more art is controlled, limited, worked over, the more it is free" ([31, p. 63]). Thus, for him, the limitation is a crucial aspect of the compositional process:

> As for myself, I experience a sort of terror when, at the moment of setting to work and finding myself before the infinitude of possibilities that present themselves, I have the feeling that everything is permissible to me. If everything is permissible to me, the best and the worst; if nothing offers me any resistance, then any effort is inconceivable, and I cannot use anything as a basis, and consequently every undertaking becomes futile.
>
> [...]

[12] In section III, I introduce a way of measuring this complexity in terms of quantity of available choices.

> Let me have something finite, definite matter that can lend itself to my operation only insofar as it is commensurate with my possibilities. And such matter presents itself to me together with its limitations. I must in turn impose mine upon it. [31, pp. 63-64].

In fact, the constraint of creative variables is an effective way of stimulating the composer's imagination as it decreases the number of variables to be chosen. This constraining strategy has been a recurrent procedure throughout the history of Western Classical Music. In the study of common-practice species counterpoint, for example, the complexity increases along with the species. This means the various melodic and harmonic constraints are gradually introduced to students so their comprehension is parsimonious, that is, before learning a new set of rules, the student shall master the previous ones.[13] By doing so, the student shall gradually improve the ability to deal with different musical aspects, such as the independence of voices, the preparation and resolution of dissonances, the harmonic progression holding the tonal sense, the melodic directionality, the vertical (harmonic) resultant of the superposition of voices, and so forth. If all rules were presented at once the study of counterpoint would be probably daunting due to the number of its restrictions. It is interesting to consider that good contrapuntal writing urges from a set of limitations so the composer shall master those rules to accordingly create inventive pieces. Concerning the relation between freedom of choice and constraint of possibilities, Stravinsky states that "we find freedom in strict submission to the object" ([31, p. 76]).

> Let us take the best example: the fugue, a pure form in which the music means nothing outside itself. Doesn't the fugue imply the composer's submission to the rules? And is it not within those strictures that he finds the full flowering of his freedom as a creator? Strength, says Leonardo da Vinci, is born of constraint and dies in freedom. ([31, p. 76])

In tonal practices, composing a fugue in a given tonality compels the composer to attend to some invariable features for any fugue, as it involves a specific type of polyphonic writing based on imitative principles. Not following this principle means the final result is anything but a fugue. Yet, despite its limitations, the composition of a fugue is sufficiently flexible to provide rooms for exploring creativity in various ways. Thus, creativity is not necessarily a matter of the number of available compositional choices, but how he/she deals with any possible limitation by inventing creative solutions to them.

It is up to the composer to define the size of the set of variables that he/she is more comfortable in dealing with or that is more suitable to his/her compositional goals and skills. Such a decision may involve either a wide set of choices, leaving rooms for inventive solutions, or a more easily manageable set of possibilities with a limited number of variables. A skilled composer would probably see a wide set of choices in both instances as a wealthy territory to explore his/her imaginative mind. On the other hand, as discussed above, choice overload may baffle composers given the overwhelming amount of available variables to be chosen.

In order to choose a set of variables, the composer shall be aware of the universe of possibilities that are available to him/her. Therefore, mapping how many possibilities are available for the realization of a given compositional choice is a crucial database for musical composition as it gives the composer the freedom to choose in advance the degree of complexity in the continuum he/she would like to manage in the construction of his/her music. This degree of complexity can be measured by what I call *compositional entropy*.

[13] Note that in this strategy, the complexity of compositional choices is also progressive, gradually increasing the number of possibilities.

## III. Compositional Entropy

In the article "A Mathematical Theory of Communication", published in 1948, Claude Elwood Shannon ([30]) proposes the theoretical foundations for what would be known as the Theory of Information. According to Shannon ([30, p. 379]):

> The fundamental problem of communication is that of reproducing at one point either exactly or approximately a message selected at another point. Frequently the messages have meaning; that is they refer to or are correlated according to some system with certain physical or conceptual entities. These semantic aspects of communication are irrelevant to the engineering problem. The significant aspect is that the actual message is one selected from a set of possible messages. The system must be designed to operate for each possible selection, not just the one which will actually be chosen since this is unknown at the time of design.

Assume, for example, that $A$ and $B$ are two different messages one should choose to transmit to someone else. Both messages are equivalent in terms of choice, as any of them can be chosen. Nevertheless, to transmit messages $A$ and $B$, it is necessary to optimize the transmission process for the most economical as possible. For example, if message $A$ is a sequence of zeros repeated thirty thousand times, then it is not necessary to send the entire sequence of message $A$, but only two pieces of information: number zero and thirty thousand. With that, the receiver will be able to understand the message $A$. Now, if message $B$ is, for example, a random noise, this is information is essentially incompressible, which means the entire sequence contained in B must be sent. This means message $B$ is more complex than message $A$ given the amount of information it contains.[14] In a general sense, a message can be understood as the random realization of successive autonomous variables so that to measure the amount of information within a random message, Shannon proposes the idea of *entropy*. The term was borrowed from thermodynamics, where it refers to a molecular system, that is, it measures the degree of disorder of a given system. In terms of information, entropy is a metric that quantifies the degree of predictability of a given message happening in a certain context. This measurement depends on the amount of information so that the greater the amount of information, the greater the entropy, that is, the message is more chaotic or unpredictable (message $B$). Similarly, a low value of entropy indicates more predictable information with less variables to be considered, which evokes a sense of simplicity (message $B$). In his words:

> Suppose we have a set of possible events whose probabilities of occurrence are $p_1, p_2, ..., p_n$. These probabilities are known but that is all we know concerning which event will occur. Can we find a measure of how much "choice" is involved in the selection of the event or of how uncertain we are of the outcome?
>
> If there is such a measure, say $H(p_1, p_2, ..., p_n)$, it is reasonable to require of it the following properties:
>
> 1. $H$ should be continuous in the $p_i$.
>
> 2. If all the $p_i$ are equal, $p_i = \frac{1}{n}$, then $H$ should be a monotonic increasing function of $n$. With equally likely events there is more choice, or uncertainty, when there are more possible events.
>
> 3. If a choice be broken down into two successive choices, the original $H$ should be the weighted sum of the individual values of $H$. ([30, p. 379])

[14] See ([28]).

Based on this idea, I shall define the compositional entropy as the measurement of the amount of freedom a composer has in creating his/her music considering how many compositional options are available to him/her in the scope of a given musical parameter. To put it differently, it expresses how complex a compositional choice can be by considering the number of variables involves in it. This means the higher the compositional entropy, the more complex is the compositional choice as there will be a high number of possibilities to be chosen. On the other hand, a low degree of compositional entropy implies a limited set of compositional possibilities.[15]

The compositional entropy may be used to measure the complexity for compositional choices in both instances so that it expresses the complexity of choosing within the set of possibilities. Therefore, exhaustive taxonomies are crucial for computing compositional entropy as it defines the number of all variables to be considered the compositional choices. For example, let $K$ be a set of musical objects of a given nature. The compositional entropy of $K$, denoted by $H(K)$, is given by the weighted average of the logarithm of the probability of each realization of the objects of $K$, where the weights are the respective probabilities themselves. This is in accordance with Shannon's definition of entropy in Information Theory [30], and this quantity is computed as (Equation 2)[16]:

$$H(K) = -\sum_{i=1}^{n} p_i \log p_i, \quad \text{where } n \text{ is the number of possibilities within } K. \tag{2}$$

Now, consider $G$ is a set of possible realizations for a given musical object defined in $K$. Then, $H(G)$ measures the compositional entropy of available choices in a compositional choice of the second instance. In order to differ $H(K)$ from $H(G)$, henceforth, I shall refer to them as, respectively, *compositional entropy of choices*, that concerns the process of choosing a musical object, and a *compositional entropy of realization*, in which is taken into consideration all available ways to realize the chosen object as music. Therefore, compositional entropy of realization is conditionally related to the nature of the musical object previously defined. Such a formulation is analogous to the idea of conditional entropy, in which the computation of the entropy of a random variable is conditioned to the entropy of another random variable (see [28]).

During the compositional process, all variables have the same probability to be chosen by a composer within both instances of compositional choices. Hence, there are no preferred variables a priori from the set of possibilities.[17] Thus, the formula may be simplified as the following by setting each $p_i$ as equal to $1/n$ (Equation 3)[18]:

$$H(K) = \log(n) \tag{3}$$

[15]This article does not intend to discuss in depth the concept of entropy developed by Shannon [30], but only to use it as a quantifier of uncertainty in compositional choices based on the number of available options. Moreover, the proposal presented here differs from those concerning the application of entropy in musical contexts. Leonard Meyer ([17][16]) was the first to introduce the concept of entropy in music. Other musical applications can be seen, for example, in [13] [10] [11][32][33][12]

[16]It is noteworthy that his is the unique function that satisfies the involved axioms.

[17]Note that a non-uniform probability to choose a variable would involve either a total awareness of the composer's idiosyncratic preferences, which would probably require a compositional maturity, or a statistical observation to catalog the most recurrent variables in the compositional practices.

[18]In the present work, all logarithm base are equal to 2, but for concision, it is omitted. Entropy is usually measured in a logarithm base of 2 in information theory due to the output is in bits—the unit. Nevertheless, any base greater than 1 may be used.

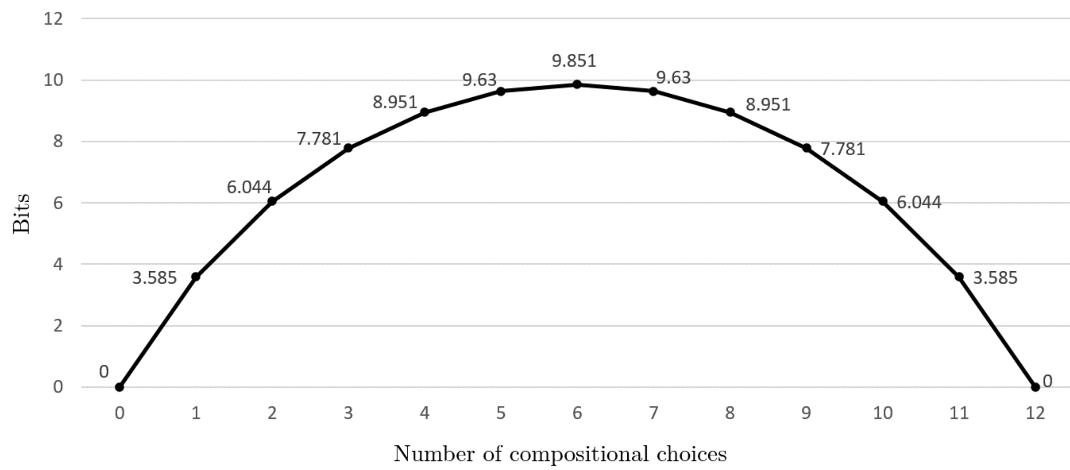


**Figure 4:** *Curve of compositional entropy defined by the number of possibilities demonstrated in Table 1 defined by the formula* $\log(n)$, *where n is the number of possibilities to be chosen from the set of twelve pitch classes.*

If $K$ holds five musical objects, for example, and a composer wants to choose one element of $K$. Then, the compositional entropy of this compositional choice of $K$ is equal to 2.322 (i.e., $H(K) = \log(\binom{5}{1} = 5) = 2.322$). This means the complexity of the compositional choice in dealing with $K$ is relatively low for a situation of choosing a single component. Now, if a composer is compelled to choose two elements of $K$, then the compositional entropy is equal to 3.322 ($H(K) = \log(\binom{5}{2} = 10)$ = 3.322). Of course, for any musical object chosen from K there is also a compositional entropy of realization (i.e., $H(G)$).

Figure 4 provides the curve of compositional entropy for the set of possibilities presented in Table 1. As one would imagine, choosing the $\mathbb{U}$ of available choices constitutes the minimal degree of compositional entropy since it does not involve a choice properly speaking. Also, the highest degree of entropy is related to the set with more available compositional choices to be made (six out of twelve). Even so, in terms of compositional entropy, the difference of its value compared to the others is not as much greater than its difference in terms of possibilities provided in Table combinationspc. Based on the degree of the compositional entropy, the composer may decide the amount of complexity he/she would prefer to manage in their music.

In the following sections I will present the exhaustive taxonomy of musical textures by considering their relations to integer partition, a proposal first introduced by Pauxy Gentil-Nunes[6]. From this definition, I will discuss a way of mapping their possible realization as music, enabling, thereby, the calculus of the degree of compositional entropy of realization in a textural perspective. For this computation, I will assume the premise that the probability of compositional choices is uniform, so the process will be as simple as counting the possibilities by using the formula log $(n)$, where $n$ is either the number of musical objects to be chosen (first instance) or the number of available realizations of a given object (second instance).[19]

[19] As aforementioned, the consideration of non-uniform probabilities is out of the scope of the present work. Such an investigation would demand examining which are the musical textures or their realizations that are more likely to be chosen by a composer, considering his/her compositional expertise or awareness of most traditional compositional practices.

## IV. Musical Texture

Musical texture has been of great interest for composers since at least the late eighteenth century as an increasingly important musical component to articulate musical syntax and its hierarchical relations. Such poietic development has influenced not only manuals for composition and orchestration, but also the way texture (and music) is perceived. During the twentieth century, texture became even more significant to compositional practices as the urge to overcome the traditional syntax of the tonal system led various composers to address texture and its potential to unfold the musical form. In fact, the way music evolves over time can be conceived as changes of texture and therefore functions to advance the shape or trajectory of a composition.

The definition of the term "texture" is not consensual among musicians and theorists since it has been associated with different musical aspects throughout history. Anne Trenkamp states that "despite problems of definition, the musician recognizes texture as a definable element." [34, p. 14]. Based on the vocabulary used to describe musical texture, the concept of texture seems to converge into two main approaches[3]: a) texture as a musical attribute that concerns the organization of musical materials, which is usually described by traditional labels, such as *monophony*, *homophony*, *polyphony*, and *heterophony*; and b) texture as sonority, whose vocabulary describes the aural perception of registral activities, combinations of timbre, variations on dynamics, and the like, using metaphorical descriptive words, such as thin, thick, dark, light, ethereal, sparse, dense, and so on.

In the present work, texture is understood as an organizational attribute defined by the number of simultaneous vocal or instrumental parts therein (quantitative aspect) and the way they interact to one another to assemble what most people would refer to as the “layers” of texture (qualitative aspect). Formally, a given texture (or textural configuration) may be defined as the organization of $n$ simultaneous musical threads[20] into $m$ textural layers.[21] The criteria for defining how many layers a textural configuration holds are contextual and argumentative according to its specific musical context. If two or more threads share one or more characteristics, such as pitch (or pitch-class), rhythm (exact duration), register span, timbre, dynamics, and the like, they assemble the same layer. Otherwise, each thread stands for a different layer with a thickness of 1.[22]

A piece of music can hold either a single or multiple textural configurations, and the way they are diachronically arranged within the piece is intrinsically related to the perception of musical flow. That is, depending on the contrast between two contiguous textural configurations, it may imply a rupture in the sense of continuity. Of course, this rupture may be associated with a structural segmentation of musical form. Parsimonious motion, on the other hand, may contribute to a smooth musical flow without undermining the sense of unity. In order to discuss these textural properties, each texture may be described by a different integer partition.[23] A partition is a way of representing a number by summing other numbers. Number four, for example, holds five different partitions namely: [4], [1+3], [2+2], [1+1+2], and [1+1+1+1].

Figure 5 demonstrates how each one of the partitions of four may address a specific textural

[20] A musical thread (or simply thread) is the minimal constituent of a texture, which may be a single note (or sound), a series of notes positioned in a register, a pitch within a chord, etc. See [21] for further information.

[21] This definition has a theoretical ground on the seminal work of Wallace Berry ([2]), Pauxy Gentil-Nunes’ proposal called *Partitional Analysis* that relates texture with the Theory of Partitions ([6]), and on some of my previous works ([21]).

[22] In a general sense, the thickness of a layer is determined by the number of threads therein. Based on its thickness, a layer may be classified as either a *line* (a single thread) or a *block* (a group of two or more threads).

[23] The use of numbers to depict textural configurations were first introduced by Berry [2], but its association with partitions was proposed by Gentil-Nunes, which enables a series of further developments, as the access of the exhaustive taxonomy of textural configurations for a given number of threads, as well as their topological relations defined by partitional operators. See [6].

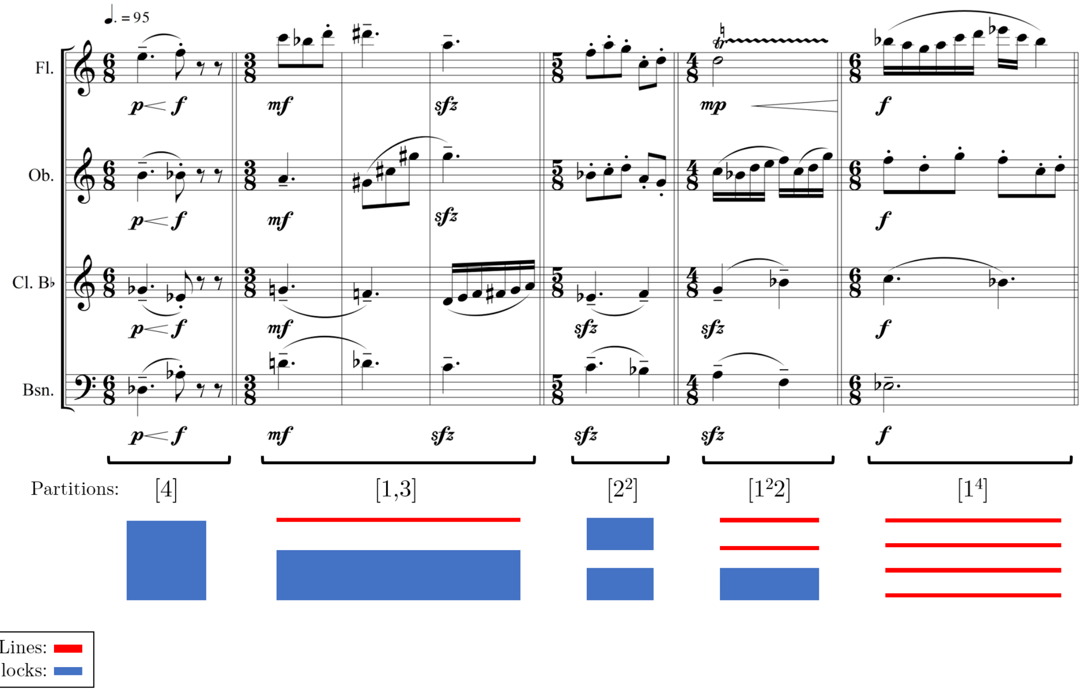


**Figure 5:** *Partitions of four addressing textural configurations. Parts are defined by rhythmic coincidences.*

configuration.[24] Each number stands for a layer and their respective absolute value expresses the number of musical threads therein (thickness). Textural parts can be classified as either a *line* (a unique thread indicated by number 1) or a block (the assemble of multiple threads expressed by any number equal to or greater than 2).[25] Parts are defined by rhythmic coincidences, that is, threads in "rhythmic unison" (where their duration are strictly aligned) assemble the same part. Note that the criterion for segregating textural parts is also related to the diversity of elements for the sake of the analysis. This means that the number of parts reveals the rhythmic diversity of the texture. Similarly, if the criterion were timbral differentiation, the number of parts would reveal the number of different timbres therein. Below each partition, there is an iconic model that portrays the number of parts according to their classification (either lines or blocks).

In the partition [1,3], the line (part 1) shifts its position over the register, moving from one instrument to another within the ensemble. First, the line is presented in the flute. Then, it moves to the oboe, concluding in the clarinet. This change creates internal variations of the textural configuration, without, however, disturbing its morphology—the organization still preserves the superposition of a line (part 1) and a block (part 3). Thus, the variance implies a change in the

[24]Gentil-Nunes presents a series of original concepts and tools concerning the relation between musical elements and integer partitions, which enables a refined analysis of texture, as well as its systematic manipulation for compositional purposes. [6]

[25]For the sake of clarity and conciseness, partitions are noted within brackets in their abbreviated notation, in which the repetition of a number is indicated by a superscript. Also, in order to avoid notational ambiguities, each part is separated by either a comma or by the superscript of the previous part.

spatial order of parts.[26] A partition is, in essence, an unordered set of numbers, which means the order of parts is irrelevant; partitions [1,3] and [3,1] are equivalent. Nevertheless, the inclusion of an ordering factor may portray the registral span of parts, which would perfectly suit to describe the variations of partition [1,3] in Figure 5. [27]

Each combinatorial permutation of the parts within a partition corresponds to an ordered partition. Obviously, the partition must have more than one part; otherwise, the ordered partition is redundant to the partition. The number of ordered partition for a positive integer $n$ is equal to $2^{n-1}$. The five partitions of number four, for example, may be combined into eight ordered partitions: <4>, <1,3>, <3,1>, <$2^2$>, <$1^2$2>, <1,2,1>, <2, $1^2$>, and <$1^4$>.[28] The order factor indicates the general registral placement of parts so that the left-to-right notation corresponds to the top-to-down position of them in the register. Consider $x$ and $y$ as two different textural parts of a texture. If part $x$ is higher than $x$ in the register, then the ordered partition is written as $< x, y >$; otherwise, the ordered partition is $< y, x >$. Of course, this relation is not absolute since it depends on a comparative observation among all parts. This means the highest part written in the leftmost position within an ordered partition does not necessarily is located in a high register, but it is the highest among all parts.

One may notice that ordered partitions are not suitable to describe *interpolated textures*, i.e., textures in which "threads of a part are interlaced (or interwoven) with threads of another." ([21, p. 81]). To put it differently, ordered partition can only describe textural realizations in which the non-overlapping parts are perfectly stacked to one another. In order to portray the spatial organization of those situations, I have introduced a specific for texture notation that I call *thread-word* (see [21]). A thread-word maps the way threads are posed in the register, conveying their organization into parts. It is based on a one-to-one correspondence between parts, their threads, and letters. For each thread within the texture, is ascribed a letter in such a way that all threads that assemble the same part receive the same letter. The number of different letters corresponds to the number of different parts of the texture, and the sum of all equivalent letters reveals its thickness. The notational principle of thread-words is identical to ordered partitions, so the left-to-right order of letters maps their top-down disposition in the register. For example, partition [$1^3$] can be represented as any of the following thread-words: <*abc*>, <*bcd*>, <*xyz*>, and so on. In the same way, a thread-word noted as <$aba^2$> indicates a texture holding a single line (written as "b") and a block of thickness of three (indicated by the sum of letters "a").[29] Note that this thread-word is equivalent to partition [1,3], expressing an organization an ordered partition is not able to.[30] Figure 6 shows the comparison between ordered partitions and thread-words describing all spatial organizations of threads of partition [1,3]. Textural parts differ from one another in their rhythmic articulations.

Each ordered partition or thread-word provides what I call *textural layout*—a possible compositional choice of the second instance available for the composer to realize a given textural configuration as music, considering the combinatorial permutation of its component parts in the register. Note that the textural realization discussed here deals only with textural factors, that is, it

[26] By spatial order, I refer to the general vertical distribution of threads and/or parts without dealing with their actual registral span—a conception from the same realm of Theory of Musical Contour (see [23][14]). Therefore, this proposal differs from Berry's *texture-space* in which the spatial factor of textures consists of measuring the number of semitones between the outer parts to observe expansions and contractions of register through a textural sequence (see [2, pp. 195-199]).

[27] In mathematics, an ordered partition is called *composition*, but this word can be confusing in musical contexts. For this reason, in the present work, I shall refer to a partition where the order matter as ordered partition. For further discussions on ordered partitions and their features, see [1].

[28] In this article, ordered partitions are enclosed with "< >" to differ them from unordered partitions.

[29] As in partitional notation, thread-words can use a superscript to express the multiplicity of letters.

[30] See [21] for further discussion on thread-words, their particularities, and applications.


| | | | | |
|---|---|---|---|---|
| Unordered Partitions: | [1,3] | [1,3] | [1,3] | [1,3] |
| Ordered Partitions: | <1,3> | - | - | <3,1> |
| Thread-words: | <ab$^3$> | <aba$^2$> | <a$^2$ba> | <a$^3$b> |

**Figure 6:** *Comparison between ordered partitions and thread-words to describe the spatial organizations of threads of partition [1,3]. Parts are defined by rhythmic coincidences.*

is exclusively defined by aspects regarding textural morphology, regardless of the particularities of musical materials that undergird it. In terms of realization, considering materials involved to realize a given texture as music would intricate the compositional process by increasing considerably the number of involved variables, which would, consequently, impact the degree of compositional entropy. Concerning compositional choices, a textural layout refers to the second instance while partitions are defined in the first instance. Therefore, the relation between a partition and its possible textural layouts is of the same nature of the relation between pitch classes and pitches so that the set of all layouts of a given textural configuration enables to do constitutes the exhaustive taxonomy of its spatial realizations.

Prior to the discussion on compositional entropy in a textural perspective, it is necessary to further examine the very notation of thread-word to compute the number of available possibilities, what I call *exhaustive taxonomy of textural layouts*. Moreover, it is important to introduce some concepts regarding textural layouts that are crucial for such a discussion.

## i. Exhaustive Taxonomy of Textural Layouts

In the notation of thread-words, the letters are arbitrary, that is, any letter can be associated with any textural part. Even so, it is possible to attribute a specific letter to a given textural part based on an endemic characteristic of the materials that underlie it.[31] By doing so, it is possible to track the diachronic transformations of textural parts along with contiguous textures. For example, in Figure 7, each letter refers to a different part of the thickness of two that combined forms partition [$2^2$]. The criterion for segmentation is both timbre and pitch-class content as indicated in the score. As letters "a" and "b" are invariably associated with the same parts, thread-letters spell out how threads permute within partition [$2^2$], revealing its textural layout. This shows partition [$2^2$] provides a high degree of compositional entropy as the composer may explore each one of these possible layouts in various creative ways during the compositional process.

[31] According to Berry, this prominent characteristic of textural parts emerges "when materials are of such distinctive textural cast, and when the particular qualities of texture are so vital a factor in identity and interest of thematic-motivic material." [2, p. 254]

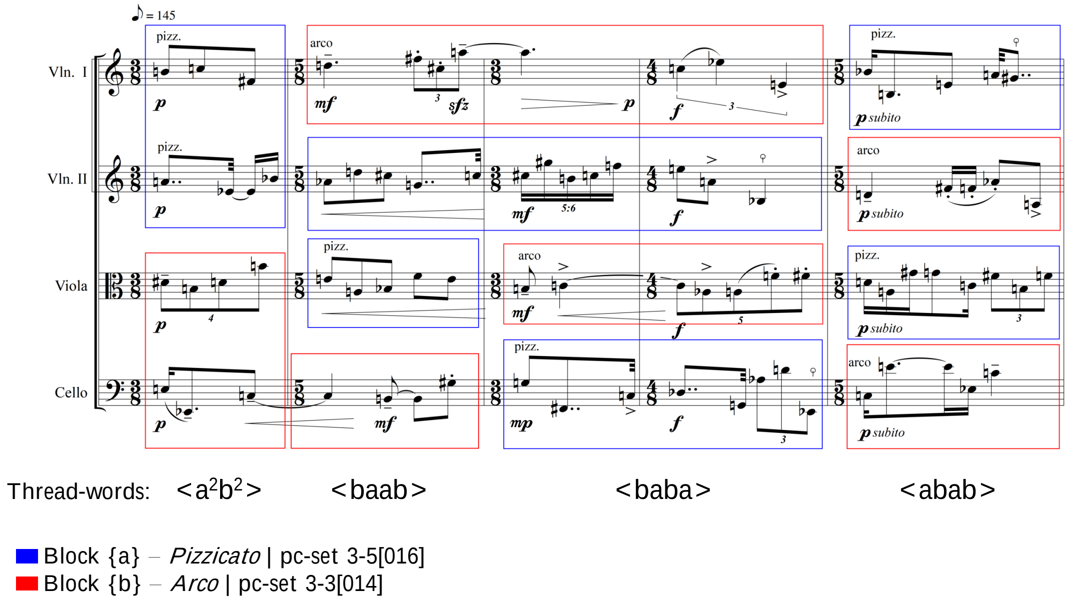


**Figure 7:** *Thread-words portraying different spatial organizations of threads within partition* [$2^2$]*. Parts are defined by both timbre (either* pizzicato *or* arco*) and pitch class content. [21, p. 99]*

Although differing from one another in their notation, the last two thread-words in Figure 7 can be understood as equivalent to each other as they consist of an alternation of threads from both parts. In fact, they correspond to the same *thread-word class* (*tw-class*). A tw-class is a notational convention akin to normal form on pitch-class set theory. It assembles all thread-classes that share the same spatial organization. By convention, the first letter of a tw-class is always "a" and for each new part is ascribed a new letter following the alphabetical order. For example, the thread-word class <$ab^2ac$> comprises thread-words all of the following: <$ba^2bd$>, <$xy^2xz$>, <$ca^2cb$>, etc. Similarly, a thread-word <$x^3yxz^2$> is rewritten as tw-class <$a^3bac^2$>. The main goal of tw-classes is to reduce notational redundancies thereby providing the accurate number of potential realizations of a partition—crucial information for calculating compositional entropy. This property may be clear with the following example. Consider thread-word <*abcd*>. How many permutations is it possible to form from it? A simple factorial of the number of elements reveals thread-word <*abcd*> can hold 24 possible permutations (i.e. $4! = 24$). Thread-words <*dcba*>, <*bacd*>, and <*cbda*> are some of these permutations. However, they are essentially identical in terms of organization: four independent threads (lines) stacked to one another. In this case, these notational variations are only reasonable in a context where each letter express a different musical idea, as demonstrated in Figure 7, which does not concern the spatial organization of texture. Thus, by considering tw-classes, partition [$1^4$] has a unique partition-layout possibility of realization expressed by the tw-class <*abcd*>.

In Figure 7, the tw-class <$a^2b^2$> in the first measure is unique that is equivalent to unordered and ordered partitions. For that reason, tw-class <$a^2b^2$> is called the *thread-word prime class* (or simply *tw-prime*). A tw-prime is a textural adaptation for the concept of prime form in pitch-class set theory. It may be defined as a tw-class where non-intermingled parts are expressed in increasing order so that all layouts of a given textural configuration (partition) are, in fact, permutations (or *anagrams*) of the tw-prime that stands for it. Each one of these anagrams stands

**Table 2:** *Anagrams of tw-prime <$a^2b^2$> and their respective notation in tw-class showing redundancies.*

| Anagrams | Tw-classes |
|---|---|
| aabb | <$a^2b^2$> |
| baab | <$ab^2a$> |
| abab | <$abab$> |
| baba | <$abab$> |
| abba | <$ab^2a$> |
| bbaa | <$a^2b^2$> |

for a different textural layout.

In order to eliminate redundancies, all anagrams from a tw-prime must be rewritten in the form of a tw-class. Consider, for example, the thread-word <$b^2a$>. It is an anagram of tw-prime <$ab^2$>. Yet it is not a tw-class given its first letter is "b" instead of "a". In order to access all textural layouts of tw-prime <$ab^2$>, the anagram <$b^2a$>must be rewritten as the tw-class <$a^2b$>. Note that their spatial organization is invariable, but this rewriting process is necessary to compute the number of textural layouts. Table 2 shows this relation between anagrams and their notations as tw-classes to map all layouts of partition <$2^2$>. Despite threads can be arranged into six different anagrams, they provide only three tw-classes, namely: <$a^2b^2$>, <$ab^2a$>, and <$abab$>.

To calculate how many textural layouts are available for a composer from a given tw-prime, it is necessary to compute the possible permutation of its component parts, eliminating possible redundancies. The number of component parts is defined by its *density-number*(DN).[32] So the number of permutations can be accessed by its factorial (i.e., $TL = DN!$, where $TL$ is the quantity of available textural layouts). Nevertheless, not all permutation shall be computed as commuting repeated letters is useless in terms of textural layout. For example, the DN of tw-prime <$a^3$> is equal to 3 so that there are six possible permutations of it ($DN! = 6$). Since all of these permutations are identical the number of textural layouts is one, represented by tw-prime <$a^3$> itself. To eliminate the redundancies of these cases, it is necessary to divide the number of permutations by the factorial of each part (expressed by the sum of each letter). This is expressed in the following formula, where $\{p_1, ..., p_i\}$ are the parts of a partition P and $i \in \mathbb{Z}_+$ (Equation 4):

$$TL = \frac{DN!}{p_1!...p_i!}. \tag{4}$$

For example, the number of layouts of tw-prime <$a^2b^3$> is equal to $\frac{5!}{2!3!} = 10$. Similarly, tw-prime <$ab^2c^3$> provides 60 textural layouts ($TL = \frac{6!}{1!2!3!} = 60$). Note that if the tw-prime comprises a unitary part, the number of layouts will be always one since the thickness of the part is equal to the density-number ($TL = \frac{DN!}{P!} = 1$, where $P = DN$).

None of these examples includes tw-primes with duplicated parts, that is, textural configuration with two or more parts holding the same thickness. These textural configurations are those written with a superscript in the form of ordered partitions. For example, in thread-word <$ab$>, both parts hold a thickness of 1. Given the notational convention of tw-class, this is significant information that impacts the output computation. After all, although <$ab$> and <$ba$> are possible anagrams,

[32]This term was coined by Berry [2] to refer to the sum of all threads within a textural configuration. In the realm of partitions, it corresponds to the positive integer $n$ that is partitioned into various ways.

they are members of the same tw-class <*ab*>. Thus, it is necessary to exclude permutations between and among parts with the same thickness from the computation of textural layouts. To so, the formula presented in Equation 4 shall be divided by the factorial of the number of duplicate parts, as demonstrated in the following formula[33] (Equation 5):

$$TL = \frac{\left(\frac{DN!}{p_1!...p_i!}\right)}{d!}, \tag{5}$$

where $d$ is the quantity of duplicated parts within the textural configuration. For example, in the tw-prime <$abc^2d^2$> there are four duplicated parts: $a$ and $b$, and $c^2$ and $d^2$ so that it provides 45 textural layouts as ($TL = \frac{6!}{1!1!2!2!}/4! = 45$). In the same way, tw-prime <*abcd*> holds a unique partition layout since the density-number is equal to the number of duplicated parts ($TL = \frac{4!}{1!1!1!1!}/4! = \frac{4!}{4!} = 1$). Considering the factorial of zero is equal to one ($0! = 1$), this formula is applicable to any tw-prime.

Table 3 provides the exhaustive taxonomy of textural layouts in the form of tw-prime, as well as their respective anagrams in the form o tw-class, for the partition lexical set for $n = 4$. Each partition can be classified into three distinct types: a) massive partitions (type M), formed by a single block ([2], [3], and [4]); b) polyphonic partitions (type P), whose the number of lines therein is equal to the density-number ([1], [$1^2$], [$1^3$] and [$1^4$]); and c) mixed partitions (type X), when blocks and lines are combined ([1,2], [1,3], [$2^2$], and [$1^2$2]). While type X contains the textural configurations with more textural layouts, in both types M and P the number of textural layouts is equal to their respective tw-prime. In this lexical set, tw-prime <$abc^2$> provides more textural layouts (6).

Figure 8 shows the topological relation among adjacent tw-class within the lexical set for $n = 4$, organized in a structure called *thread-word classes Young Lattice* (TYL). TYL can be understood as an ordered version of Gentil-Nunes' Partitional Young Lattice ([6, pp. 50-51]. It provides the exhaustive taxonomy of textural layouts in the form of tw-classes for $n$ threads. Each square is a different tw-classes so that abreast squares are textural configurations of the same density-number.[34] Lines indicate a different transformational operation that connects them. In summary, *resizing* (m) consists of the increment or decrement of the thickness of a given part by including a singular thread in it (indicated by full blue lines). *Revariance* (v) describes the operation of adding or excluding a unitary part (line) from the textural configuration (indicated by dotted green lines). Finally, *permutation* (T) is an operation where threads swap their spatial position within the textural configuration thereby preserving the density-number (indicated by double red lines).[35] All these operators involves a parsimonious change from a tw-class to another, which means a composer may choose a sequence of textures by moving along edges to ensure a smooth textural flow if this is his/her goal.[36]

[33]Despite its logical construction, this equation is still a conjecture. The proof of its infallibility will be left for future works.

[34]Note that below each tw-class there is a pair of numbers separated by a comma. They indicate the calculus of the binary relations among all threads to express whether they hold a relation of either agglomeration (in cooperation to assemble a textural part) or dispersion (non-congruence or divergence). This pair of indices provides crucial information for the very definition of textural parts. See [6, pp. 33-38].

[35]Permutation is an operation I formulate in my Ph.D. Dissertation, while both resizing and revariance were proposed by Gentil-Nunes to deal with the topological relations among partitions. Hence, their applications in the present article consist of a simple translation of their principles to the realm of thread-word classes. For further information regarding these operations, see [6] [8] and [21].

[36]Obviously, this sense of continuity deals only with textural factors since an actual smooth flow also depends on the materials the composer choose to realize the tw-classes.

**Table 3:** *Exhaustive taxonomy of textural layouts for the partition lexical set for $n = 4$. Revised version from [21, p. 100].*

| Type | Partitions | Density-number | Tw-prime | Tw-classes of Textural Layouts | Total of Textural Layouts |
|---|---|---|---|---|---|
| P | [1] | 1 | $<a>$ | $<a>$ | 1 |
| M | [2] | 2 | $<a^2>$ | $<a^2>$ | 1 |
| P | $[1^2]$ | 2 | $<ab>$ | $<ab>$ | 1 |
| M | [3] | 3 | $<a^3>$ | $<a^3>$ | 1 |
| X | [1,2] | 3 | $<ab^2>$ | $< ab^2>$, $<aba>$, and $<a^2b >$ | 3 |
| P | $[1^3]$ | 3 | $<abc>$ | $<abc>$ | 1 |
| M | [4] | 4 | $<a^4>$ | $<a^4>$ | 1 |
| X | [1,3] | 4 | $<ab^3>$ | $<ab^3>$, $<aba^2>$, $<a^2ba>$, and $<a^3b>$ | 4 |
| X | $[2^2]$ | 4 | $<a^2b^2>$ | $<a^2b^2>$, $<ab^2a>$, and $<abab>$ | 3 |
| X | $[1^22]$ | 4 | $<abc^2>$ | $<abc^2>$, $<abcb>$, $<ab^2c>$, $<abca>$, $<abac>$, and $<a^2bc>$ | 6 |
| P | $[1^4]$ | 4 | $<abcd>$ | $<abcd>$ | 1 |

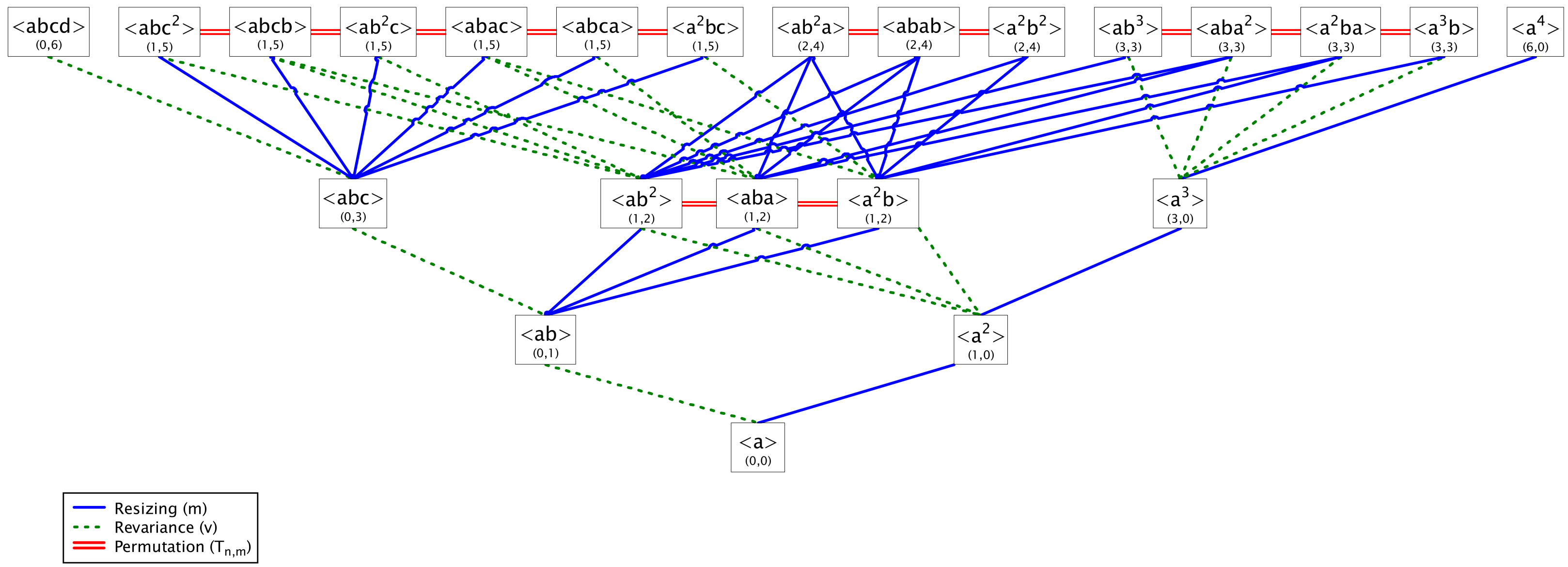


**Figure 8:** *Thread-word Young Lattice (TYL): thread-word classes associated to the lexical set of partitions for $n = 4$. Edges stands for transformational operations. Dotted green lines indicates revariance, full blue lines shows resizing, and double red lines shows permutations. [21, p. 129]*

The number of textural layouts may also consider the number of available threads. For example, suppose a composer wants to use the partition [1,2] in an instrumental set comprising up to four threads. This means that one thread will necessarily be in silence. So the composer may ask how many ways the composer can realize it within the compositional process? To answer this question, it is necessary to compute the combination of the number of available threads (T) and the density-number of the textural configuration (DN) multiplied by the number of its textural layouts (L), as can be observed in the formula (Equation 6):

$$TL = \binom{T}{DN} L = \frac{T!}{DN!(I-DN)!} L, \text{ for } \ T \geqslant DN. \tag{6}$$

Returning to the question below, the number of partition layout for [1,2] is three (i.e., $L = 3$) so that it can be realized in twelve different ways in a context of four of threads by equation$\binom{4}{3}3 = 12$ (Figure 9). In order to indicate which thread is at rest in the realization, I have included the number zero in the notation of tw-class. In this case, instead of mapping the general register span of threads, the spatial order within tw-class is associated with the timbral distribution of threads according to the way they are written in the score so that, the first position within the tw-class corresponds to the first violin, the second position indicates the second violin, and so on. Needless to say that the number of possibilities may increase significantly in contexts where the difference between the number of available parts and the density-number of the texture is greater. Textural parts are defined by timbre (either *pizzicato* or *arco*), rhythmic coincidence, and dynamics.

As discussed above, when the tw-class is a single block (type M), its realization comprises a unique textural layout each that corresponds to their respective tw-prime. Nevertheless, from the analysis of the repertoire, it is possible to observe that blocks usually assume various forms of articulation other than common sense. In my Ph.D. thesis, I proposed five modes of textural realization that probably "cover the most recurrent textural realizations of concert music inferred by various analyses" ([21, p. 166]). One of the modes, called *evolving realization*, deals specifically with the possible realizations of blocks.[37] The main premise of evolving realization is a block can be realized by "the successive superposition of its constituting parts" in such a way that "the block is not understood as such until it is fully constructed; or one can say it is retrospectively defined by the stationary motion of staggered entrance of sustaining notes." ([21, p. 174]).[38] This means the block evolves over time from a polyphonic (or mixed) to a massive presentation. An opposite effect of this construction may involve the "dilution" of blocks by gradually removing its constituting parts, as a "filtering" process. The difference between gradual *construction* and *dilution* of blocks is their onsets and offsets (endpoints). In the first, all threads assembling the block hold the same offset, but differing on their onsets. The latter, in turn, has exactly the opposite relation, with threads aligned on their onsets, but shifting their offsets. On the other hand, the traditional sense of blocks is that where the onsets and offsets of all threads match in time.

Figure 10 exemplifies all three ways of realizing a block with three notes.[39] Below each realization there is an iconic model that portrays their onset/offset relation as described above. Note that both gradual construction and dilution emerge from the cumulative superimposition of

[37]This mode is, in part, based on the proposal of Bernardo Ramos in the analysis of the ways thereby blocks can be articulated in the guitar. See [26].

[38]To better understand the principle of evolving realization, it is important to consider a wide *window of observation*, that is, the temporal frame whereby all components therein are understood as assembling the same texture. This In a wider window of observation

[39]Needless to say that the idea of block is not necessarily restricted to notes as instruments with undefined pitches can also assemble blocks according to given criteria.

<ab$^2$0>

<a0b$^2$>

<ab0b>

<0ab$^2$>

<ab0a>

<aba0>

<0aba>

<a$^2$b0>

<a0ba>

<a$^2$0b>

<a0ab>

<0a$^2$b>

**Figure 9:** *Exhaustive taxonomy of textural layouts for partition [1,2] in a context of four threads. Parts defined by both timbre, rhythmic coincidence, and dynamics.*

each thread, which would be understood, in a strict perspective, as the combination of three lines (partition [$1^4$]).[40]

A block with a thickness $n$ can be presented in evolving realization as any of the partitions of $n$. This means a block with three threads can be presented as any of the following: a) a standard realization of a block (partition [3]), b) three lines ([$1^3$]), and c) a block of two and a line ([1,2]). Just those possibilities are of great significance to composers; however, the spatial factor may be considered in order to increase significantly the set of compositional choices. Hence, the spatial organization of threads within a block, as well as the order of their onsets/offsets may

[40]The music of Edgar Varèse is full of examples of both gradual construction and dilution of blocks. See, for example, the various blocks of *Déserts* (1950/1954)—more specifically mm. 21-26 and mm. 171-174 where this strategy is clearer.


**Figure 10:** *Three ways of realizing a block with three notes considering the evolving realization: standard realization, gradual construction, and gradual dilution. Parts are defined by their stationary motion.*

include textural layouts for textures of type M, expanding, thereby, the set of layouts in tw-classes discussed above.

To compute all possible layouts for a block with a thickness of $n$, it is necessary to sum all layouts of all partitions of $n$. This may be accessed by using the same formula provided in Equation 5, but multiplying the output by $i!$, where $i$ is the number of parts within the partition, and then by two. The first multiplication is meant to consider permutations of threads in all available time-points while the second includes the computation of both gradual constructions and dilutions, (Equation 7)[41]:

$$L(K) = 2\frac{\left(\frac{DN!}{p_1!\ldots p_i!}\right)}{d!}j!\ . \tag{7}$$

With this formula, the computation of all layouts is just a matter of summing all outputs for all partitions of $n$, subtracting one from it to exclude the duplicated standard realization of the block (where $K = n$). This is given by the formula, where $n$ is the thickness of the block to be realized (Equation 8):

$$TL_{(n)} = \sum_{j=1}^{K} L(K_j) - 1. \tag{8}$$

For example, a block of three can be presented in 25 different textural layouts considering evolving realization, as demonstrated below (Equation 9):

[41]In the formula $K$ is a partition of $n$, $p_i$ is its parts, $i$ is the number of parts, and $d$ is the quantity of duplicated parts within the partition.

$$\begin{aligned} P_{(3)} &= \{[3],[1,2],[1^3]\} \\ L([3]) &= 2^{\frac{(\frac{3!}{3!})}{0!}}1! = 2 \\ L([1,2]) &= 2^{\frac{(\frac{3!}{1!2!})}{0!}}2! = 12 \\ L([1^3]) &= 2^{\frac{(\frac{3!}{1!1!1!})}{3!}}3! = 12 \\ TL_n &= 2+12+12-1 = 25 \end{aligned} \tag{9}$$

Figure 11 shows a set of iconic models to portray each one of these 25 textural layouts, relating them with their respective partition.[42] Letter (a) shows the standard realization of [3], with both onsets and offsets of all threads strictly aligned. This corresponds to the textural layout for tw-prime <$a^3$> presented in Table 3. The non-coincidence of either onsets or offsets produces, as mentioned above, gradual constructions (Figure 11b and c) and dilutions (Figure 11d and e), respectively. Below each iconic model has its description as a *contour in duration space* (see [15, pp. 150-167]).

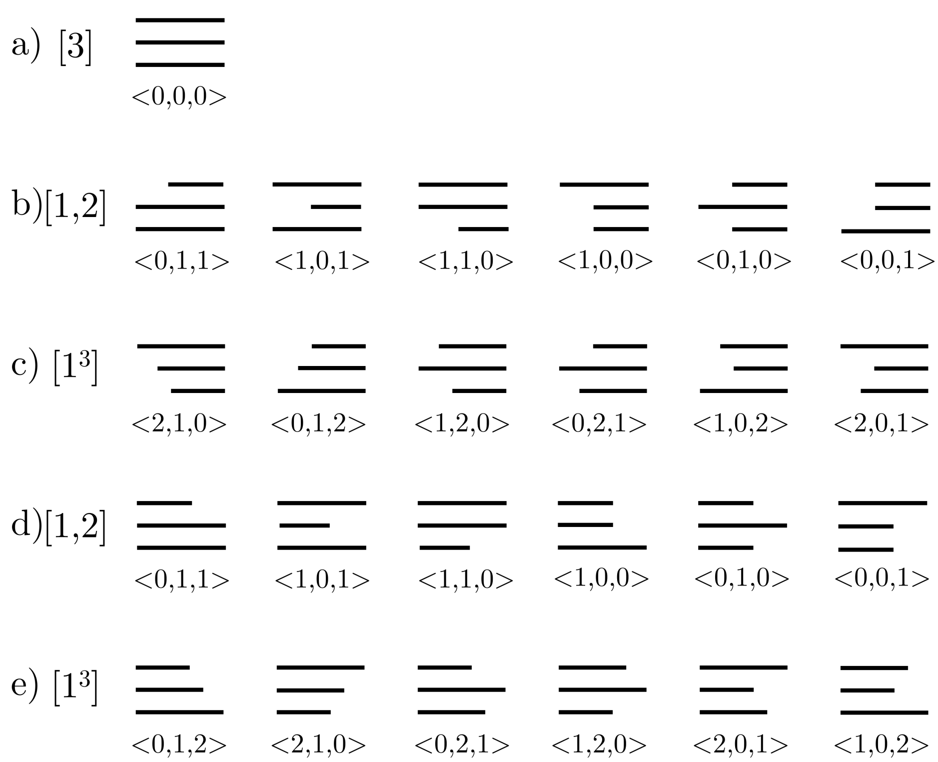


**Figure 11:** *Exhaustive taxonomy of textural layouts of three threads to assemble a block of a thickness of three considering the evolving realization: onset and offset alignment (a); offset alignment (b and c); and onset alignment (d and e). Revised version of [21, p. 176].*

[42] These iconic models are based on Robert Morris organization of sequential tones in time and space (see [22, pp. 295-299][24, p. 346])

Each thread in this textural layout corresponds to a sequence of attack points in the register so the contour expresses their spatial order of presentation according to their relative duration. In the contour, each thread is identified by a number from zero (the shortest duration) to $n - 1$ (the longest duration), where $n$ stands for the number of different durations therein.[43] Also, within the contour, the first element refers to the relative duration of the highest note, the second element indicates the duration of the note contiguously above that, and so on, so that the left-to-right order in the contour depicts the top-to-down disposition of threads in the register. Note that all contours in gradual construction have a version in gradual dilution as they are the very retrogradation of each other. In the iconic models, they may be understood as the rotational operation from one another.

During the compositional process, a composer may be compelled to combine gradual constructions and dilutions to produce more complex articulations of the block of three in evolving realization. In this case, there are 144 possible combinations available for the composer. This wealth amount of possibilities may be a fruitful compositional inventory for composers to explore their creativity in dealing with blocks. As could not be otherwise, the greater the thickness of the block, the higher the number of its textural layouts to be considered in evolving realization.

From the computation of an exhaustive taxonomy of textural layout presented here, it is possible to measure the compositional entropy of both the set of musical textures and textural layouts. This is the subject of the next section.

## V. Compositional Entropy Applied to Musical Texture

A compositional entropy applied to texture consists of the measurement of the amount of choices a composer may have to choose both textural configurations in the form of partitions for a given number of threads (the first instance of compositional choices—compositional entropy of choice) and the variety of textural layouts a chosen partition holds (second instance of compositional choices—compositional entropy of realization). In any case, the higher the number of options available for the composer, the higher is the compositional entropy. In order to proceed in this discussion, let me exemplify the relation between textural entropy in both instances: partitions and textural layouts.

By examining all partitions from 1 to $n$, one may notice that the higher the number of threads, the greater will be the set of available partitions for it, and, consequently, the more complex will be the choosing process expressed by the compositional entropy. If a composer chooses to create a piece for a wind quintet, for example, despite multiphonics and other extended techniques, it involves a set of five available threads. Then, there are eighteen different partitions to be chosen in the construction of his/her piece (Table 4).

The compositional entropy involved in the choosing process of a single partition from this set is equal to $\log(18) = 4.17$. Nevertheless, it is not common for an entire piece to comprise only a single partition, except, perhaps, in cases of monophonic pieces as it comprises a single thread. Therefore, the number of compositional entropy may increase depending on the number of partitions to be chosen. A compositional choice involves five different partitions of the set presented in Table 4 denotes a compositional entropy equal to 13.065. Figure 12 provides the curve of compositional entropy of the lexical set of partition for $n = 5$ given a number of compositional choices ranging from one to eighteen so that the composer may decide on how much freedom he/she would like to manage during the compositional process based on the degree of compositional entropy.

[43] In musical contour, the relation among its internal elements is not absolute, but relative, so that two distinct musical structures (e.g., melodies or rhythms) can be depicted by the same contour. Thus, the duration discussed here is based only on the relative proportion among threads, without considering their actual duration.

**Table 4:** *Lexical set of partitions (exhaustive taxonomy) for $n = 5$.*

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| Partitions of 1 | [1] | | | | | | |
| Partitions of 2 | [2] | $[1^2]$ | | | | | |
| Partitions of 3 | [3] | [1,2] | $[1^3]$ | | | | |
| Partitions of 4 | [4] | [1,3] | $[2^2]$ | $[1^2 2]$ | $[1^4]$ | | |
| Partitions of 5 | [5] | [1,4] | [2,3] | $[1^2 3]$ | $[1, 2^2]$ | $[1^3 2]$ | $[1^5]$ |

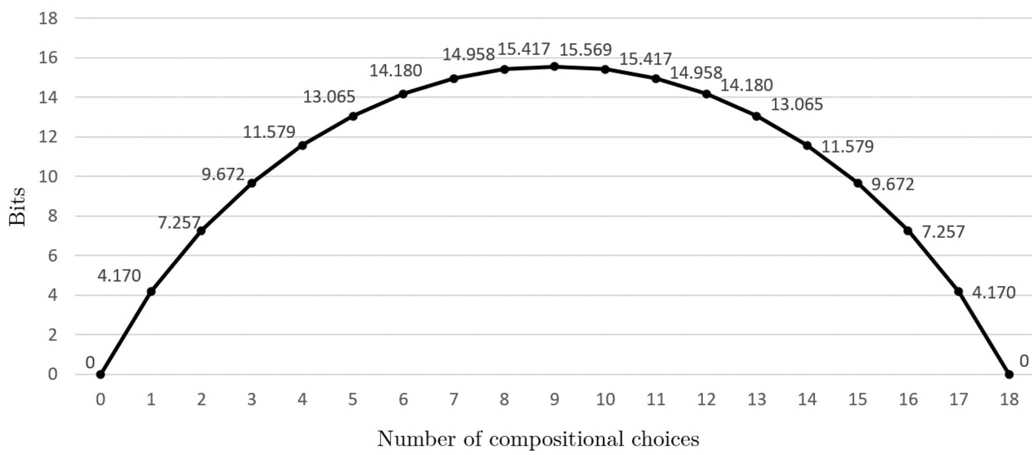


**Figure 12:** *Curve of compositional entropy applied to the 18 partitions provided in of Table 4 considering the number of compositional choices involved.*

Most composers are not aware of all textural possibilities provided in Table 4. Indeed, despite its importance:

> the compositional potential of texture is a topic not often discussed or analyzed in the theoretical literature. Even recent composers (most often) do not explicitly present in their writings on their own music ideas about textural organization; nor do they display any concern about a systematic approach to texture. If their music seems to be constructed out of various textural configurations, these are conceived intuitively, as the outcome of the interaction of the other musical parameters ([21, p. 58]).

Thus, by accessing the exhaustive taxonomy of partitions of a given number of threads is significant to musical compositional as it provides all possible creative paths a composer may take This means that exhaustive taxonomies of partitions enable the composer to be aware of the most recurrent textural configurations in his/her music, thereby bringing texture to a more conscious zone of the creative process. Furthermore, it may allow the composer to explore textural configurations that would probably not otherwise be used in his/her music.

A short musical example of the realization of the sequence of partitions <$[2^2][2,3][4][1^2][2]$> made from partitions of Table 4 may contribute to this still embryonic discussion on both compositional entropy applied to the texture (Figure 13). Textural parts are defined by the rhythmic


| | | | | | |
|---|---|---|---|---|---|
| Partitions: | $[2^2]$ | [2,3] | [4] | $[1^2]$ | [2] |
| Number options within DN: | 5 | 7 | 4 | 2 | 2 |
| Number of available textural layouts: | 3 | 10 | 149 | 1 | 5 |

**Figure 13:** *A possible realization of the sequence of partitions <$[2^2]$[2,3][4]$[1^2]$[2]> and the number of possible choices involved in both their very definition, considering its probability of being chosen within partitions of its density-number, and the number of their possible textural layouts. Parts are defined by rhythmic coincidence.*

coincidence of threads. Despite the instrumental mean (piano solo) provides the possibility to use more than five simultaneous threads, for the sake of simplicity, the maximum value of $n$ is five. Below each realization is its corresponding partition, the number of possible choices involved in their definition (first instance), considering its probability of being chosen within partitions of its density-number, and the number of their respective possible textural layouts (second instance). Such information is decisive for computing textural entropy of partitions and their respective realizations. Not all available density-numbers were used in this example; only partitions for $n$ equal to either two, four, or five.

A graph can be plotted to compare the difference of compositional entropy of choice and realization (Figure 14). The graph shows each compositional entropy is independent of one another and their value can diverge considerably. This means a partition with a low degree of compositional entropy may imply a high entropy in terms of its realization and vice-versa. Take, for example, the last two partitions of the sequence $[1^2]$ and [2]. Both constitute all partitions of $n = 2$ so their compositional entropy of choice is equivalent as they have the same probability to be chosen. Yet, while $[1^2]$ has the minimal degree of compositional entropy of realization (zero), which means there is only one way of realizing it, partition [2] has an entropy of realization of 2.322.

A striking feature may be observed in partition [4]. It holds a compositional entropy of choice equals to 2, but provides an entropy of realization equals to 7.219 given its 149 possible realizations, which includes the standard realization of the block of four (the very realization presented in the score) and all its possible gradual constructions and dilutions in evolving realization. In face of that, one may logically conclude that for any value of $n$, a block with a thickness of $n$ is, by far, the most complex situation in terms of realization measured by the compositional entropy. This is quite curious information given the simplicity of the idea of a block may evoke in textural aspects. Indeed, blocks hold the lowest degree of textural complexity within their density-number.[44] In fact,

[44]The complexity of a given textural configuration can be measured by the evaluation of the degree of independence of threads so that polyphonic textural organizations tend to be understood as more complex than massive ones. Of course, this definition is based only in regards of textural aspects. See [18], [20], and [21].

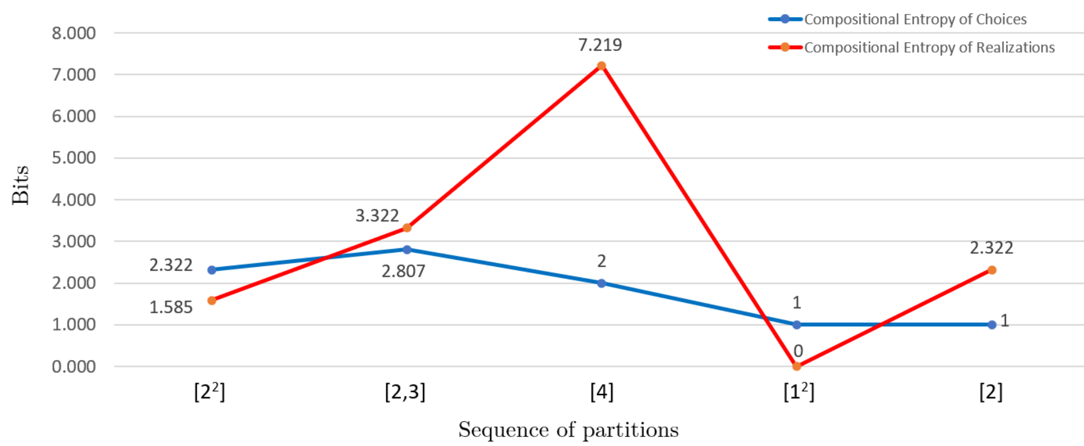


**Figure 14:** *Compositional entropy of choice and compositional entropy of realization for the sequence of partitions* <$[2^2][2,3][4][1^2][2]$>.

the compositional practices of texture show that blocks are commonly associated with opening or cadential gestures, situations in which they are more likely to be understood as simpler than textural those configurations with a higher sense of polyphony (multiple parts).[45] Nevertheless, the compositional entropy brings to lights that blocks are way more complex to be implemented as music than one could imagine, by simply considering their combinatorial permutation of threads in evolving realization. This complexity increases astronomically by combining gradual construction and dilutions.

## VI. Concluding remarks

This article introduces the idea of compositional entropy—a proposal for measuring the amount of freedom a given object in the scope of a musical parameter or attribute provides for compositional purposes so that the more possibilities (variables) to be chosen are given by the object, the higher will be its compositional entropy. The main interest of such a formulation is to discuss compositional choices in a view of probability and combinatorial permutation, considering the number of available compositional choices a composer may operate during the creative process. Further applications in various musical parameters and compositional situations shall be considered in future works to verify all potentialities of this still embryonic proposal.

In order to apply this proposal in the realm of musical texture to verify its potentialities in terms of compositional choices, a series of concepts and mathematical tools regarding textural morphology, as well as their possible spatial organizations, were introduced. This can contribute to the development of the textural field from both analytical and compositional perspectives. Moreover, the exhaustive taxonomy of textural layouts presented in this article opens avenues for compositional and analytical—which shall be further developed in future works.

All examples presented in this article, but those in evolving realization, were given according to a standard mode of presentation, which may be understood as "a simple articulation of all parts of a textural configuration in a strict way within the same time span", constituting, therefore,

[45] See [6].

"a strict one-to-one relation with the referential configuration and its musical realization." ([21, p. 167]). The inclusion of the other modes in the perspective proposed here may expand the number of possible realizations of a given partition. For example, Gentil-Nunes [8] elaborates the idea of a *partitional complex*, in which small deviations of textural parts may produce subsets and other derived parts of the referential partition. Unlike evolving mode, in partitional complex, polyphonic partitions provide a wider set of possible realizations. Hence, taking the formulation of partitional complexes into consideration may expand the number of possible realizations for all partitions, thereby impacting the computation of compositional entropy. Another possibility to be considered is the application of the principles of evolving realization in any block in contexts of multiple parts. For example, each block of partitions [$2^2$] and [2,3] in Figure **??** could be realized as the cumulative superimposition of threads therein. All these possibilities discussed here are a wealthy territory to be explored in future works.

The elaboration of computational tools for automatizing the process of computing the number of possibilities and their compositional entropy may be devised to further improve this discussion. Also, the automatic elaboration of the list of exhaustive taxonomies for textural layouts of a given partition may facilitate their application within the compositional process, also contributing to further theoretical developments from it. Finally, the automatic generation of graphs from a MIDI file may enable analytical applications in a more systematic way. All these computational implementations are left for future works.

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