# MusMat

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# Foreword

The MusMat Research Group is very pleased to announce the release of the first number of the fifth volume of **MusMat • Brazilian Journal of Music and Mathematics**. This issue comprises seven exciting papers, that present the results of original and innovative research in the field. The number opens with a work by Robert Peck, where he applies the Power Group Enumeration Theorem to extend the theory of beat-class sets by also considering rhythms with more than one voice. Next, Gideon Effiong shows how quasigroups can be used as a unifying framework to describe musical objects and events, such as chord inversions, n-tone composition charts, and melodic motions. Ciro Visconti then discusses how graph theory can be used to describe all classes of trichords and tetrachords in Neo-Riemannian theory, expanding beyond triads and seventh chords. Francisco Aragão presents an application of Kripke Semantics to identify if a sequence of chords constitutes or not a tonal progression, which can be used to create a software to benefit students that do not have easy access to a harmony teacher. Subsequently, Juan Sebastián Arias-Valero and Emilio Lluis-Puebla develop deep relations and philosophical reflections between Gesture Theory and Category Theory. Next, Silvio Ferraz describes a series of patches in Max/MSP environment tailored to aid musical analysis and composition and exemplifies it with the composition of a piece based on Brahms Op. 119. The issue closes with Daniel Moreira introducing the concept of compositional entropy, which deals with the amount of freedom of a composer when dealing with compositional choices, and such a concept is demonstrated in musical texture.

Hugo Tremonte de Carvalho

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# Contents

# Beat-Class Set Classes and the Power Group Enumeration Theorem

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***Abstract:*** *The theory of beat-class sets originates in the work of Milton Babbitt, who demonstrates a correspondence between modular pitch-class spaces and metric spaces in the framework of total serialism. Later authors, particularly Richard Cohn, John Roeder, and Robert Morris, apply similar concepts to a variety of analytical situations, drawing on concepts and procedures from pitch-class set theory. In light of the correspondence between these theories, the universe of beat-class sets for a given modulus may be partitioned into equivalence classes similar to pitch-class set classes. This study investigates processes of enumerating these equivalence classes.*

*We consider extensions to the theory of beat-class sets by including rhythms with more than one voice. Specifically, we examine equivalence classes of multiple-voiced beat-class sets using the Power Group Enumeration Theorem (PGET) of Frank Harary and Edgar M. Palmer. The PGET allows us to determine the numbers of equivalence classes of beat-class sets as determined by various groups of transformations: metric shift, retrogradation, and voice permutation, among others. Our results have implications for further applications in pitch-class set theory, serial theory, and transformational theory.*

***Keywords****: Beat-class set. Equivalence class. Combinatorics. Enumeration. Power group.*

## I. Introduction

Consider rhythms (a)-(c) in Figure 1. Each rhythm displays six eighth notes, distributed among three voices. Rhythms (a) and (b) relate by a metric shift of three eighth notes. They are, accordingly, members of an orbit of the cyclic group of metric shifts, acting on the set of rhythms in the meter $\frac{6}{8}$. In contrast, rhythm (c) does not relate by metric shift to the previous two. Consequently, it is a member of a different orbit of this group; hence, it is not a member of the same equivalence class.

Such equivalence classes are similar to the $T_n$ set classes of pitch-class set theory. They are orbits of cyclic groups: the cyclic transposition group of order 12, in the case of the traditional, pitched $T_n$ classes, and the cyclic metric-shift group of order 6, in the case of these rhythms. As with $T_n$ classes, these rhythmic equivalence classes vary in size, depending on the degree of symmetry of their members. The first two rhythms here display no rotational symmetry. As a result, the equivalence class to which they belong contains six such rhythms. The third rhythm,

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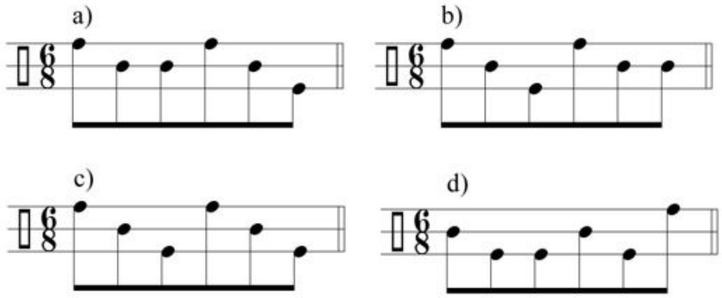


**Figure 1:** *Four three-voiced rhythms in* $\frac{6}{8}$.

on the other hand, is symmetrical under a metric shift of three eighth notes. As a result, its equivalence class contains only three rhythms.

Next, compare rhythm (d) in Figure 1 to (a) and (b). Rhythm (d) does not obtain from (a) by an operation on metric positions, but these rhythms do relate by voice permutation. They are members of a different equivalence class, an orbit of the group of voice permutations. Likewise, rhythms (b) and (d) are members of an *equivalence class*: an orbit of a group that acts both on metric positions and voices. Such a group is an example of a power group, which we investigate in greater depth below.

This study considers these types of equivalence classes. In particular, we are concerned with the enumeration of equivalence classes of rhythmic sets that relate by metric shifts and voice permutations. This task is a music-theoretical application of concepts and procedures from the mathematical field of combinatorics. Some of our work will incorporate classical combinatorial techniques, such as Burnside's Lemma and Pólya's Enumeration Theorem. Other methods derive from more recent results, including de Bruijn's Theorem and the Power Group Enumeration Theorem.

Throughout this study, we consider examples of rhythms in $\frac{6}{8}$ under the action of groups of metric shifts and voice permutations. Several of these examples incorporate three voices, as in Figure 1. The theory presented here is not limited to these particular parameters. It is sufficiently general to apply to rhythms in any meter with any number of voices under the action of any relevant group. Similarly, for simplicity, the examples included here contain only beat-level onsets. The theory may apply as well to rhythms that include subdivision-level onsets.

## i. Beat-Class Sets

We situate our findings in the context of beat-class set theory. The theory of beat-class sets originates in the work of Milton Babbitt [1], who refers to sets of time points in terms of a correspondence between modular pitch-class spaces and metric spaces in the framework of total serialism. Benjamin Boretz [2], John Rahn [3], and David Lewin [4] investigate this connection further. Later authors, including Richard Cohn [5], Robert Morris [6], and John Roeder [7], apply similar concepts to a variety of analytical situations, such as in the phase music of Steve Reich.

A beat-class set is a rhythmic analogue of a pitch-class set, where the modular pitch-class space of the latter is exchanged with a modular space of metric positions in the former. For instance, an eighth-note rhythm in $\frac{6}{8}$ has six potential beat-level onsets, which we label in the integers modulo 6 (where the downbeat is equal to 0). The rhythm in Figure 2 features onsets at metric positions 0, 2, and 3, thereby constituting the beat-class set $\{0, 2, 3\}$. As with pitch-class sets, we may perform

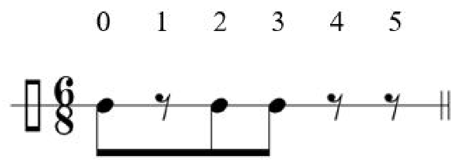


**Figure 2:** *Beat-class set* $\{0, 2, 3\}$.

various operations to beat-class sets. Unit metric shift generates a cyclic group that acts on the set of metric positions, similar to the group of transposition operators that acts on pitch-classes. Retrogradation agrees with the pitch-class operation of inversion. Adjoining this operation to the group of metric shifts yields a dihedral group of operators, corresponding to the pitch-class transposition-and-inversion group. The respective actions of these groups on the set of beat-class sets within a particular meter partition that set into equivalence classes that parallel the $T_n$ and $T_n/T_nI$ set classes of pitch-class set theory.

In this study, we consider not only beat-class sets with a single voice, like the set $\{0, 2, 3\}$ above, but also multiple-voice sets, such as the three-voice examples we considered initially. Operations on metric positions—such as metric shift and/or retrogradation—partition the set of multiple-voice beat-class sets into equivalence classes. However, these operations do not situate beat-class sets that are related merely by a permutation of their voices into the same equivalence class. Nevertheless, such sets are equivalent in a rhythmically generic sense. For instance, both the rhythms in Figure 1 (a) and (d) above may be described as follows: an initial voice presents an onset at metric position 0; two onsets follow in another voice at positions 1 and 2; the initial voice presents another single onset at position 3; the second voice has one onset at position 4; and a third voice concludes the rhythm with an onset at position 5. By virtue of this generic rhythmic identity, we adjoin voice permutations to the previous groups to form a new category of groups, the orbits of which constitute equivalence classes of multiple-voice beat-class sets.

In addition to the multiple-voiced rhythms in Figure 1, which contain an onset on every beat of the measure, we also consider multiple-voiced rhythms with some number of beats that contain no onsets (e.g., beats with rests or which simply contain no onset). One solution to this problem would be merely to assign the beats with no onsets to a particular voice. However, if the number of onsets is to remain constant, then we should not allow the voice with no onsets to permute with the voices with onsets. As we will see, the Power Group Enumeration Theorem offers a method for achieving this result.

## ii. Enumeration Applications in Music Theory

Considering that pitch-class set theory as a discipline did not emerge until the 1960s and early 1970s, the enumeration of what are essentially pitch-class set classes has a surprisingly long tradition, beginning in the late 19th and early 20th centuries. Jonathan Bernard [8] and Catherine Nolan [9, 10] discuss the history of early efforts in this endeavor, citing the work of Heinrich Vincent, Anatole Loquin, and Ernst Bacon. Julian Hook [11] presents a detailed tutorial of classical combinatorial enumeration techniques, including Burnside's Lemma and Pólya's Enumeration Theorem, as applied to a host of music-theoretical topics, including the counting of $T_n$, $T_n/T_nI$, and $T_n/T_nI/T_nM$ set classes in modular pitch-class spaces of various sizes, row classes of twelve-tone series, equivalence classes of beat-class sets, and the like.

Other music-theoretical research that incorporates enumeration applies these techniques to various problems. Joel Haack [12] uses combinatorial methods to determine the number of beat-class sets that have the same properties as—and that could substitute for—the rhythm that serves as the basis for Steve Reich's 1972 composition *Clapping Music*. Ronald C. Read [13] considers the size of the set of all possible performances of Stockhausen's *Klavierstück XI*. He calculates precisely the number of paths that may be taken through 19 fragments that constitute the score to that piece, approximately equal to $1.7 \times 10^{41}$. Harald Fripertinger [14, 15] discusses the enumeration of a variety of musical objects, such as intervals and chords, patterns of rhythms and motives, as well as tone rows and patterns of tropes. Fripertinger and Peter Lackner [16] examine in greater depth the enumeration of the latter two topics in an article that constitutes an entire special issue of *Journal of Mathematics and Music*. The enumeration of tropes is of particular relevance to this study, as it involves the action of a power group: a group that acts on hexachordal partitions of 12-tone rows by transposing or inverting the pitch-classes in the hexachords and/or by permuting the hexachords themselves.

## II. Enumeration of Equivalence Classes of Beat-Class Sets

### i. Single-Voice Sets

The enumeration of single-voice beat-class sets, such as the example of the beat-class set $\{0, 2, 3\}$ above, is equivalent to counting the number of subsets of beats classes in a single measure with $n$ beats. For a given subset, two possibilities exist for any beat class in the measure: the subset may include an onset at that beat or it may not. Hence, there is a total of $2^n$ possible subsets. The number of equivalence classes to which these subsets belong, however, is not merely $2^n$ divided by the order of the group that acts on the beat classes, as certain sets may possess symmetry.

Burnside's Lemma gives the number of orbits as an average. Each member $\alpha$ of the group $A$ fixes some number of elements of the set $X$ on which the group acts, the set $X^\alpha$. The number of orbits for $A$, then, is the average of these numbers of fixed elements as $\alpha$ varies within $A$:

$$|A/G| = \frac{1}{|A|} \sum_{\alpha \in A} |X^\alpha| . \tag{1}$$

In the case of $\frac{6}{8}$, we observe that there exist $2^6 = 64$ possible beat class sets (including the empty set and the set that includes onsets in all six metric positions). As we noted above, some of these sets possess symmetry. To calculate the number of orbits into which the 64 sets partition under the action of the group of metric shifts, we need to know how many of these sets are fixed by translation by one metric position, two metric positions, etc., through six metric positions. Then, the number of equivalence classes is the average of these tallies.

To ascertain the numbers of sets that are fixed by any particular member of a group, it is useful to examine the disjoint cycle decomposition of each element of the group. Let $A$ be a permutation group with an action on a set $X$ of size $d$. Each permutation in $A$ may be written as a product of disjoint cycles. For every $k$ from 1 to $d$, let $j_k$ be a function that counts the number of cycles of length $k$ among the disjoint cycles in the decomposition of the permutation. For example, our group of metric shifts in $\frac{6}{8}$ acts on the set of 6 metric positions, labeled 0 (downbeat) through 5. Let $T_n$ be a metric shift of $n$ eighth notes. Then, the members of $A$ have the disjoint cycle

**Table 1:** *Values of $j_k(T_n)$.*

| $T_n$ | $j_1$ | $j_2$ | $j_3$ | $j_4$ | $j_5$ | $j_6$ |
|---|---|---|---|---|---|---|
| $T_0$ | $\mathbf{j_1(T_0) = 6}$ | $j_2(T_0) = 0$ | $j_3(T_0) = 0$ | $j_4(T_0) = 0$ | $j_5(T_0) = 0$ | $j_6(T_0) = 0$ |
| $T_1$ | $j_1(T_1) = 0$ | $j_2(T_1) = 0$ | $j_3(T_1) = 0$ | $j_4(T_1) = 0$ | $j_5(T_1) = 0$ | $\mathbf{j_6(T_1) = 1}$ |
| $T_2$ | $j_1(T_2) = 0$ | $j_2(T_2) = 0$ | $\mathbf{j_3(T_2) = 2}$ | $j_4(T_2) = 0$ | $j_5(T_2) = 0$ | $j_6(T_2) = 0$ |
| $T_3$ | $j_1(T_3) = 0$ | $\mathbf{j_2(T_3) = 3}$ | $j_3(T_3) = 0$ | $j_4(T_3) = 0$ | $j_5(T_3) = 0$ | $j_6(T_3) = 0$ |
| $T_4$ | $j_1(T_4) = 0$ | $j_2(T_4) = 0$ | $\mathbf{j_3(T_4) = 2}$ | $j_4(T_4) = 0$ | $j_5(T_4) = 0$ | $j_6(T_4) = 0$ |
| $T_5$ | $j_1(T_5) = 0$ | $j_2(T_5) = 0$ | $j_3(T_5) = 0$ | $j_4(T_5) = 0$ | $j_5(T_5) = 0$ | $\mathbf{j_6(T_5) = 1}$ |

decompositions shown here:

$$T_0 := (0)(1)(2)(3)(4)(5) \tag{2}$$
$$T_1 := (0, 1, 2, 3, 4, 5) \tag{3}$$
$$T_2 := (0, 2, 4)(1, 3, 5) \tag{4}$$
$$T_3 := (0, 3)(1, 4)(2, 5) \tag{5}$$
$$T_4 := (0, 4, 2)(1, 5, 3) \tag{6}$$
$$T_5 := (0, 5, 4, 3, 2, 1). \tag{7}$$

Table 1 shows the values of $j_k$ for each $T_n$ (non-zero values appear in bold).

Call $q(T_n)$ the sum of all values $j_k$ for any particular $T_n$. Then, we use $2^{q(T_n)}$ to determine the number of beat-class sets in $\frac{6}{8}$ that are symmetrical under the element $T_n$ of the group of metric shifts. For example, since $q(T_0) = 6$, $T_0$ fixes all $2^6 = 64$ beat-class sets. Similarly, since $q(T_1) = 1$, $T_1$ fixes 2 beat-class sets, and so forth. The total number of beat-class sets fixed by the elements of the group is 84, which, divided by 6 (the size of the group), equals 14—the number of $T_n$ equivalence classes (or orbits of this particular group).

It will be useful below to keep track of the disjoint cycle decomposition of the elements of a group $A$. We may do this by means of the cycle index of $A$, $Z(A)$, a polynomial in the variables $s_1, s_2, ..., s_d$:[1]

$$Z(A) = \frac{1}{|A|} \sum_{\alpha \in A} \prod_{k=1}^{d} s_k^{j_k(\alpha)}. \tag{8}$$

For example, consider the cycle index for the group $A$ of metric shifts in $\frac{6}{8}$:

$$Z(A) = \frac{1}{6}(s_1^6 + s_2^3 + 2s_3^2 + 2s_6). \tag{9}$$

Note that there exists one element in the group, $T_0$, that contains six cycles of length 1, and 1 is the coefficient of $s_1$ in $Z(A)$; one element, $T_3$, contains three cycles of length 2, and 1 is the coefficient of $s_2$; two elements, $T_2$ and $T_4$, contain two cycles of length 3, and 2 is the coefficient of $s_3$; finally, two elements, $T_1$ and $T_5$, contain one cycle of length 6, and 2 is the coefficient of $s_6$.

Next, we wish to determine the number of equivalence classes that exist for sets of the same size. Let $A$ be a group of order $m$ that acts on a set $X$ of size $d$; $\alpha \in A$. Further, let $B$ be a group that acts on a set $Y$ of size $e \geq 2$ into which we map the elements of $X$; $\beta \in B$.

[1] The variable $s$ here does not mean anything; it is a placeholder.

**Definition 1.** Call $B^A$ the power group, which has an action of the set $Y^X$ of all functions $f : X \to Y$. The members of $B^A$ are the ordered pairs $(\alpha; \beta)$. Any function $f$ in $Y^X$ under $(\alpha; \beta)$, then, consists of the mapping $((\alpha; \beta)f)(x) = \beta f(\alpha x)$.

For now, however, we will assume that $B$ is the identity group—that is, its single element does not permute elements of $Y$.

Let $w$ be a weight function that maps $Y$ into the set of non-negative integers, $0 \leq k \leq \infty$, and let

$$c^k = \left|w^{-1}(k)\right| \tag{10}$$

be the number of number of configurations with weight $k$. Then, we call the series in the indeterminant $x$,

$$c\,(x) = \sum_{k=0}^{\infty} c_k x^k, \tag{11}$$

the "configuration counting series". Because all functions in the same orbit of $B^A$ have the same weight, there exists a finite number of orbits of any particular weight. Therefore, let $C_k$ be the size of the set of orbits of weight $k$. Then, the series in the indeterminant $x$,

$$C\,(x) = \sum_{k=0}^{\infty} C_k x^k, \tag{12}$$

is the "function counting series". This leads to the weighted version of Pólya's Enumeration Theorem. Let $Z(A, c(x))$ be an abbreviated denotation of $Z(A; c(x), c(x^2), c(x^3), ...)$.

*Theorem* 1 (Pólya). We substitute $c(x^k)$ in the configuration counting series for each variable $s_k$ in $Z(A)$ to determine the function counting series $C(x)$. That is,

$$C(x) = Z(A, c(x)). \tag{13}$$

Let us again consider the example of beat-class sets in $\frac{6}{8}$. Let $A$ be the group of metric shifts, acting on the set of six beat classes, and let $B$ be the identity group with an action on a single voice. The configuration counting series for $A$ appears here:

$$c(x) = (1 + x) + (1 + x^2) + (1 + x^3) + (1 + x^4) + (1 + x^5) + (1 + x^6). \tag{14}$$

We recall the cycle index of $A$ from Equation 9, and we substitute $(1 + x^k)$ for $s_k$ in $Z(A)$ to determine the function counting series:

$$C(x) = \frac{1}{6}((1 + x)^6 + (1 + x^2)^3 + 2(1 + x^3)^2 + 2(1 + x^6)). \tag{15}$$

Expanding the above yields the following polynomial,

$$\begin{aligned} C(x) = \frac{1}{6}((1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6) + (1 + 3x^2 + 3x^4 + x^6) \\ +(2 + 4x^3 + 2x^6) + (2 + 2x^6)), \end{aligned} \tag{16}$$

which we simplify as follows:

$$C(x) = 1 + x + 3x^2 + 4x^3 + 3x^4 + x^5 + x^6. \tag{17}$$

The coefficients of $x^k$ in the polynomial determine the number of equivalence classes for beat-class sets of size $k$. Therefore, there exists one equivalence class of sets of size 0, one equivalence class of sets of size 1, three of sets of size 2, and so on, for a total of 14.

## ii. Multiple-Voice Sets

We may also use Pólya's Enumeration Theorem to find the numbers of equivalence classes of multiple-voiced beat-class sets. For $e$ voices, we substitute an expression with $e$ variable summands in place of the indeterminants in the cycle index of $A$. For instance, if we wish to enumerate the equivalence classes of the $3^6 = 729$ three-voice beat-class sets in $\frac{6}{8}$, we could replace $s_k$ in $Z(A)$ with $(1 + x^k + y^k + z^k)$. Then, the coefficients of $x^m y^n z^p$ in $C(x, y, z)$ are the numbers of equivalence classes of sets with $m$, $n$, and $p$ onsets in each respective voice.

The method above for partitioning the set of $e$-voiced beat-class sets in a meter with $d$ beats does not include sets that relate to one another by a permutation of the voices in the same equivalence class. If we wish to determine the numbers of these larger equivalence classes, we utilize the Power Group Enumeration Theorem of Harary and Palmer (PGET), which generalizes Pólya's Enumeration Theorem [17].[2] The PGET comes in two forms: the constant form, which counts the total number $N$ of orbits of a power group; and the polynomial form, which enumerates the number $N(x)$ of equivalence classes of sets of the same cardinality.

We define the power group $B^A$ as above, where $A$ is a group of metric shifts, acting on a set $X$ of $d$ beats in a given meter. In the previous examples, however, we let $B$ be merely an identity group, acting trivially on the set of $e = 1$ voices. Now, we let $B$ have a non-trivial action on the set of $e > 1$ voices. Then, the PGET in its constant form is as follows:

*Theorem* 2 ([17], p. 163).

$$N = \frac{1}{|B|} \sum_{\beta \in B} Z\left(A; m_1(\beta), m_2(\beta), ..., m_d(\beta)\right), \tag{18}$$

where

$$m_k(\beta) = \sum_{s|k} s j_s(\beta). \tag{19}$$

Let us take the case of beat-class sets in $\frac{6}{8}$ that include up to three voices. We are interested in determining the number of equivalence classes of these sets that are inclusive both of metric shifts and permutations of the voices. Then, $A$ is a cyclic group of order 6, acting on the set of six beats, and $B$ is the symmetric group of degree three, the set of all permutations of three voices. We gave the cycle index of $A$ above (see Equation 9); the cycle index of $B$ appears below:

$$Z(B) \ = \ \frac{1}{6}\left(s_1^3 + 3 s_1 s_2 + 2 s_3\right). \tag{20}$$

There exists one element of $B$ that consists of three cycles of length 1; three elements of $B$ that contain one cycle of length 1 and one cycle of length 2; and two elements of $B$ that have one cycle of length 3.

We can calculate the tuple $(m_1(\beta), m_2(\beta), ..., m_d(\beta))$ for a member $\beta \in B$ as follows. Let $\beta$ be one of the three elements $s_1 s_2$ of the group of voice permutations that has one cycle of length 1 and one cycle of length 2. Table 2 demonstrates that the tuple $(m_1(\beta), m_2(\beta), ..., m_d(\beta))$ for

[2] The PGET is itself a generalization of an earlier result, de Bruijn's Theorem, which enumerates equivalence classes of weighted functions from one set to another [18].

**Table 2:** *Calculating* $(m_1(\beta), m_2(\beta), ..., m_6(\beta))$ *for* $\beta = s_1 s_2$.

| $m_k$ | $k$ | $s$ | $j_s(\beta)$ | value of $m_k(\beta)$ |
|---|---|---|---|---|
| $m_1$ | 1 | 1 | $j_1(\beta) = 1$ | $(1 \cdot 1) = 1$ |
| $m_2$ | 2 | 1 | $j_1(\beta) = 1$ | |
| | | 2 | $j_2(\beta) = 1$ | $(1 \cdot 1) + (2 \cdot 1) = 3$ |
| $m_3$ | 3 | 1 | $j_1(\beta) = 1$ | |
| | | 3 | $j_3(\beta) = 0$ | $(1 \cdot 1) + (3 \cdot 0) = 1$ |
| $m_4$ | 4 | 1 | $j_1(\beta) = 1$ | |
| | | 2 | $j_2(\beta) = 1$ | |
| | | 4 | $j_4(\beta) = 0$ | $(1 \cdot 1) + (2 \cdot 1) + (4 \cdot 0) = 3$ |
| $m_5$ | 5 | 1 | $j_1(\beta) = 1$ | |
| | | 5 | $j_5(\beta) = 0$ | $(1 \cdot 1) + (5 \cdot 0) = 1$ |
| $m_6$ | 6 | 1 | $j_1(\beta) = 1$ | |
| | | 2 | $j_2(\beta) = 1$ | |
| | | 3 | $j_3(\beta) = 0$ | |
| | | 6 | $j_6(\beta) = 0$ | $(1 \cdot 1) + (2 \cdot 1) + (3 \cdot 0) + (6 \cdot 0) = 3$ |

this particular member of $B$ is $(1, 3, 1, 3, 1, 3)$. In this way, we can calculate the tuples for all the members of $B$:

- The tuple for the one element $s_1^3$ of $B$ is $(3, 3, 3, 3, 3, 3)$.
- The tuple for the three elements $s_1 s_2$ of $B$ is $(1, 3, 1, 3, 1, 3)$.
- The tuple for the two elements $s_3$ of $B$ is $(0, 0, 3, 0, 0, 3)$.

Substituting each of the above for $s_k$ in $Z(A)$ and summing the results yields

$$(1 \cdot 130) + (3 \cdot 6) + (2 \cdot 4) = 156,$$

which, when divided by the order of $B = 6$, gives 26 equivalence classes that incorporate up to three voices in $\frac{6}{8}$.

This number, however, also includes those equivalence classes that incorporate one and two voices. Therefore, to determine the number of equivalence classes that use exactly three voices, we may repeat the procedure, substituting the symmetric group of degree 2 for $B$, yielding 8, and subtracting that result from 26. Hence, we find precisely 18 classes of three-voiced beat class sets in $\frac{6}{8}$ that are equivalent under metric shift and voice permutation. Figure 3 displays one representative rhythm from each of these equivalence classes (up to metric shift and voice permutation).

## iii. Beat-Class Sets of Varying Cardinalities within a Meter

The previous process considers only multiple-voiced rhythms that feature onsets on every beat in a measure, as in our initial examples in Figure 1. If we wish to consider beat-class sets of variable cardinalities within a meter, we incorporate the polynomial form of the PGET.

*Theorem* 3 ([17], p. 166). The polynomial which enumerates according to weight the equivalence classes of functions in $Y^X$ determined by the power group $B^A$ is

$$N(x) \ = \ \sum_{\beta \in B} Z\left(A;\ m_1\left(\beta,\ x\right),\ m_2\left(\beta, x\right), ..., m_d\left(\beta, x\right)\right), \tag{21}$$

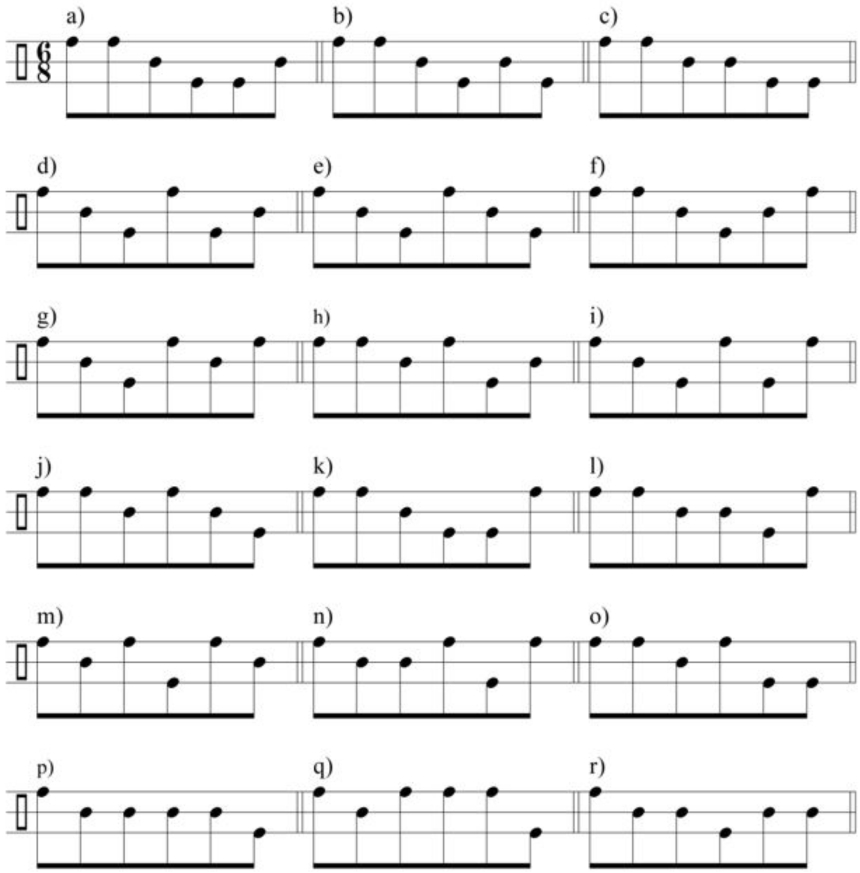


**Figure 3:** *Representatives from each equivalence class of 3-voiced rhythms in* $\frac{6}{8}$.

where

$$m_k\left(\beta,\ x\right) \ = \sum_{t=0}^{r} \left( \sum_{s|k} s j_s \left(\beta_t\right) \right) x^{kt}. \tag{22}$$

In doing so, we assign to each voice a particular weight. We assign the beats that include no onsets to a voice with a weight of $t = 0$—essentially disabling its ability to permute with the other voices—and all other voices to a weight of $t = 1$.

Let us return to our previous example of three-voiced, eighth-note rhythms in $\frac{6}{8}$ under the action of the power group $B^A$, where $A$ is the group of metric shifts and $B$ is the group of voice permutations. We now use Theorem 3 to discover how many equivalence classes exist for measures that incorporate varying numbers of beats with onsets, from 0 to 6. In doing so, we apply the cycle index of $A$ to the tuple $(m_1(\beta, x), m_2(\beta, x), ..., m_6(\beta, x))$ for each $\beta \in B$. Table 3 demonstrates the calculation of $(m_1(\beta, x), m_2(\beta, x), ..., m_6(\beta, x))$ for an arbitrary member $\beta \in B$ with the form $s_1 s_2$. Accordingly,

$$(m_1(\beta, x), m_2(\beta, x), ..., m_6(\beta, x)) = (1 + x, 1 + 3x^2, 1 + x^3, 1 + 3x^4, 1 + x^5, 1 + 3x^6). \tag{23}$$

In this way, we determine the following values for each of the members of $B$.

- The tuple for the one element $s_1^3$ of $B$ is $(1+3x, 1+3x^2, 1+3x^3, 1+3x^4, 1+3x^5, 1+3x^6)$.
- The tuple for the three elements $s_1s_2$ of $B$ is $(1+x, 1+3x^2, 1+x^3, 1+3x^4, 1+x^5, 1+3x^6)$.
- The tuple for the two elements $s_3$ of $B$ is $(1, 1, 1+3x^3, 1, 1, 1+3x^6)$.

Substituting each of these values for $s_k$ in the cycle index of $A$ and summing those results for the remaining members of $B$ yields the following polynomial, where the coefficients of $x^k$ show the number of equivalence classes of measures that incorporate $k$ beats that contain onsets (i.e., sets of cardinality $k$):

$$1 + 1x + 6x^2 + 18x^3 + 38x^4 + 41x^5 + 26x^6. \tag{24}$$

The polynomial indicates that there exists 1 equivalence class with 0 beats that contain an onset (the empty measure), 1 equivalence class with 1 onset, 6 with 2 onsets, 18 with 3 onsets, 38 with 4 onsets, 41 with 5 onsets, and 26 (the same number we found above) with 6 onsets.

The above totals include rhythms with *up to* three voices. If we wish to determine the number of equivalence classes in $\frac{6}{8}$ that include *exactly* three voices, we need to determine the number of equivalence classes with up to two voices (i.e., for $e = 2$) for the sets of varying cardinality and subtract those values from the corresponding values above:

$$\begin{array}{rccccccccccccc} & 1 & + & 1x & + & 6x^2 & + & 18x^3 & + & 38x^4 & + & 41x^5 & + & 26x^6 \\ - & 1 & + & 1x & + & 6x^2 & + & 14x^3 & + & 22x^4 & + & 16x^5 & + & 8x^6 \\ \hline & 0 & + & 0x & + & 0x^2 & + & 4x^3 & + & 16x^4 & + & 25x^5 & + & 18x^6 \end{array} \tag{25}$$

The coefficients in the difference indicate 0 equivalence classes of beat-class sets with 0 onsets, 1 onset, or 2 onsets (as it is not possible to have exactly three voices for sets of cardinalities $0 \le k \le 2$); 2 equivalence classes of sets with 3 onsets; 16 with 4 onsets; 25 with 5 onsets; and 18 with 6 onsets (again, agreeing with our previous result).

### iv. Results for $d \le 6$, $e \le 6$

Table 4 presents the numbers of equivalence classes for all beat-class sets in $\frac{6}{8}$ of cardinalities $k$, $1 \le k \le 6$, that contain exactly $e$ voices, $0 \le e \le 6$. In each case, we determine the polynomials $N(x)$ as above for each value $k$, and subtract from the coefficients in any one polynomial the corresponding coefficients of the polynomial for the value $k-1$. We note that no beat class sets are possible with $e$ voices where $e < k$ (with the exception of $e = 1$, a single voice, wherein we count the trivial case of $k = 0$, the empty measure).

## III. Conclusions

The Power Group Enumeration Theorem permits us to count objects on which exist two group actions. Such situations occur frequently in musical contexts, particularly as events—which may relate to one another by means of group actions on a space, such as transposition or inversion—also relate to one another in time, in which we define other operations. Additionally, it allows us to complete this task merely on the basis of the cycle indices of the respective group actions—information that is readily available. Whereas it requires a considerable amount of computation, it is significantly more efficient than counting these objects by hand, particularly as there exists a combinatorial explosion as the cardinalities of the sets on which the constituent groups act increases.

**Table 3:** *Calculating* $(m_1(\beta, x), m_2(\beta, x), ..., m_6(\beta, x))$ *for* $\beta = s_1 s_2$

| $m_k$ | $t$ | $s$ | $j_s(\beta_t)x^{kt}$ | polynomial for $m_k(\beta, x)$ |
|---|---|---|---|---|
| $m_1$ | 0 | 1 | $1x^{1\cdot 0}$ | $1$ |
| | 1 | 1 | $1x^{1\cdot 1}$ | $+1x^1$ |
| $m_2$ | 0 | 1 | $1x^{2\cdot 0}$ | $1$ |
| | | 2 | $0x^{2\cdot 0}$ | $+0$ |
| | 1 | 1 | $1x^{2\cdot 1}$ | $+1x^2$ |
| | | 2 | $1x^{2\cdot 1}$ | $+2x^2$ |
| $m_3$ | 0 | 1 | $1x^{3\cdot 0}$ | $1$ |
| | | 3 | $0x^{3\cdot 0}$ | $+0$ |
| | 1 | 1 | $1x^{3\cdot 1}$ | $+1x^3$ |
| | | 3 | $0x^{3\cdot 1}$ | $+0x^3$ |
| $m_4$ | 0 | 1 | $1x^{4\cdot 0}$ | $1$ |
| | | 2 | $0x^{4\cdot 0}$ | $+0$ |
| | | 4 | $0x^{4\cdot 0}$ | $+0$ |
| | 1 | 1 | $1x^{4\cdot 1}$ | $+1x^4$ |
| | | 2 | $1x^{4\cdot 1}$ | $+2x^4$ |
| | | 4 | $0x^{4\cdot 1}$ | $+0x^4$ |
| $m_5$ | 0 | 1 | $1x^{5\cdot 0}$ | $1$ |
| | | 5 | $0x^{5\cdot 0}$ | $+0$ |
| | 1 | 1 | $1x^{5\cdot 1}$ | $+1x^5$ |
| | | 3 | $0x^{5\cdot 1}$ | $+0x^5$ |
| $m_6$ | 0 | 1 | $1x^{6\cdot 0}$ | $1$ |
| | | 2 | $0x^{6\cdot 0}$ | $+0$ |
| | | 3 | $0x^{6\cdot 0}$ | $+0$ |
| | | 6 | $0x^{6\cdot 0}$ | $+0$ |
| | 1 | 1 | $1x^{6\cdot 1}$ | $+1x^6$ |
| | | 2 | $1x^{6\cdot 1}$ | $+2x^6$ |
| | | 3 | $0x^{6\cdot 1}$ | $+0x^6$ |
| | | 6 | $0x^{6\cdot 1}$ | $+0x^6$ |

**Table 4:** *Numbers of equivalence classes for beat-class sets in* $\frac{6}{8}$ *of cardinality k with e voices*

| | $k=0$ | $k=1$ | $k=2$ | $k=3$ | $k=4$ | $k=5$ | $k=6$ |
|---|---|---|---|---|---|---|---|
| $e=1$ | 1 | 1 | 3 | 4 | 3 | 1 | 1 |
| $e=2$ | 0 | 0 | 3 | 10 | 19 | 15 | 7 |
| $e=3$ | 0 | 0 | 0 | 4 | 16 | 25 | 18 |
| $e=4$ | 0 | 0 | 0 | 0 | 3 | 10 | 13 |
| $e=5$ | 0 | 0 | 0 | 0 | 0 | 1 | 3 |
| $e=6$ | 0 | 0 | 0 | 0 | 0 | 0 | 1 |

One area for further related work is the enumeration of equivalence classes of multiple-voiced beat class sets in which we wish to keep track of how many beats are articulated in each of the respective voices. It is possible to count the sets themselves using standard combinatorial methods, but I have yet to find an efficient method for counting their equivalence classes.

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# Musical Quasigroups

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***Abstract:*** *A musical quasigroup is a musical groupoid in which all its left and right translation mappings are permutations. Some quasigroups of chords with a left (right, middle) identity element have been investigated. It is noted that the left (right, middle) identity element of a musical quasigroup is often associated with the root note of a musical chord. It is shown that chord inversions can be displayed by quasigroups. Examples of musical sequence of triads are constructed by using quasigroups. It is shown that a twelve-tone matrix can be created by using a quasigroup. Some examples of an n-tone composition chart using quasigroup are constructed. In particular, some charts showing a circle of fourths and fifths have been obtained by musical quasigroup. Some examples of ascending, descending, disjunct, and conjunct motions respectively have been described using quasigroups. Also, some examples of contrary, strict contrary, oblique, similar, and parallel motions have been given using quasigroups. The bass, treble, and grand staves have been described by a quasigroup. Examples on motion of a single melody is given having both conjunct and disjunct motions. Also, an example of an oblique motion in which one of its melodies is static while the other moves into conjunct and disjunct motions is demonstrated by a quasigroup. Some examples of a subquasigroup for pitch classes are constructed and verified with some musical examples. By the concept of a normal subquasigroup, a melodic motion which is disjunct is described to have a sub-melodic motion which is conjunct. It is shown that there are paired melodies which are not in contrary motion to each other but have paired sub-melodies which are in contrary (or strict-contrary) motion.*

***Keywords****: Quasigroup. Subquasigroup. Normal subquasigroup. n-tone composition chart. Chord inversion. Melodic motion.*

## I. Introduction

Mathematics in a number of ways has been applied to music theory in the past centuries. One of such interdisciplinary studies on music was carried out by Morris [10]. The relationship exhibited between mathematical reasoning and musical creativity has gained attention in recent years due to an increasing human interest in both subjects. Ben [4] studied the integral relationship between frequencies of tones, while Lewin [9] applied group theory to music by the concept of transformation theory using music intervals, and Wright [13] discusses on some abstract group relationship of musical structures. There are many research results showing the application of group theory to music, but when quasigroup theory and its relationship in music is

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mentioned, little or no direct result is obtained. From this study, we have seen that some musicians have been applying some concepts of quasigroup in one way or the other in their compositions. Therefore, this study seeks to examine a formal approach to music through quasigroups.

## i. Preliminaries

Let $Q$ be a set, then by binary operation $*$ on $Q$, we mean a mapping

$$* : Q \times Q \longrightarrow Q. \tag{1}$$

Then, the pair $(Q, *)$ is called a groupoid or a Magma. In this paper, we use the terminology adopted in Bruck [5].

Let $(Q, *)$ be a groupoid. If, for all $x, y \in Q,\ x * y = y * x$, then $(Q, *)$ is called a commutative groupoid. A groupoid $(Q, *)$ has an identity element $e \in Q$ if

$$\text{for every } x \in Q,\ x * e = e * x = x. \tag{2}$$

The order of a groupoid $(Q, *)$ denoted by $|Q|$ is the cardinality of $Q$. A groupoid $(Q, *)$ is said to be of finite order (that is, a finite groupoid ) if $|Q|$ is a finite number, otherwise it is called an infinite groupoid.

We write $xy$ instead of $x * y$ and designate that $*$ has a lower priority than juxtaposition among factors to be multiplied. For instance, $x * yz$ represents $x(yz)$. Let $(Q, *)$ be a groupoid, and let $x$ be a fixed element in $(Q, *)$. Then the left and right translation maps of $Q$, $L_x$, $R_x : Q \longrightarrow Q$ are defined respectively by

$$yL_x = x * y \ \text{ and } \ yR_x = y * x, \tag{3}$$

for all $x, y \in Q$. In the above definition, we adopt the definition of left and right translation mappings used in Jaiyeola [8]. Let $(Q, *)$ be a groupoid. An element $a \in Q$ is said to satisfy the left (right) cancellation law if, for all $x, y \in Q,\ x = y$ if and only if

$$a * x = a * y \ (x * a = y * a). \tag{4}$$

In a groupoid $(Q, *)$, the left and the right translation mappings need not be permutations. A groupoid $(Q, *)$ is said to be associative if for all $a, b, c \in Q,\ (ab)c = a(bc)$. An associative groupoid is a semigroup. For more studies on binary operation, integers and groupoids, readers may check [1, 6, 7, 5]. Let $(Q, *)$ be a groupoid. If each of the equations

$$a * x = b \ \text{ and } \ y * a = b \tag{5}$$

has unique solutions in $Q$ for $x$ and $y$ respectively, then $(Q, *)$ is called a *quasigroup*. Thus, a groupoid $(Q, *)$ is a quasigroup if its left and right translation mappings are permutations. An element $u$ is a left identity element of a quasigroup $(Q, *)$ if for all $x \in Q,\ u * x = x$. An element $v$ is a right identity element of a quasigroup $(Q, *)$ if for all $x \in Q,\ x * v = x$. An element $w$ is a middle identity element of a quasigroup $(Q, *)$ if for all $x \in Q,\ x * x = w$. For more on Quasigroup, readers may check [2, 3, 8, 12].

Let $\mathbb{Z}$ be the set of integers and consider $x, y, z \in \mathbb{Z}$. The positive integer $z$ is called the greatest divisor of integers $x$ and $y$ if any divisor of $x$ and $y$ is also a divisor of $z$, and $z$ is a divisor of both $x$ and $y$. An integer $p > 1$ is called a prime if its divisors are $\pm p$ and $\pm 1$ only.

Two integers $x$ and $y$ are said to be relatively prime if their greatest common divisor is 1. As a consequence, the equation

$$1 = ax + by \tag{6}$$

holds for some integers $a, b \in \mathbb{Z}$.

Let $\mathbb{Z}$ be a set of integers and let $m, n \in \mathbb{Z}$. By modulo operation, abbreviated as *mod*, on $\mathbb{Z}$ we mean, $n$ divides $m$ and has a remainder $r \in \mathbb{Z}$, written as $m \ mod \ n = r$. For a fixed $n \in \mathbb{Z}$, the set of all $r$ satisfying $m \ mod \ n = r$ for all $m \in \mathbb{Z}$ is denoted by $\mathbb{Z}_n$. The notation $\mathbb{Z}_n$ is called the set of integers modulo $n$. Also, the product of two elements $g$ and $h$ in $\mathbb{Z}_n$ can be defined as $gh \ (mod \ n)$. On a music scale, modulo operation is a computation which restarts the music notes once a certain value is attained.

We denote the ordered sequence of pitch classes in the angle brackets $\langle \ \rangle$. The function of a musical note can be changed by the use of flats and sharps. When two notes sound the same but are named as different notes, they are said to be enharmonic equivalent to each other. Scales, chords, keys and intervals can as well be enharmonically named.

## II. Main Results

In this section, some concepts and examples of quasigroups are applied to music. Some quasigroups of chords with a left (right, middle) identity element are investigated. It is noted that the left (right, middle) identity element of a finite quasigroup is often associated with the root note of the musical chord. It is obtained that quasigroups with a left (right, middle) identity element provide options for harmonizations and progressions. Using finite quasigroups, triads, tetrads, pentads, and hexads with their inversions can be described respectively. Examples of musical sequence of triads are constructed by quasigroups. It has been shown that an $n$-tone row chart can be obtained by a musical quasigroup. The chart showing a circle of fourths and circle of fifths have been obtained by a musical quasigroup. Some examples describing the descending, ascending, disjunct and conjunct motions respectively have been given. By the motion of a single melody, the structure of a musical staff using quasigroups is investigated. Similarly, some examples of contrary, strict contrary, oblique, similar, and parallel motions have been described using quasigroups. Examples of subquasigroups have been constructed for pitch classes and are verified with some musical examples. Also, the concept of a normal subquasigroup is applied to motion of melodies. It is shown that there are paired melodies which are not in contrary motion to each other but have paired sub-melodies which are in contrary (or strict-contrary) motion.

### i. Musical Quasigroups

In this subsection, some concepts of quasigroups are defined in music, and some examples are given. We demonstrated some orders of progressions involving $I, IV$, and $V$ using quasigroups.

A set is a collection of well-defined objects. An object of a set is called a member or element of the given set. If a set is a collection of well-defined musical objects then these objects are also called musical elements or members of the set. Examples of sets are $F$-major scale and $C$-major chord. Each object in $F$-major scale and $C$-major chord is a musical element and it is also called a note.

**Definition 1.** Let us consider $Q$ a set of musical notes. If the product of any two musical notes of $Q$ under an operation is also a member of $Q$, then such an operation is called a binary operation.

For an example, consider a set of musical notes of $G$-major chord. An operation on $G$-major chord will be called a binary operation if the product of any two members of $G$-major chord is also a member of $G$-major chord.

**Definition 2.** A set $Q$ of musical elements on which a binary operation $*$ is defined is called a musical quasigroup if, for all $u, v \in Q$, $u * v \in Q$ and there exist unique $x, y \in Q$ satisfying $u * x = v$ and $y * u = v$.

We denote the ordered sequence of musical elements in the angle brackets $\langle\ \rangle$. As applied below, if $\{I, IV, V\}$ is a set of pitch classes from a given scale, then $\langle I, IV, V\rangle$ is an ordered sequence from $\{I, IV, V\}$. We also assigned a sequence of pitch classes to musical elements of a given set. As used in Example 1, $\langle I, IV, V\rangle = \langle 0, 1, 2\rangle$ implies that $I, IV$ and $V$ represent $0, 1$ and $2$ respectively, without a change in position.

*Example* 1. Let $Q = \{0, 1, 2\}$ be a set of musical elements. Define the binary operation $*$ on $Q$ as $a * b = 2a + b\ (mod\ 3)$ then $(Q, *)$ is a musical quasigroup with a left identity element (Table 1).

**Table 1:** *A musical quasigroup with a left identity element 0.*

| $*$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 1 | 2 |
| 1 | 2 | 0 | 1 |
| 2 | 1 | 2 | 0 |

Let $Q = \{I, IV, V\}$ be the set of notes of a major scale in a progression, and let $\langle 0, 1, 2\rangle = \langle I, IV, V\rangle$. Then we have (Table 2):

**Table 2:** *A quasigroup of progression with a left identity element I.*

| $*$ | $I$ | $IV$ | $V$ |
|---|---|---|---|
| $I$ | $I$ | $IV$ | $V$ |
| $IV$ | $V$ | $I$ | $IV$ |
| $V$ | $IV$ | $V$ | $I$ |

The order of progression from left to right of the first, second and third rows are $I - IV - V$, $V - I - IV$, and $IV - V - I$ respectively.

*Example* 2. Let $Q = \{0, 1, 2\}$ be a set of musical elements. Define the binary operation $*$ on $Q$ as $a * b = a + 2b\ (mod\ 3)$ then $(Q, *)$ is a musical quasigroup with a right identity element (Table 3).

**Table 3:** *A musical quasigroup with a right identity element 0.*

| $*$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 2 | 1 |
| 1 | 1 | 0 | 2 |
| 2 | 2 | 1 | 0 |

Let $Q = \{I, IV, V\}$ be the set of notes of a major scale in a progression, and let $\langle 0, 1, 2\rangle = \langle I, IV, V\rangle$. Then we have (Table 4):

**Table 4:** *A quasigroup of progression with a right identity element I*

| $*$ | $I$ | $IV$ | $V$ |
|---|---|---|---|
| $I$ | $I$ | $V$ | $IV$ |
| $IV$ | $IV$ | $I$ | $V$ |
| $V$ | $V$ | $IV$ | $I$ |

The order of progression from left to right of the first, second and third rows are $I - V - IV$, $IV - I - V$, and $V - IV - I$ respectively.

*Example* 3. Let $Q = \{0, 1, 2\}$ be a set of musical elements. Define the binary operation $*$ on $Q$ as $a * b = 2a + b + 1 \ (mod\ 3)$ then $(Q, *)$ is a musical quasigroup with a middle identity element (Table 5).

**Table 5:** *A musical quasigroup with a middle identity element 1*

| $*$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 1 | 2 | 0 |
| 1 | 0 | 1 | 2 |
| 2 | 2 | 0 | 1 |

Let $Q = \{I, IV, V\}$ be the set of notes of a major scale in a progression, and let $\langle 0, 1, 2 \rangle = \langle I, IV, V \rangle$. Then we have (Table 6):

**Table 6:** *A quasigroup of progression with a middle identity element IV*

| $*$ | $I$ | $IV$ | $V$ |
|---|---|---|---|
| $I$ | $IV$ | $V$ | $I$ |
| $IV$ | $I$ | $IV$ | $V$ |
| $V$ | $V$ | $I$ | $IV$ |

The order of the progression from left to right of the first, second and third rows are $IV - V - I$, $I - IV - V$, and $V - I - IV$ respectively.

## ii. Quasigroups of triads

In this subsection, we present some triads using a quasigroup of order three. The quasigroup tables obtained show some triad inversions for major triad when viewed in rows or columns. We also present the order in which a minor, diminished and augmented triads and their inversions can be displayed respectively by a quasigroup of order three.

Let $Q = \{I, III, V\}$ be the set of notes of the major scale in a triad. By Example 1, let $\langle I, III, V \rangle$ represents $\langle 0, 1, 2 \rangle$. Then the Table 7 of major triad inversions is obtained.

**Table 7:** *C-Major chord from a Quasigroup with a left identity element*

| $*$ | $C$ | $E$ | $G$ |
|---|---|---|---|
| $C$ | $C$ | $E$ | $G$ |
| $E$ | $G$ | $C$ | $E$ |
| $G$ | $E$ | $G$ | $C$ |

*Remark* 1. The left identity element of a musical quasigroup of triads is assigned the root note of the triad. From Table 1, $(Q, \cdot)$ is a quasigroup with a left identity element and Table 7 is such that the root note occupies the left identity element position whenever $(Q, \cdot)$ has a left identity element. Example, for $C$-Major triad, $C$ is the root note.

Let $Q = \{I, III, V\}$ be the set of notes of the major scale in a triad. By Example 2, let $\langle I, III, V\rangle$ represents $\langle 0, 1, 2\rangle$. Then the Table 8 of major triad inversions is obtained.

**Table 8:** *C-Major chord from a Quasigroup with a right identity element.*

| $*$ | $C$ | $E$ | $G$ |
|---|---|---|---|
| $C$ | $C$ | $G$ | $E$ |
| $E$ | $E$ | $C$ | $G$ |
| $G$ | $G$ | $E$ | $C$ |

*Remark* 2. The right identity element of a musical quasigroup of triads is assigned the root note of the triad. From Table 3, $(Q, \cdot)$ is a quasigroup with a right identity element and Table 8 is such that the root note occupies the right identity element position whenever $(Q, \cdot)$ has a right identity element.

*Remark* 3. Let $Q = \{I, III, V\}$ be the set of notes of the major scale in a triad. Examples 1 and 2 can also be applied for a minor, diminished, and augmented triads as follows:

For a minor triad, let $\langle I, III^\flat, V\rangle$ represents $\langle 0, 1, 2\rangle$.
For a diminished triad, let $\langle I, III^\flat, V^\flat\rangle$ represents $\langle 0, 1, 2\rangle$.
For an augmented triad, let $\langle I, III, V^\sharp\rangle$ represents $\langle 0, 1, 2\rangle$.

*Remark* 4. Quasigroup of triads with a middle identity element is similarly deduced.

## iii. Quasigroup with examples of musical sequences from triads

Let $Q = \{I, III, V\}$ be the set of notes of the major scale in a triad. By Examples 1 and 2, let $\langle I, III, V\rangle$ represents $\langle 0, 1, 2\rangle$. Consider Tables 9, 10, and 11 of major triad inversions:

**Table 9:** *C-Major.*

| $*$ | $C$ | $E$ | $G$ |
|---|---|---|---|
| $C$ | $C$ | $E$ | $G$ |
| $E$ | $G$ | $C$ | $E$ |
| $G$ | $E$ | $G$ | $C$ |

**Table 10:** *F-major.*

| $*$ | $F$ | $A$ | $C$ |
|---|---|---|---|
| $F$ | $F$ | $C$ | $A$ |
| $A$ | $A$ | $F$ | $C$ |
| $C$ | $C$ | $A$ | $F$ |

**Table 11:** *G-Major*

| $*$ | $G$ | $B$ | $D$ |
|---|---|---|---|
| $G$ | $G$ | $D$ | $B$ |
| $B$ | $B$ | $G$ | $D$ |
| $D$ | $D$ | $B$ | $G$ |

In Figure 1, examples (i), (ii), and (iii) are obtained from Tables 7, 10, and 11, by using their 1st row-1st column-3rd column respectively, then the process is repeated in each case. Example (iv) is obtained from Tables 7 by using its 1st row-3rd column-1st row, then the process is repeated.

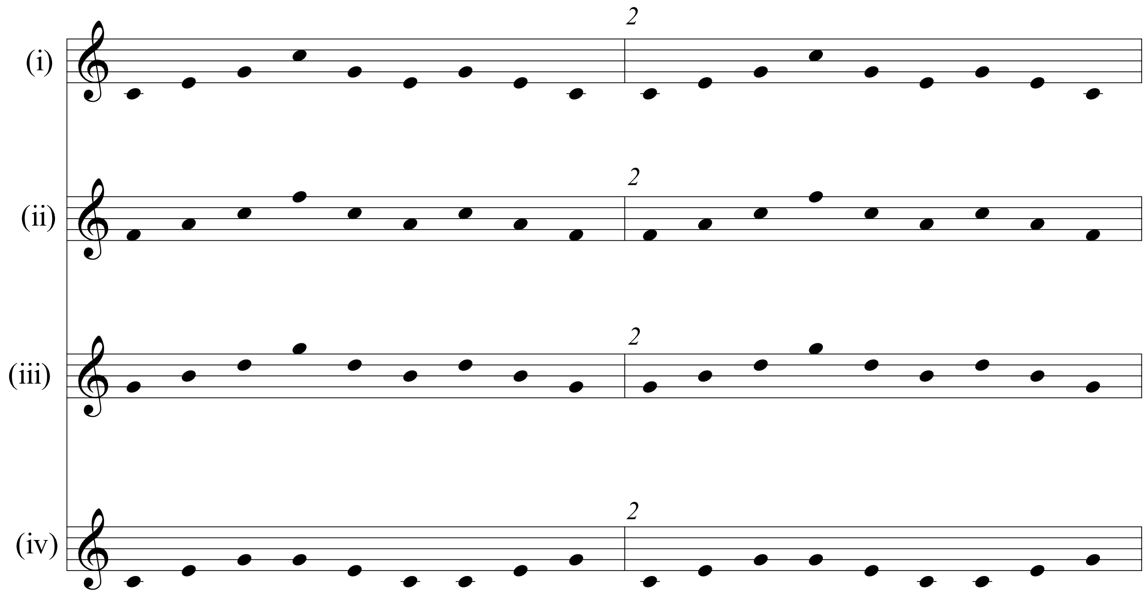


**Figure 1:** *Examples of musical sequences obtained via quasigroups.*

## iv. A Twelve-Tone Matrix

In this subsection, we present the connection between $n$-tone composition chart with quasigroup. It is shown that a twelve-tone matrix can be created by using a quasigroup.

*Example* 4. (Wright [13]) Let $a_n \in \mathbb{Z}_{12}$ and let the ordered pair $(i, j)$ be the position at row $i$ and column $j$, such that the entries of the original row are labelled: $a_1 = [0]; a_2 = [3]; a_3 = [2]; a_4 = [5]; a_5 = [4]; a_6 = [8]; a_7 = [1]; a_8 = [10]; a_9 = [11]; a_{10} = [9]; a_{11} = [7]$; and $a_{12} = [6]$. Then the first column inversion in $\mathbb{Z}_{12}$ are given as : $-a_1 = [0]; -a_2 = [9]; -a_3 = [10]; -a_4 = [7]; -a_5 = [8], -a_6 = [4]$ and each position of the chart with the elements of $\mathbb{Z}_{12}$ corresponding to the appropriate note class is given as $a_j - a_i$. Then a twelve-tone matrix is obtained.

*Remark* 5. Example 4 is a known construction to musicians. It is called a quasigroup if it is presented in the form of a Cayley table. See Table 12 of Example 5 for an example.

*Example* 5. Let $a_n \in \mathbb{Z}_{12}$ and let the ordered pair $(i, j)$ be the position at row $i$ and column $j$. We assign the subscripts $a_0, a_1, a_2, \ldots, a_n$ of the entries thus: $a_0 = [0]; a_1 = [3]; a_2 = [2]; a_3 = [5]; a_4 = [4]; a_5 = [8]; a_6 = [1]; a_7 = [10]; a_8 = [11]; a_9 = [9]; a_{10} = [7]$; and $a_{11} = [6]$. Then

(i) the first column inversion in $\mathbb{Z}_{12}$ are given as :$-a_0 = [0]; -a_1 = [9]; -a_2 = [10]; -a_3 = [7]; -a_4 = [8], -a_5 = [4]; -a_6 = [11]; -a_7 = [2]; -a_8 = [1]; -a_9 = [3]; -a_{10} = [5]; -a_{11} = [6]$; and

(ii) the pair $(\mathbb{Z}_{12}, *)$ such that $*(i, j) = a_j - a_i \in \mathbb{Z}_{12}$ for all $a_i, a_j \in \mathbb{Z}_{12}$ is a quasigroup. The entry in the position $(9, 6)$ is $*(9, 6) = a_6 - a_9 = [1] - [9] = -[8] = [4]$. Therefore, $9 * 6 = 4$.

For more details, see Tables 12 and 13.

**Table 12:** *Multiplication Table for* $(\mathbb{Z}_{12}, *)$ *by Example 5.*

| $*$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 3 | 2 | 5 | 4 | 8 | 1 | 10 | 11 | 9 | 7 | 6 |
| 1 | 9 | 0 | 11 | 2 | 1 | 5 | 10 | 7 | 8 | 6 | 4 | 3 |
| 2 | 10 | 1 | 0 | 3 | 2 | 6 | 11 | 8 | 9 | 7 | 5 | 4 |
| 3 | 7 | 10 | 9 | 0 | 11 | 3 | 8 | 5 | 6 | 4 | 2 | 1 |
| 4 | 8 | 11 | 10 | 1 | 0 | 4 | 9 | 6 | 7 | 5 | 3 | 2 |
| 5 | 4 | 7 | 6 | 9 | 8 | 0 | 5 | 2 | 3 | 1 | 11 | 10 |
| 6 | 11 | 2 | 1 | 4 | 3 | 7 | 0 | 9 | 10 | 8 | 6 | 5 |
| 7 | 2 | 5 | 4 | 7 | 6 | 10 | 3 | 0 | 1 | 11 | 9 | 8 |
| 8 | 1 | 4 | 3 | 6 | 5 | 9 | 2 | 11 | 0 | 10 | 8 | 7 |
| 9 | 3 | 6 | 5 | 8 | 7 | 11 | 4 | 1 | 2 | 0 | 10 | 9 |
| 10 | 5 | 8 | 7 | 10 | 9 | 1 | 6 | 3 | 4 | 2 | 0 | 11 |
| 11 | 6 | 9 | 8 | 11 | 10 | 2 | 7 | 4 | 5 | 3 | 1 | 0 |

**Table 13:** *The chart of pitch classes from key F by Example 5.*

| $*$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | $F$ | $G^\sharp$ | $G$ | $B^\flat$ | $A$ | $C^\sharp$ | $F^\sharp$ | $D^\sharp$ | $E$ | $D$ | $C$ | $B$ |
| 1 | $D$ | $F$ | $E$ | $G$ | $F^\sharp$ | $B^\flat$ | $D^\sharp$ | $C$ | $C^\sharp$ | $B$ | $A$ | $G^\sharp$ |
| 2 | $D^\sharp$ | $F^\sharp$ | $F$ | $G^\sharp$ | $G$ | $B$ | $E$ | $C^\sharp$ | $D$ | $C$ | $B^\flat$ | $A$ |
| 3 | $C$ | $D^\sharp$ | $D$ | $F$ | $E$ | $G^\sharp$ | $C^\sharp$ | $B^\flat$ | $B$ | $A$ | $G$ | $F^\sharp$ |
| 4 | $C^\sharp$ | $E$ | $D^\sharp$ | $F^\sharp$ | $F$ | $A$ | $D$ | $B$ | $C$ | $B^\flat$ | $G^\sharp$ | $G$ |
| 5 | $A$ | $C$ | $B$ | $D$ | $C^\sharp$ | $F$ | $B^\flat$ | $G$ | $G^\sharp$ | $F^\sharp$ | $E$ | $D^\sharp$ |
| 6 | $E$ | $G$ | $F^\sharp$ | $A$ | $G^\sharp$ | $C$ | $F$ | $D$ | $D^\sharp$ | $C^\sharp$ | $B$ | $B^\flat$ |
| 7 | $G$ | $B^\flat$ | $A$ | $C$ | $B$ | $D^\sharp$ | $G^\sharp$ | $F$ | $F^\sharp$ | $E$ | 9 | $C^\sharp$ |
| 8 | $F^\sharp$ | $A$ | $G^\sharp$ | $B$ | $B^\flat$ | $D$ | $G$ | $E$ | $F$ | $D^\sharp$ | $C^\sharp$ | $C$ |
| 9 | $G^\sharp$ | $B$ | $B^\flat$ | $C^\sharp$ | $C$ | $E$ | $A$ | $F^\sharp$ | $G$ | $F$ | $D^\sharp$ | $D$ |
| 10 | $B^\flat$ | $C^\sharp$ | $C$ | $D^\sharp$ | $D$ | $F^\sharp$ | $B$ | $G^\sharp$ | $A$ | $G$ | $F$ | $E$ |
| 11 | $B$ | $D$ | $C^\sharp$ | $E$ | $D^\sharp$ | $G$ | $C$ | $A$ | $B^\flat$ | $G^\sharp$ | $F^\sharp$ | F |

*Remark* 6. Table 12 is not associative. For instance, $4 * (5 * 6) = 4$ and $(4 * 5) * 6 = 9$. Thus, $4 * (5 * 6) \neq (4 * 5) * 6$.

*Remark* 7. Example 5 shows that a twelve-tone matrix can be obtained by using a quasigroup.

*Remark* 8. Wright [13] explored on creating an $n$-tone row chart using modular Arithmetic. From Examples 4 and 5 we note that, given an original row $a_0 = [0], a_1, a_2, \cdots, a_n$ from $\mathbb{Z}_n$ for some $n \in \mathbb{Z}_n^+$, the $n \times n$ chart constructed by taking the ordered pair entry $(i, j)$ such that $*(i, j) = a_j - a_i$ where $a_i, a_j \in \mathbb{Z}_n$, is an n-tone musical quasigroup.

## v. Construction of an $n$-Tone Composition Chart

In this subsection, we construct some examples of an $n$-tone composition chart using quasigroups. We apply Muktibodh [11] in the construction. In particular, some charts showing circles of fourths and fifths have been obtained by musical quasigroups.

**Definition 3.** Let $\mathbb{Z}_n = \{0, 1, 3, \cdots, n-1\}$ for $n \geq 3$, be the set of pitch classes. Define a binary operation $*$ on $\mathbb{Z}_n$ as $a * b = xa + yb(mod\ n)$ where $x, y$ are two distinct elements in $\mathbb{Z}_n \backslash \{0\}$ which are primes such that $(x, y) = 1$, and $n = x + y$ and $+$ the addition of integers under mod $n$. An $n$-tone musical quasigroup $(\mathbb{Z}_n(x, y), *)$ is obtained.

*Example* 6. Let $Q = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$ be the set of pitch classes. Define the binary operation $*$ on $Q$ as $a * b = 7a + 5b(mod\ 12)$ then $(Q, *)$ is a twelve-tone musical quasigroup (see Tables 14 and 15).

**Table 14:** *Multiplication Table for* $(\mathbb{Z}_{12}, *)$ *by Example 6.*

| $*$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 5 | 10 | 3 | 8 | 1 | 6 | 11 | 4 | 9 | 2 | 7 |
| 1 | 7 | 0 | 5 | 10 | 3 | 8 | 1 | 6 | 11 | 4 | 9 | 2 |
| 2 | 2 | 7 | 0 | 5 | 10 | 3 | 8 | 1 | 6 | 11 | 4 | 9 |
| 3 | 9 | 2 | 7 | 0 | 5 | 10 | 3 | 8 | 1 | 6 | 11 | 4 |
| 4 | 4 | 9 | 2 | 7 | 0 | 5 | 10 | 3 | 8 | 1 | 6 | 11 |
| 5 | 11 | 4 | 9 | 2 | 7 | 0 | 5 | 10 | 3 | 8 | 1 | 6 |
| 6 | 6 | 11 | 4 | 9 | 2 | 7 | 0 | 5 | 10 | 3 | 8 | 1 |
| 7 | 1 | 6 | 11 | 4 | 9 | 2 | 7 | 0 | 5 | 10 | 3 | 8 |
| 8 | 8 | 1 | 6 | 11 | 4 | 9 | 2 | 7 | 0 | 5 | 10 | 3 |
| 9 | 3 | 8 | 1 | 6 | 11 | 4 | 9 | 2 | 7 | 0 | 5 | 10 |
| 10 | 10 | 3 | 8 | 1 | 6 | 11 | 4 | 9 | 2 | 7 | 0 | 5 |
| 11 | 5 | 10 | 3 | 8 | 1 | 6 | 11 | 4 | 9 | 2 | 7 | 0 |

**Table 15:** *The chart of pitch classes from key F by Example 6.*

| $*$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | $F$ | $B^\flat$ | $D^\sharp$ | $G^\sharp$ | $C^\sharp$ | $F^\sharp$ | $B$ | $E$ | $A$ | $D$ | $G$ | $C$ |
| 1 | $C$ | $F$ | $B^\flat$ | $D^\sharp$ | $G^\sharp$ | $C^\sharp$ | $F^\sharp$ | $B$ | $E$ | $A$ | $D$ | $G$ |
| 2 | $G$ | $C$ | $F$ | $B^\flat$ | $D^\sharp$ | $G^\sharp$ | $C^\sharp$ | $F^\sharp$ | $B$ | $E$ | $A$ | $D$ |
| 3 | $D$ | $G$ | $C$ | $F$ | $B^\flat$ | $D^\sharp$ | $G^\sharp$ | $C^\sharp$ | $F^\sharp$ | $B$ | $E$ | $A$ |
| 4 | $A$ | $D$ | $G$ | $C$ | $F$ | $B^\flat$ | $D^\sharp$ | $G^\sharp$ | $C^\sharp$ | $F^\sharp$ | $B$ | $E$ |
| 5 | $E$ | $A$ | $D$ | $G$ | $C$ | $F$ | $B^\flat$ | $D^\sharp$ | $G^\sharp$ | $C^\sharp$ | $F^\sharp$ | $B$ |
| 6 | $B$ | $E$ | $A$ | $D$ | $G$ | $C$ | $F$ | $B^\flat$ | $D^\sharp$ | $G^\sharp$ | $C^\sharp$ | $F^\sharp$ |
| 7 | $F^\sharp$ | $B$ | $E$ | $A$ | $D$ | $G$ | $C$ | $F$ | $B^\flat$ | $D^\sharp$ | $G^\sharp$ | $C^\sharp$ |
| 8 | $C^\sharp$ | $F^\sharp$ | $B$ | $E$ | $A$ | $D$ | $G$ | $C$ | $F$ | $B^\flat$ | $D^\sharp$ | $G^\sharp$ |
| 9 | $G^\sharp$ | $C^\sharp$ | $F^\sharp$ | $B$ | $E$ | $A$ | $D$ | $G$ | $C$ | $F$ | $B^\flat$ | $D^\sharp$ |
| 10 | $D^\sharp$ | $G^\sharp$ | $C^\sharp$ | $F^\sharp$ | $B$ | $E$ | $A$ | $D$ | $G$ | $C$ | $F$ | $B^\flat$ |
| 11 | $B^\flat$ | $D$ | $G^\sharp$ | $C^\sharp$ | $F^\sharp$ | $B$ | $E$ | $A$ | $D$ | $G$ | $C$ | $F$ |

*Remark* 9. Table 15 above displays a circle of fourths' progression when it is read from left to right horizontally, and displays a circle of fifths' progression, when it is read from right to left horizontally.

**Definition 4.** Let $\mathbb{Z}_n = \{0, 1, 3, \cdots, n-1\}$ for $n \geq 3$, be the set of pitch classes and let $p$ be a prime number such that $\mathbb{Z}_p(x, y)$ is a groupoid and $x + y = p,\ (x, y) = 1$. A $p$-tone musical quasigroup $\mathbb{Z}_p(x, y)$ is obtained.

*Example* 7. Let $Q = \{0, 1, 2, 3, 4, 5, 6\}$ be the set of pitch classes. Define the binary operation $*$ on $Q$ as $a * b = 2a + 5b (mod\ 7)$ then $(Q, *)$ is a seven-tone musical quasigroup (see Tables 16 and 17).

**Table 16:** *Multiplication Table for $(Q, *)$ by Example 7.*

| $*$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 5 | 3 | 1 | 6 | 4 | 2 |
| 1 | 2 | 0 | 5 | 3 | 1 | 6 | 4 |
| 2 | 4 | 2 | 0 | 5 | 3 | 1 | 6 |
| 3 | 6 | 4 | 2 | 0 | 5 | 3 | 1 |
| 4 | 1 | 6 | 4 | 2 | 0 | 5 | 3 |
| 5 | 3 | 1 | 6 | 4 | 2 | 0 | 5 |
| 6 | 5 | 3 | 1 | 6 | 4 | 2 | 0 |

**Table 17:** *The chart of pitch classes from key F by Example 7.*

| $*$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | $F$ | $B^\flat$ | $G^\sharp$ | $F^\sharp$ | $B$ | $A$ | $G$ |
| 1 | $G$ | $F$ | $B^\flat$ | $G^\sharp$ | $F^\sharp$ | $B$ | $A$ |
| 2 | $A$ | $G$ | $F$ | $B^\flat$ | $G^\sharp$ | $F^\sharp$ | $B$ |
| 3 | $B$ | $A$ | $G$ | $F$ | $B^\flat$ | $G^\sharp$ | $F^\sharp$ |
| 4 | $F^\sharp$ | $B$ | $A$ | $G$ | $F$ | $B^\flat$ | $G^\sharp$ |
| 5 | $G^\sharp$ | $F^\sharp$ | $B$ | $A$ | $G$ | $F$ | $B^\flat$ |
| 6 | $B^\flat$ | $G^\sharp$ | $F^\sharp$ | $B$ | $A$ | $G$ | $F$ |

**Definition 5.** Let $\mathbb{Z}_n = \{0, 1, 3, \cdots, n-1\}$ for $n \geq 3$, be the set of pitch classes. Define a binary operation $*$ on $\mathbb{Z}_n$ as $a * b = xa + yb (mod\ n)$ where $x, y$ are elements in $\mathbb{Z}_n \backslash \{0\}$ and $x = y$. For a fixed prime $n$ and varying $x$ and $y$, an $n$-tone musical quasigroup $(\mathbb{Z}_n, *)$ is obtained.

*Example* 8. Let $Q = \{0, 1, 2, 3, 4, 5, 6\}$ be the set of pitch classes. Define the binary operation $*$ on $Q$ as $a * b = 2a + 2b (mod\ 7)$ then $(Q, *)$ is a seven-tone musical quasigroup (see Tables 18 and 19).

**Table 18:** *Multiplication Table for $(Q, *)$ by Example 8.*

| $*$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 2 | 4 | 6 | 1 | 3 | 5 |
| 1 | 2 | 4 | 6 | 1 | 3 | 5 | 0 |
| 2 | 4 | 6 | 1 | 3 | 5 | 0 | 2 |
| 3 | 6 | 1 | 3 | 5 | 0 | 2 | 4 |
| 4 | 1 | 3 | 5 | 0 | 2 | 4 | 6 |
| 5 | 3 | 5 | 0 | 2 | 4 | 6 | 1 |
| 6 | 5 | 0 | 2 | 4 | 6 | 1 | 3 |

**Table 19:** *The chart of pitch classes from key F by Example 8.*

| $*$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | $F$ | $G$ | $A$ | $B$ | $F^\sharp$ | $G^\sharp$ | $B^\flat$ |
| 1 | $G$ | $A$ | $B$ | $F^\sharp$ | $G^\sharp$ | $B^\flat$ | $F$ |
| 2 | $A$ | $B$ | $F^\sharp$ | $G^\sharp$ | $B^\flat$ | $F$ | $G$ |
| 3 | $B$ | $F^\sharp$ | $G^\sharp$ | $B^\flat$ | $F$ | $G$ | $A$ |
| 4 | $F^\sharp$ | $G^\sharp$ | $B^\flat$ | $F$ | $G$ | $A$ | $B$ |
| 5 | $G^\sharp$ | $B^\flat$ | $F$ | $G$ | $A$ | $B$ | $F^\sharp$ |
| 6 | $B^\flat$ | $F$ | $G$ | $A$ | $B$ | $F^\sharp$ | $G^\sharp$ |

**Definition 6.** Let $\mathbb{Z}_n = \{0, 1, 3, \cdots, n-1\}$ for $n \geq 3$, be the set of pitch classes. Define a binary operation $*$ on $\mathbb{Z}_n$ as $a * b = xa + yb(mod\ n)$ where $x, y$ are elements in $\mathbb{Z}_n \backslash \{0\}$ and $x = 1$ and $y = n - 1$. For a fixed integer $n$ and varying $x$ and $y$, an $n$-tone musical quasigroup $(\mathbb{Z}_n, *)$ is obtained.

*Example* 9. Let $Q = \{0, 1, 2, 3, 4, 5, 6\}$ be the set of pitch classes. Define the binary operation $*$ on $Q$ as $a * b = a + 6b(mod\ 7)$ then $(Q, *)$ is a seven-tone musical quasigroup (see Tables 20 and 21).

**Table 20:** *Multiplication Table for $(Q, *)$ by Example 9.*

| $*$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 6 | 5 | 4 | 3 | 2 | 1 |
| 1 | 1 | 0 | 6 | 5 | 4 | 3 | 2 |
| 2 | 2 | 1 | 0 | 6 | 5 | 4 | 3 |
| 3 | 3 | 2 | 1 | 0 | 6 | 5 | 4 |
| 4 | 4 | 3 | 2 | 1 | 0 | 6 | 5 |
| 5 | 5 | 4 | 3 | 2 | 1 | 0 | 6 |
| 6 | 6 | 5 | 4 | 3 | 2 | 1 | 0 |

**Table 21:** *The chart of pitch classes from key F by Example 9.*

| $*$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | $F$ | $B$ | $B^\flat$ | $A$ | $G^\sharp$ | $G$ | $F^\sharp$ |
| 1 | $F^\sharp$ | $F$ | $B$ | $B^\flat$ | $A$ | $G^\sharp$ | $G$ |
| 2 | $G$ | $F^\sharp$ | $F$ | $B$ | $B^\flat$ | $A$ | $G^\sharp$ |
| 3 | $G^\sharp$ | $G$ | $F^\sharp$ | $F$ | $B$ | $B^\flat$ | $A$ |
| 4 | $A$ | $G^\sharp$ | $G$ | $F^\sharp$ | $F$ | $B$ | $B^\flat$ |
| 5 | $B^\flat$ | $A$ | $G^\sharp$ | $G$ | $F^\sharp$ | $F$ | $B$ |
| 6 | $B$ | $B^\flat$ | $A$ | $G^\sharp$ | $G$ | $F^\sharp$ | $F$ |

**Definition 7.** Let $\mathbb{Z}_n = \{0, 1, 3, \cdots, n-1\}$ for $n \geq 3$, be the set of pitch classes. Define a binary operation $*$ on $\mathbb{Z}_n$ as $a * b = xa + yb(mod\ n)$ where $x, y$ are elements in $\mathbb{Z}_n \backslash \{0\}$ and $(x, y) = 1,\ x + y = n$ and $|x - y|$ is a minimum. For a fixed integer $n$ and varying $x$ and $y$, an $n$-tone musical quasigroup $(\mathbb{Z}_n, *)$ is obtained.

*Example* 10. Let $Q = \{0,1,2,3,4,5,6,7\}$ be the set of pitch classes. Define the binary operation $*$ on $Q$ as $a * b = 5a + 3b(mod\ 8)$ then $(Q,*)$ is an eight-tone musical quasigroup (see Tables 22 and 23).

**Table 22:** *Multiplication Table for* $(Q,*)$ *by Example 10.*

| $*$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 3 | 6 | 1 | 4 | 7 | 2 | 5 |
| 1 | 5 | 0 | 3 | 6 | 1 | 4 | 7 | 2 |
| 2 | 2 | 5 | 0 | 3 | 6 | 1 | 4 | 7 |
| 3 | 7 | 2 | 5 | 0 | 3 | 6 | 1 | 4 |
| 4 | 4 | 7 | 2 | 5 | 0 | 3 | 6 | 1 |
| 5 | 1 | 4 | 7 | 2 | 5 | 0 | 3 | 6 |
| 6 | 6 | 1 | 4 | 7 | 2 | 5 | 0 | 3 |
| 7 | 3 | 6 | 1 | 4 | 7 | 2 | 5 | 0 |

**Table 23:** *The chart of pitch classes from key F by Example 10.*

| $*$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 0 | $F$ | $G^\sharp$ | $B$ | $F^\sharp$ | $A$ | $C$ | $G$ | $B^\flat$ |
| 1 | $B^\flat$ | $F$ | $G^\sharp$ | $B$ | $F^\sharp$ | $A$ | $C$ | $G$ |
| 2 | $G$ | $B^\flat$ | $F$ | $G^\sharp$ | $B$ | $F^\sharp$ | $A$ | $C$ |
| 3 | $C$ | $G$ | $B^\flat$ | $F$ | $G^\sharp$ | $B$ | $F^\sharp$ | $A$ |
| 4 | $A$ | $C$ | $G$ | $B^\flat$ | $F$ | $G^\sharp$ | $B$ | $F^\sharp$ |
| 5 | $F^\sharp$ | $A$ | $C$ | $G$ | $B^\flat$ | $F$ | $G^\sharp$ | $B$ |
| 6 | $B$ | $F^\sharp$ | $A$ | $C$ | $G$ | $B^\flat$ | $F$ | $G^\sharp$ |
| 7 | $G^\sharp$ | $B$ | $F^\sharp$ | $A$ | $C$ | $G$ | $B^\flat$ | $F$ |

## vi. Motion of a single melody by Quasigroups

In this subsection, some motions obtained from three consecutive semitones in a chromatic scale are described by a qausigroup of order three. Some motions of a five-tone sequence are demonstrated by a quasigroup of order five. It is shown that each row and column of a finite musical quasigroup is a melodic motion on its own right. These examples of quasigroups of musical sequences considered, demonstrated the descending, ascending, disjunct, and conjunct melodic motions. We describe the order of the notes on the bass, treble, and grand staves using a quasigroup.

Melodic motions between three consecutive semitones in a chromatic scale are demonstrated by quasigroup table of order three as follows: We consider $C, C^\sharp, D$; $A, A^\sharp, B$ and $B^\flat, B, C$ as examples. By Example 2, let $\langle 0,1,2\rangle = \langle C, C^\sharp, D\rangle$, $\langle 0,1,2\rangle = \langle A, A^\sharp, B\rangle$ and $\langle 0,1,2\rangle = \langle B^\flat, B, C\rangle$ respectively. Then we have Tables 24, 25, and 26:

**Table 24:** *From Key C*

| $*$ | $C$ | $C^\sharp$ | $D$ |
|---|---|---|---|
| $C$ | $C$ | $D$ | $C^\sharp$ |
| $C^\sharp$ | $C^\sharp$ | $C$ | $D$ |
| $D$ | $D$ | $C^\sharp$ | $C$ |

**Table 25:** *From Key* $B^\flat$

| $*$ | $B^\flat$ | $B$ | $C$ |
|---|---|---|---|
| $B^\flat$ | $B^\flat$ | $C$ | $B$ |
| $B$ | $B$ | $B^\flat$ | $C$ |
| $C$ | $C$ | $B$ | $B^\flat$ |

**Table 26:** *From Key A*

| $*$ | $A$ | $A^\sharp$ | $B$ |
|---|---|---|---|
| $A$ | $A$ | $B$ | $A^\sharp$ |
| $A^\sharp$ | $A^\sharp$ | $A$ | $B$ |
| $B$ | $B$ | $A^\sharp$ | $A$ |

From the these tables we note the followings:

(i) The first column of each of these tables displays an ascending motion when read from the top to the bottom; and shows a descending motion when read from the bottom to the top.

(ii) Each of the columns and rows from these tables displays a conjunct motion.

*Example* 11. Let $Q = \{0, 1, 2, 3, 4\}$ be the set of pitch classes. Define the binary operation $*$ on $Q$ as $a * b = 2a + 2b(mod\ 5)$ then $(Q, *)$ is a five-tone musical quasigroup (see Table 27).

**Table 27:** *A quasigroup table by Example 11*

| $*$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 2 | 4 | 1 | 3 |
| 1 | 2 | 4 | 1 | 3 | 0 |
| 2 | 4 | 1 | 3 | 0 | 2 |
| 3 | 1 | 3 | 0 | 2 | 4 |
| 4 | 3 | 0 | 2 | 4 | 1 |

Let $F, G, A, B^\flat, C$ be musical notes from key F and let $\langle 0, 1, 2, 3, 4\rangle = \langle F, G, A, B^\flat, C\rangle$. Then by Table 27, we have Table 28:

**Table 28:** *The chart of pitch-classes from key F by Table 27*

| $*$ | $F$ | $G$ | $A$ | $B^\flat$ | $C$ |
|---|---|---|---|---|---|
| $F$ | $F$ | $A$ | $C$ | $G$ | $B^\flat$ |
| $G$ | $A$ | $C$ | $G$ | $B^\flat$ | $F$ |
| $A$ | $C$ | $G$ | $B^\flat$ | $F$ | $A$ |
| $B^\flat$ | $G$ | $B^\flat$ | $F$ | $A$ | $C$ |
| $C$ | $B^\flat$ | $F$ | $A$ | $C$ | $G$ |

From Table 28, we note the followings:

(i) The main diagonal of the table shows a descending motion when read from the top to the bottom; and shows an ascending motion when read from the bottom to the top.

(ii) The table shows a disjunct melodic motion for each row and column.

*Example* 12. Let $Q = \{0, 1, 2, 3, 4\}$ be the set of musical notes in a five-tone sequence. Let $*$ be a binary operation defined on $Q$ as in Table 29.

**Table 29:** *A five-tone quasigroup with a left identity element* 0

| $*$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 |
| 1 | 4 | 0 | 1 | 2 | 3 |
| 2 | 3 | 4 | 0 | 1 | 2 |
| 3 | 2 | 3 | 4 | 0 | 1 |
| 4 | 1 | 2 | 3 | 4 | 0 |

Then $(Q, *)$ is a quasigroup with a left identity element. Clearly, $(Q, *)$ is not a commutative groupoid and it is not a group. Let $F, G, A, B^\flat, C$ be musical notes from key F and let $\langle 0,1,2,3,4\rangle = \langle F, G, A, B^\flat, C\rangle$. Then by Table 29, we have Table 30:

**Table 30:** *A five-tone chart by Table 29 from key F*

| $*$ | $F$ | $G$ | $A$ | $B^\flat$ | $C$ |
|---|---|---|---|---|---|
| $F$ | $F$ | $G$ | $A$ | $B^\flat$ | $C$ |
| $G$ | $C$ | $F$ | $G$ | $A$ | $B^\flat$ |
| $A$ | $B^\flat$ | $C$ | $F$ | $G$ | $A$ |
| $B^\flat$ | $A$ | $B^\flat$ | $C$ | $F$ | $G$ |
| $C$ | $G$ | $A$ | $B^\flat$ | $C$ | $F$ |

From Table 30, we note the followings:

(i) Each row and column of this table is a melodic motion on its own right.

(ii) The first row displays an ascending melodic motion when read from left to right; and displays a descending melodic motion when read from right to left.

(iii) The first row and the last column of this table display a conjunct motion respectively.

(iv) The third row and the second column of this table display both conjunct and disjunct motions respectively.

*Example* 13. Let $Q = \{0,1,2,3,4,5,6\}$ be the set of notes in a diatonic scale with 0 as the root note. From $C$ major scale, let $\langle C, D, E, F, G, A, B\rangle = \langle 0,1,2,3,4,5,6\rangle$. Then by Table 18, we have Table 31:

**Table 31:** *Musical staves by Quasigroup*

| $*$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | $C$ | $E$ | $G$ | $B$ | $D$ | $F$ | $A$ |
| 1 | $E$ | $G$ | $B$ | $D$ | $F$ | $A$ | $C$ |
| 2 | $G$ | $B$ | $D$ | $F$ | $A$ | $C$ | $E$ |
| 3 | $B$ | $D$ | $F$ | $A$ | $C$ | $E$ | $G$ |
| 4 | $D$ | $F$ | $A$ | $C$ | $E$ | $G$ | $B$ |
| 5 | $F$ | $A$ | $C$ | $E$ | $G$ | $B$ | $D$ |
| 6 | $A$ | $C$ | $E$ | $G$ | $B$ | $D$ | $F$ |

From Table 31, we note the followings:

(i) The first row from left to right shows the order of musical notes on the lines of the treble staff from the middle $C$ upward to the first ledger line above the staff occupied by $A$.

(ii) The second row from right to left shows the order of musical notes on the lines of the bass staff from the middle $C$ downward to the first ledger line below the staff occupied by $E$.

(iii) The second row (third row) from left to right shows the order of notes on the treble (bass) staff starting from the first line of the staff upward.

(iv) The sixth row (seventh row) from left to right shows the order of notes on the spaces of the treble (bass) staff starting from the first space of the staff upward.

(v) To accommodate more extra lower or higher notes, two or more rows (columns) are considered in the same direction provided the next row (column) begins with the note that ended the immediate row (column). For instance, the first and last rows from left to right of this table extend the lines for treble staff from the first row, by adding six ledger lines upward, if the two rows are joined at $A$.

(vi) From (i), (ii), (iii) and (v) above, this table describes the order of notes on a grand staff.

## vii. Motion of two melodies by Quasigroups

In this subsection, some motions obtained by two melodies from some finite quasigroups are considered. We described some motions obtained from a diatonic scale by a quasigroup of order seven. Examples of two melodies with a contrary, parallel, and oblique motions respectively are shown by a quasigroup of order seven. By a quasigroup of order four, an example of a similar motion is given.

Consider the musical notes from F-major scale and let $\langle 0,1,2,3,4,5,6\rangle = \langle F, G, A, B^\flat, C, D, E\rangle$. Then by Table 20, we have Table 32:

**Table 32:** *The chart of pitch classes from key F by Example 9.*

| $*$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | $F$ | $E$ | $D$ | $C$ | $B^\flat$ | $A$ | $G$ |
| 1 | $G$ | $F$ | $E$ | $D$ | $C$ | $B^\flat$ | $A$ |
| 2 | $A$ | $G$ | $F$ | $E$ | $D$ | $C$ | $B^\flat$ |
| 3 | $B^\flat$ | $A$ | $G$ | $F$ | $E$ | $D$ | $C$ |
| 4 | $C$ | $B^\flat$ | $A$ | $G$ | $F$ | $E$ | $D$ |
| 5 | $D$ | $C$ | $B^\flat$ | $A$ | $G$ | $F$ | $E$ |
| 6 | $E$ | $D$ | $C$ | $B^\flat$ | $A$ | $G$ | $F$ |

From Table 32, we note the followings:

(i) The first row from the left is in a descending motion and the first column from the top is in an ascending motion. Thus, these melodies departing from index $(1,1)$ into row and column are in a contrary motion.

(ii) The melodic motion described by first row (column) and last row (column) is a contrary motion.

(iii) Any two melodies departing from index $(i,i)$ into row and column of this table form a contrary motion.

(iv) Any two melodies departing from index $(i,i)$ into the diagonal and row, or, the diagonal and column respectively, form an oblique motion. Clearly, the diagonal is occupied by $F$ while the rows descend from left to right, and columns ascend from the top to the bottom.

Consider the notes of the F-major scale and let $\langle 0,1,2,3,4,5,6\rangle = \langle F,G,A,B^\flat,C,D,E\rangle$. Then by Table 18, we have Table 33:

**Table 33:** *The chart of pitch classes from key F by Example 8.*

| $*$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | $F$ | $A$ | $C$ | $E$ | $G$ | $B^\flat$ | $D$ |
| 1 | $A$ | $C$ | $E$ | $G$ | $B^\flat$ | $D$ | $F$ |
| 2 | $C$ | $E$ | $G$ | $B^\flat$ | $D$ | $F$ | $A$ |
| 3 | $E$ | $G$ | $B^\flat$ | $D$ | $F$ | $A$ | $C$ |
| 4 | $G$ | $B^\flat$ | $D$ | $F$ | $A$ | $C$ | $E$ |
| 5 | $B^\flat$ | $D$ | $F$ | $A$ | $C$ | $E$ | $G$ |
| 6 | $D$ | $F$ | $A$ | $C$ | $E$ | $G$ | $B^\flat$ |

From Table 33, we note the followings:

(i) The first row (column) contains both the ascending and descending motions. Therefore, both conjunct and disjoint motions occur in the first row (column).

(ii) The first column and row departing from index $(1,1)$ are both ascending with the same interval. Thus, these two melodies form a parallel motion.

(iii) Any two melodies departing from index $(i,i)$ into its row and column of this table respectively form a parallel motion.

*Example* 14. Let $Q = \{0,1,2,3\}$ be the set of four consecutive notes from a diatonic scale with 0 as the root note. Let $*$ be a binary operation defined on $Q$ as in Table 34.

**Table 34:** *A musical quasigroup with a right identity element* 0.

| $*$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 2 | 3 | 1 |
| 1 | 1 | 0 | 2 | 3 |
| 2 | 2 | 3 | 1 | 0 |
| 3 | 3 | 1 | 0 | 2 |

Let $F,G,A,B^\flat$ be the four consecutive notes from F-major scale and let $\langle 0,1,2,3\rangle = \langle F,G,A,B^\flat\rangle$. Then by Table 34, we have the below table:

**Table 35:** *A musical quasigroup chart from key F*

| $*$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | $F$ | $A$ | $B^\flat$ | $G$ |
| 1 | $G$ | $F$ | $A$ | $B^\flat$ |
| 2 | $A$ | $B^\flat$ | $G$ | $F$ |
| 3 | $B^\flat$ | $G$ | $F$ | $A$ |

From Table 35, we note that at index $(1,1)$, the row and column depart into ascending motions with different intervals, but the first row finished with a descending motion. This implies that, the row and column melodies departed from index $(1,1)$ initially formed a similar motion with each other, but finished with a contrary motion.

## viii. Motion between melodies in chords by Quasigroups

In this subsection, some motions in chords obtained by melodies from a finite quasigroup are considered. We considered quasigroup of terads, pentads and hexads with an example for each of them. It is noted that a musical chord of $n$ notes and its inversions can be viewed by a quasigroup of order $n$. Quasigroup of triads discussed in Section 2.2 shows ascending motion in the first row by a quasigroup with a left identity element and descending motion in the first column by a quasigroup with a right identity element. We note that a melody from any row (column) from a quasigroup of triads is in a disjunct motion, see Section 2.2. These results obtained from quasigroups of triads are also satisfied by some chords in quasigroup of hexads. Here, we give examples of quasigroups of chords whose row (column) contains both conjunct and disjunct motions. By considering some melodic pairs from a row and column of a quasigroup of chords, we obtained examples of contrary, parallel, oblique and similar motions respectively. An example of an oblique motion which one of its melodies is static while the other moves into disjunct and conjunct motions is given.

### Motion between melodies by a Quasigroup of Tetrads

A musical chord with four notes and its inversions can be viewed by a quasigroup of order four.

*Example* 15. Let $Q = \{0,1,2,3\}$ be the set of pitch classes. Define the binary operation $*$ on $Q$ as $a * b = a + 3b (mod\ 4)$ then $(Q, *)$ is a four-tone musical quasigroup (see Table 36).

**Table 36:** *A musical quasigroup with a right identity element* 0.

| $*$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 3 | 2 | 1 |
| 1 | 1 | 0 | 3 | 2 |
| 2 | 2 | 1 | 0 | 3 |
| 3 | 3 | 2 | 1 | 0 |

Let $C, E, G, A$ be the four consecutive notes from C-Major 6th chord and let $\langle 0,1,2,3 \rangle = \langle C, E, G, A \rangle$. Then by Table 36, we have Table 37:

**Table 37:** *C-Major 6th chord*

| $*$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | $C$ | $A$ | $G$ | $E$ |
| 1 | $E$ | $C$ | $A$ | $G$ |
| 2 | $G$ | $E$ | $C$ | $A$ |
| 3 | $A$ | $G$ | $E$ | $C$ |

From Table 37, we note the followings:

(i) The first row displays a descending melodic motion when read from left to right; and the first column displays an ascending melodic motion when read from top to bottom. Thus, the two melodies departing from index $(1,1)$ into the first row and column respectively are in contrary motion.

(ii) The first row and the last row of this table display a similar motion when read from left to right.

(iii) Some rows from left to right of this table display both conjunct and disjunct motions.

(iv) Using the main diagonal and the first row starting from the left, an oblique motion is obtained; such that, one melody is static while the other melody moves into conjunct and disjunct motions.

## Motion between melodies by a Quasigroup of Pentads

A musical chord with five notes and its inversions can be viewed by a quasigroup of order five.

*Example* 16. Let $C, E, G, B, D$ be the five consecutive notes from C-Major 7th with 9 chord and let $\langle 0,1,2,3,4\rangle = \langle C, E, G, B, D\rangle$. Then by Table 27, we have Table 38:

**Table 38:** *C-Major 7th with 9 chord*

| $*$ | $C$ | $E$ | $G$ | $B$ | $D$ |
|---|---|---|---|---|---|
| $C$ | $C$ | $G$ | $D$ | $E$ | $B$ |
| $E$ | $G$ | $D$ | $E$ | $B$ | $C$ |
| $G$ | $D$ | $E$ | $B$ | $C$ | $G$ |
| $B$ | $E$ | $B$ | $C$ | $G$ | $D$ |
| $D$ | $B$ | $C$ | $G$ | $D$ | $E$ |

From Table 38, we note the followings:

(i) Parallel motion is obtained, for each pair of melodies departing from index $(i,i)$ into a row and column respectively.

(ii) The rows from left to right of this table display both conjunct and disjunct motions.

## Motion between melodies by a Quasigroup of Hexads

A musical chord with six notes and its inversions can be viewed by a quasigroup of order six.

*Example* 17. Let $Q = \{0, 1, 2, 3, 4, 5\}$ be the set of pitch classes. Define the binary operation $*$ on $Q$ as $a * b = 5a + b(mod\ 6)$ then $(Q, *)$ is a six-tone musical quasigroup (see Table 39).

**Table 39:** *Multiplication Table for* $(Q, *)$ *by Example 17.*

| $*$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 | 5 |
| 1 | 5 | 0 | 1 | 2 | 3 | 4 |
| 2 | 4 | 5 | 0 | 1 | 2 | 3 |
| 3 | 3 | 4 | 5 | 0 | 1 | 2 |
| 4 | 2 | 3 | 4 | 5 | 0 | 1 |
| 5 | 1 | 2 | 3 | 4 | 5 | 0 |

Let $C, E, G, B^{\flat}, D, F^{\sharp}$ be the six consecutive notes from C-Augmented 11th chord and let $\langle 0, 1, 2, 3, 4, 5\rangle = \langle C, E, G, B^{\flat}, D, F^{\sharp}\rangle$. Then by Table 39, we have Table 40:

**Table 40:** *C-Augmented 11th chord.*

| $*$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | $C$ | $E$ | $G$ | $B^{\flat}$ | $D$ | $F^{\sharp}$ |
| 1 | $F^{\sharp}$ | $C$ | $E$ | $G$ | $B^{\flat}$ | $D$ |
| 2 | $D$ | $F^{\sharp}$ | $C$ | $E$ | $G$ | $B^{\flat}$ |
| 3 | $B^{\flat}$ | $D$ | $F^{\sharp}$ | $C$ | $E$ | $G$ |
| 4 | $G$ | $B^{\flat}$ | $D$ | $F^{\sharp}$ | $C$ | $E$ |
| 5 | $E$ | $G$ | $B^{\flat}$ | $D$ | $F^{\sharp}$ | $C$ |

From Table 40, we note the followings:

(i) Each row and column has a disjunct motion.

(ii) Any two melodies of this table departing from index $(i, i)$ into its diagonal and row, or its diagonal and column respectively, form an oblique motion.

## ix. Musical Subquasigroup and Normal Subquasigroup

In this subsection, we define a musical subquasigroup and normal subquasigroup. We construct some examples of subquasigroup and normal subquasigroup for pitch classes, and applied them to music. It is shown that a melodic motion which is not descending (nor ascending) may contain a sub-melodic motion which is descending (or ascending). A melodic motion which is disjunct is shown to have a sub-melodic motion which is conjunct. Also, it is shown that a melodic motion may contain a sub-melodic motion whose distance between its consecutive notes are shorter than that of the melodic motion. It is shown that there are paired melodies which are not in contrary motion to each other but have paired sub-melodies which are in contrary (or strict-contrary) motion.

**Definition 8.** Let $(Q, *)$ be a musical quasigroup. A proper subset $W$ of $Q$ is said to be a musical subquasigroup of $Q$ if $W$ is a musical quasigroup on its own right under the binary operation $*$.

**Definition 9.** Let $n$ be an even integer and let $\mathbb{Z}_n = \{0, 1, 2, 3, \cdots, n-1\}$ be the set of pitch classes for $n \geq 4$. Define a binary operation $*$ on $\mathbb{Z}_n$ as $a * b = xa + yb (mod\ n)$ where $x, y$ are two distinct elements in $\mathbb{Z}_n \backslash \{0\}$ which are primes such that $(x, y) = 1,\ x + y = n$. An $n$-tone musical quasigroup $(\mathbb{Z}_n, *)$ is obtained. Let $\mathbb{Z}'_n$ be the set of all even numbers in $\mathbb{Z}_n$, we obtained a musical subquasigroup $(\mathbb{Z}'_n, *)$ of $(\mathbb{Z}_n, *)$.

*Example* 18. Let $Q = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$ be the set of pitch classes. Define the binary operation $*$ on $Q$ as $a * b = 5a + 7b (mod\ 12)$ then $(Q, *)$ is a twelve-tone musical quasigroup and the set $Q' = \{0, 2, 4, 6, 8, 10\}$ forms a musical subquasigroup under the binary operation $*$. We note that, the subquasigroup $(Q', *)$ is normal (see Tables 41, 42, 43, and 44).

**Table 41:** *Multiplication Table for $(Q, *)$ by Example 18.*

| $*$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 7 | 2 | 9 | 4 | 11 | 6 | 1 | 8 | 3 | 10 | 5 |
| 1 | 5 | 0 | 7 | 2 | 9 | 4 | 11 | 6 | 1 | 8 | 3 | 10 |
| 2 | 10 | 5 | 0 | 7 | 2 | 9 | 4 | 11 | 6 | 1 | 8 | 3 |
| 3 | 3 | 10 | 5 | 0 | 7 | 2 | 9 | 4 | 11 | 6 | 1 | 8 |
| 4 | 8 | 3 | 10 | 5 | 0 | 7 | 2 | 9 | 4 | 11 | 6 | 1 |
| 5 | 1 | 8 | 3 | 10 | 5 | 0 | 7 | 2 | 9 | 4 | 11 | 6 |
| 6 | 6 | 1 | 8 | 3 | 10 | 5 | 0 | 7 | 2 | 9 | 4 | 11 |
| 7 | 11 | 6 | 1 | 8 | 3 | 10 | 5 | 0 | 7 | 2 | 9 | 4 |
| 8 | 4 | 11 | 6 | 1 | 8 | 3 | 10 | 5 | 0 | 7 | 2 | 9 |
| 9 | 9 | 4 | 11 | 6 | 1 | 8 | 3 | 10 | 5 | 0 | 7 | 2 |
| 10 | 2 | 9 | 4 | 11 | 6 | 1 | 8 | 3 | 10 | 5 | 0 | 7 |
| 11 | 7 | 2 | 9 | 4 | 11 | 6 | 1 | 8 | 3 | 10 | 5 | 0 |

**Table 42:** *The chart of pitch classes from key F by Example 18.*

| $*$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | $F$ | $C$ | $G$ | $D$ | $A$ | $E$ | $B$ | $F^\sharp$ | $C^\sharp$ | $G^\sharp$ | $D^\sharp$ | $B^\flat$ |
| 1 | $B^\flat$ | $F$ | $C$ | $G$ | $D$ | $A$ | $E$ | $B$ | $F^\sharp$ | $C^\sharp$ | $G^\sharp$ | $D^\sharp$ |
| 2 | $D^\sharp$ | $B^\flat$ | $F$ | $C$ | $G$ | $D$ | $A$ | $E$ | $B$ | $F^\sharp$ | $C^\sharp$ | $G^\sharp$ |
| 3 | $G^\sharp$ | $D^\sharp$ | $B^\flat$ | $F$ | $C$ | $G$ | $D$ | $A$ | $E$ | $B$ | $F^\sharp$ | $C^\sharp$ |
| 4 | $C^\sharp$ | $G^\sharp$ | $D^\sharp$ | $B^\flat$ | $F$ | $C$ | $G$ | $D$ | $A$ | $E$ | $B$ | $F^\sharp$ |
| 5 | $F^\sharp$ | $C^\sharp$ | $G^\sharp$ | $D^\sharp$ | $B^\flat$ | $F$ | $C$ | $G$ | $D$ | $A$ | $E$ | $B$ |
| 6 | $B$ | $F^\sharp$ | $C^\sharp$ | $G^\sharp$ | $D^\sharp$ | $B^\flat$ | $F$ | $C$ | $G$ | $D$ | $A$ | $E$ |
| 7 | $E$ | $B$ | $F^\sharp$ | $C^\sharp$ | $G^\sharp$ | $D^\sharp$ | $B^\flat$ | $F$ | $C$ | $G$ | $D$ | $A$ |
| 8 | $A$ | $E$ | $B$ | $F^\sharp$ | $C^\sharp$ | $G^\sharp$ | $D^\sharp$ | $B^\flat$ | $F$ | $C$ | $G$ | $D$ |
| 9 | $D$ | $A$ | $E$ | $B$ | $F^\sharp$ | $C^\sharp$ | $G^\sharp$ | $D^\sharp$ | $B^\flat$ | $F$ | $C$ | $G$ |
| 10 | $G$ | $D$ | $A$ | $E$ | $B$ | $F^\sharp$ | $C^\sharp$ | $G^\sharp$ | $D^\sharp$ | $B^\flat$ | $F$ | $C$ |
| 11 | $C$ | $G$ | $D$ | $A$ | $E$ | $B$ | $F^\sharp$ | $C^\sharp$ | $G^\sharp$ | $D^\sharp$ | $B^\flat$ | $F$ |

**Table 43:** *A musical subquasigroup by Example 18.*

| $*$ | 0 | 2 | 4 | 6 | 8 | 10 |
|---|---|---|---|---|---|---|
| 0 | 0 | 2 | 4 | 6 | 8 | 10 |
| 2 | 10 | 0 | 2 | 4 | 6 | 8 |
| 4 | 8 | 10 | 0 | 2 | 4 | 6 |
| 6 | 6 | 8 | 10 | 0 | 2 | 4 |
| 8 | 4 | 6 | 8 | 10 | 0 | 2 |
| 10 | 2 | 4 | 6 | 8 | 10 | 0 |

**Table 44:** *A musical subquasigroup from key F by Example 18.*

| $*$ | 0 | 2 | 4 | 6 | 8 | 10 |
|---|---|---|---|---|---|---|
| 0 | $F$ | $G$ | $A$ | $B$ | $C^\sharp$ | $D^\sharp$ |
| 2 | $D^\sharp$ | $F$ | $G$ | $A$ | $B$ | $C^\sharp$ |
| 4 | $C^\sharp$ | $D^\sharp$ | $F$ | $G$ | $A$ | $B$ |
| 6 | $B$ | $C^\sharp$ | $D^\sharp$ | $F$ | $G$ | $A$ |
| 8 | $A$ | $B$ | $C^\sharp$ | $D^\sharp$ | $F$ | $G$ |
| 10 | $G$ | $A$ | $B$ | $C^\sharp$ | $D^\sharp$ | $F$ |

**Definition 10.** Let $(Q,*)$ be a musical quasigroup and let $(W,*)$ be a musical subquasigroup of $(Q,*)$. Then $(W,*)$ is called a musical normal subquasigroup of $(Q,*)$ if:

(i) $nW = Wn$ (ii) $y(xW) = (yx)W$ (iii) $(Wx)y = W(xy)$ for all $n, x, y \in Q$.

**Definition 11.** Let $n$ be an even integer and let $\mathbb{Z}_n = \{0,1,2,3,\cdots,n-1\}$ be the set of pitch classes for $n \geq 4$. Define a binary operation $*$ on $\mathbb{Z}_n$ as $a*b = xa+yb(mod\ n)$ where $x,y$ are elements in $\mathbb{Z}_n\backslash\{0\}$ and $(x,y)=1,\ x+y=n$ and $|x-y|$ is a minimum. For a fixed integer $n$ and varying $x$ and $y$, an $n$-tone musical quasigroup $(\mathbb{Z}_n,*)$ is obtained. Let $\mathbb{Z}'_n$ be the set of all even numbers in $\mathbb{Z}_n$, we obtain a musical subquasigroup $(\mathbb{Z}'_n,*)$ of $(\mathbb{Z}_n,*)$.

*Example* 19. Let $Q = \{0,1,2,3,4,5,6,7\}$ be the set of pitch classes. Define the binary operation $*$ on $Q$ as $a*b = 5a+3b(mod\ 8)$ then $(Q,*)$ is an eight-tone musical quasigroup and the set $Q' = \{0,2,4,6\}$ forms a musical subquasigroup under the binary operation $*$.

Example 10 gives the chart for $Q$. It is clear that $Q'$ table is given by Table 45 (see also Table 46):

**Table 45:** *A musical subquasigroup by Example 19*

| $*$ | 0 | 2 | 4 | 6 |
|---|---|---|---|---|
| 0 | 0 | 6 | 4 | 2 |
| 2 | 2 | 0 | 6 | 4 |
| 4 | 4 | 2 | 0 | 6 |
| 6 | 6 | 4 | 2 | 0 |

**Table 46:** *A musical subquasigroup chart from key F by Example 19*

| $*$ | 0 | 2 | 4 | 6 |
|---|---|---|---|---|
| 0 | $F$ | $B$ | $A$ | $G$ |
| 2 | $G$ | $F$ | $B$ | $A$ |
| 4 | $A$ | $G$ | $F$ | $B$ |
| 6 | $B$ | $A$ | $G$ | $F$ |

In Example 19, $Q'$ is an example of a normal subquasigroup of $(Q, *)$. Thus, Table 46 is a musical chart of a normal subquasigroup $Q'$ from key $F$.

## Motion of a Single Melody by a Normal Subquasigroup

(A) We consider Tables 42 and 44. Table 44 is a normal subquasigroup table obtained from Table 42.

From Table 42, we note the followings:

(i) Each row and column of this table displays melodic motion of a chromatic scale

(ii) The distance between two notes on this table is either five or seven semitones in each row and column.

(iii) Each row and column of this table displays a disjunct melodic motion.

(iv) The rows (or columns) do not display an ascending or descending motion.

From Table 44, we note the followings:

(i) The first column of this table shows a descending motion when read from the top to the bottom; and shows an ascending motion when read from the bottom to the top.

(ii) The distance between two notes on this table is two semitones in each row and column.

(iii) The motion of each single melody described by each column and row is a conjunct motion.

Comparing the results from Tables 42 and 44, it is clear that:

(i) A melodic motion which is disjunct, may contain a sub-melodic motion which is conjunct.

(ii) A melodic motion which is not ascending (descending), may contain a sub-melodic motion which is ascending (descending).

(iii) The main melodic motion may contain a sub-melodic motion whose distance between its consecutive notes are shorter than that of the main melodic motion.

(B) We consider Tables 23 and 46. Table 46 is a normal subquasigroup table obtained from Table 23.

From Table 23, we note the following:

(i) The table displays some motions of eight consecutive semitones in columns and rows from Key F.

(ii) The distance between two notes on this table is either three or five semitones in each row and column.

(iii) The motion of a single melody described by each row and column is a disjunct motion.

(iv) The columns and rows do not display the ascending or descending motions. Rather, the ascending and descending motions both occur in a single melody, this is shown by the first row.

From Table 46, we note the following:

(i) The first column and row of this table shows an ascending and descending motions respectively.

(ii) The distance between two notes on this table is either two or six semitones in each row and column.

(iii) The motion of a single melody described by each row (column) is either disjunct, or have both conjunct and disjunct motions.

Comparing the results from Tables 23 and 46, it is clear that:

(i) A melodic motion which is not ascending (descending) may contain a sub-melodic motion which is ascending (descending).

(ii) The main melodic motion may have a sub-melodic motion whose distance between its consecutive notes contains more semitones than that of the main melodic motion.

(iii) A melodic motion which is disjunct may contain a sub-melodic motion which is either disjunct, or have both conjunct and disjunct motions.

## Motion of Two Melodies by a Normal Subquasigroup

(A) From Table 44, consider the entries with index $(i, i)$:

(i) For $i = 1$, we have the index $(1, 1)$, that is, the position of $F$ on the first row. At the index $(1, 1)$ of Table 44, the column and the row are two different melodies. These two melodies departing from index $(1, 1)$ are in opposite directions. Thus, the two melodies form a strict contrary motion as they depart from the index $(1, 1)$.

(ii) Each of these melodic pairs departing from index $(i, i)$ into rows and columns of this table is seen to be in a strict contrary motion, keeping a distance of two semitones between two notes.

From Table 42, consider the entries with index $(i, i)$. Each of these melodic pairs departing from index $(i, i)$ into rows and columns of this table is not in a contrary motion, and keeps a distance of five or seven semitones between two notes.

Comparing result from Tables 42 and 44, it is clear that, there are some paired melodies which are not in contrary motion to each other but have paired sub-melodies which are in strict contrary motion.

(B) From Table 46, consider the entries with index $(i, i)$:

(i) At the index $(1, 1)$ of Table 46, the column and the row are two different melodies. These two melodies departing from index $(1, 1)$ are in opposite directions. Thus, the two melodies form a contrary motion as they depart from the index $(1, 1)$.

(ii) Each of these melodic pairs departing from index $(i, i)$ into rows and columns of this table is seen to be in a contrary motion.

From Table 23, consider the entries with index $(i, i)$. Each of these melodic pairs departing from index $(i, i)$ into rows and columns of this table is not in a contrary motion, and keeps a distance of three or five semitones between two notes.

Comparing result from Tables 23 and 46, it is clear that, there are some paired melodies which are not in contrary motion to each other but have paired sub-melodies which are in contrary motion.

## III. Conclusion

A musical quasigroup is a musical groupoid in which all its left and right translation mappings are permutations. Some examples of quasigroups have been constructed and applied to music, and it is noted that one of the functions of a given quasigroup binary operation on a set of order $n$ is to preserve the permutations of the $n$ symbols defined by the $n \times n$ multiplication. As regarding the main results (Section II), from Sections i, ii, and viii, it has been shown that quasigroup plays important roles in chord progressions and inversions. By Remarks 1, 2 and 4, and Section viii, the function of the left (right, middle) identity element of a quasigroup could be linked to the root of a given chord. The study considered examples from triad, tetrads, pentads, and hexads. In Section iii, examples of musical sequences from triads are given. In Section iv, an example of a twelve-tone matrix is described to be a quasigroup. In Section v, we construct some examples of an $n$-tone composition chart using quasigroups. By Example 6, a circle of fourths' and fifths' progressions were obtained. In Sections vi and viii, motion of a single melody is described by some quasigroups. It is shown by examples that each row and column of a finite musical quasigroup is a melodic motion on its own right. These examples were used to describe a descending, ascending, disjunct and conjunct melodic motions. Sections vii and viii give examples of musical quasigroups which demonstrate contrary, parallel, oblique and similar motions respectively. Table 37 gives an oblique motion which one of its melodies is static while the other moves into disjunct and conjunct motions. In Section ix, examples of a subquasigroup are constructed and further characterized to be normal subquasigroups. An example of a melodic motion which is disjunct is shown to have a sub-melodic motion which is conjunct. It was obtained that there are paired melodies which are not in contrary motion to each other but have paired sub-melodies which are in contrary (or strict-contrary) motion.

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# Neo-Riemannian Graphs Beyond Triads and Seventh Chords

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***Abstract:*** *This article develops theoretical concepts that allow us to adapt the graphs used for Neo-Riemannian Theory to all classes of trichords and tetrachords, beyond triads and seventh chords. To that end, it will be necessary to determine what the main features of these graphs are and the roles that the members of each set class play in them. Each graph is related to a mode of limited transposition, which contains all the pitch classes occurring in the graph. The musical examples extracted from these graphs reveal passages in which the sets are connected by a consistent voice-leading.*

***Keywords****: Neo-Riemannian Theory. Graphs. Voice leading. Contextual inversion.*

## I. Introduction

Graphs of Neo-Riemannian Theory proved to be very useful in dealing with triads and seventh chords, covering many possible connections between them. There are several graphs with these sets, including cycles, like HexaCycles, OctaCycles, and EnneaCycles; trees, like Weitzmann Regions and Boretz Regions; and unified models, like Cube Dance, Power Towers, and 4-Cube Trio.

In this paper we will develop the theoretical tools to construct all these types of neo-Riemannian graphs with any trichord or tetrachord beyond the members of sc. (037), (048), (0258), (0358), and (0369). Section II will explore the position and function that the sets have in the graphs and how important is their symmetry in this context. We divided all sets in four types: 1) target sets, 2) pivot sets, 3) bridge sets, and 4) supersets.

Section III will determine labels for the axes of contextual symmetry. The transformations P, L, R, among others, used traditionally in the Neo-Riemannian Theory, do not fit all classes of trichords and tetrachords and, given the purpose of this work to construct graphs with all the sets of these cardinality, it was necessary to introduce different terminology. Taking advantage of the

*This work on the graph-theoretic approach of Neo-Riemannian Theory has begun in 2017 when I was at CUNY under the guidance of Distinguished Professor, Joseph Straus, to whom I am immensely grateful for his many insightful ideas, which were invaluable to the writing of this article. I would also like to thank Prof. Dr. Paulo de Tarso Salles, who was my doctoral supervisor at ECA/USP and who first introduced me to Neo-Riemannian Theory and its graphs.

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relation between the transformations and the contextual inversions, this terminology was defined according to the axes of these inversions in relation to the normal form of the sets.

Section IV will explore cycles of same set class members. In the earlier part of this section the main features of the Hexatonic Cycles, components of the graph known as HexaCycles, will be listed and then, based on these features, new cycles that include all sets of trichords and tetrachords related by contextual inversion will be constructed. All the new cycles will share three of four main features of Hexatonic Cycles. The musical examples for each cycle will show passages where the sets are connected by a consistent voice-leading. This consistency results from the fact that all the sets are limited to two only sum classes. This section will also explore the cycles with symmetric sets that are not related by contextual inversion, but by transposition. All the sets in these cycles are limited to a single sum class and they are connected by pure contrary motion.

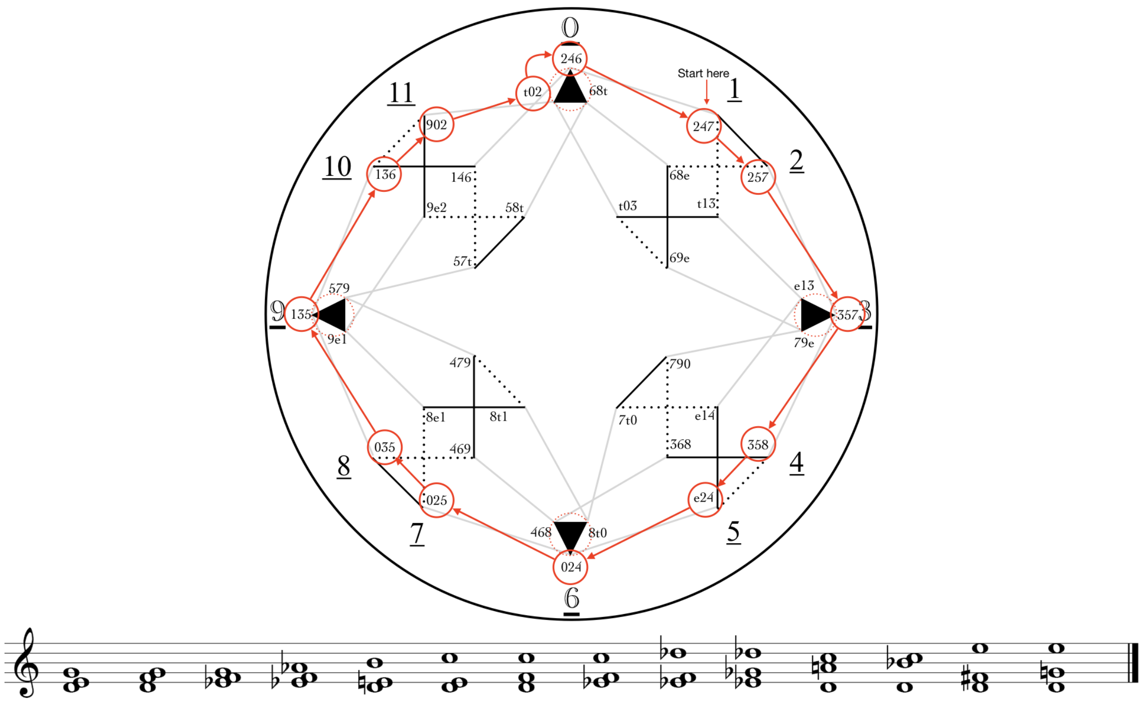
The graphs built in Section V include members of two sum classes and Weitzmann graphs, and Boretz Spiders will be used as models for that task. As with cycles, the new graphs will also share three of the four main features of their models, and the musical examples for them will also show passages with a consistent voice-leading, where the sets are limited to three sum classes.

The same strategy will be used to build the unified models in Section VI. The graphs known as Cube Dance and Power Towers will be used as models to the new graphs, which share three of the their four main features. All the cycles and graphs built in Sections IV and V will be subgraphs in these new Cube Dances and Power Towers, and the musical examples will show passages where the sets are connected by a consistent voice-leading, even if its members are distributed in all sum classes. One can see an example of this consistent voice-leading, which will be the focus of this article, in Figure 1 that shows in its upper part a Cube Dance which is built with members of sc. (024) and (025). The path drawn over the graph indicates a passage with the sets that are noted in the lower part of Figure 1. Note how all the sets are linked by a kind of voice-leading, parsimonious or not, that keeps the sets in two adjacent sum classes, the only exception being between the antepenultimate and penultimate set of the passage, in which the sets remain in the same class of sum because of the pure contrary motion. All the supplementary material for this article are available as at `https://axesofcontextualinversion.wordpress.com/` (see Figure 1).

## II. Types of Sets in the Graphs

All neo-Riemannian graphs, even those that include more than one set class, have a main set. We will refer to them as target sets. All the cycles used in Neo-Riemannian Theory have just a single set class that is the target set of the cycle, thus in the HexaCycles ([3, p. 243, Fig. 3]) and in the OctaCycles ([3, p. 247, Fig. 5]) consonant triads, sc. (037), are the target sets in each component, and in EnneaCycles ([3, p. 247, Fig. 6]) the target sets are members of sc. (0258), half-diminished and dominant seventh chords.

There is another type of graph, called tree, whose components include members of two different set classes limited to three sum class. Weitzmann Graph ([1, p. 94, Ex. 6]), the OctaTowers ([3, p. 246, Fig. 4]) and the Boretz regions ([3, p. 153, Tab. 7.2]) are examples of trees. Consonant triads are also the target sets in each component of the Weitzmann graph. They are all connected to a single augmented triad, sc (048). Members of sc. (0258) are the target sets in each component of the OctaTowers and of the Boretz regions. In the former, four pairs, made by one half-diminished chord and one dominant seventh chord, are connected to four minor seventh chords that are members of sc. (0358). In the latter, all members of sc. (0258) are connected to a single diminished seventh chord, sc. (0369). We will refer to the augmented triads in Weitzmann region and the diminished seventh chord in the Boretz regions as pivot sets, due to their quality in connecting with all the other members of the graph, and we will refer to the minor seventh chords in the




| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Sets | [2,4,7] | [2,5,7] | [3,5,7] | [3,5,8] | [e,2,4] | [0,2,4] | [0,2,5] | [0,3,5] | [1,3,5] | [1,3,6] | [9,0,2] | [t,0,2] | [2,4,6] | [2,4,7] |
| Pitch Space | G (7) | → G (7) | → G (7) | $\xrightarrow{+1}$ A♭ (8) | $\xrightarrow{+3}$ B (11) | $\xrightarrow{+1}$ C (12) | → C (12) | → C (12) | $\xrightarrow{+1}$ D♭ (13) | → D♭ (13) | $\xrightarrow{-1}$ C (12) | → C (12) | $\xrightarrow{+4}$ E (16) | → E (16) |
| | E (4) | $\xrightarrow{+1}$ F (5) | → F (5) | → F (5) | $\xrightarrow{-1}$ E (4) | → E (4) | $\xrightarrow{+1}$ F (5) | → F (5) | → F (5) | $\xrightarrow{+1}$ G♭ (6) | $\xrightarrow{+3}$ A (9) | $\xrightarrow{+1}$ B♭ (10) | $\xrightarrow{-4}$ F♯ (6) | $\xrightarrow{+1}$ G (7) |
| | D (2) | → D (2) | $\xrightarrow{+1}$ E♭ (3) | → E♭ (3) | $\xrightarrow{-1}$ D (2) | → D (2) | → D (2) | $\xrightarrow{+1}$ E♭ (3) | → E♭ (3) | → E♭ (3) | $\xrightarrow{-1}$ D (2) | → D (2) | → D (2) | → D (2) |
| P-int Sum | | +1 | +1 | +1 | +1 | +1 | +1 | +1 | +1 | +1 | +1 | +1 | 0 | +1 |
| PC Space | G (7) | → G (7) | → G (7) | $\xrightarrow{1}$ A♭ (8) | $\xrightarrow{3}$ B (11) | $\xrightarrow{1}$ C (0) | → C (0) | → C (0) | $\xrightarrow{1}$ D♭ (1) | → D♭ (1) | $\xrightarrow{11}$ C (0) | → C (0) | $\xrightarrow{4}$ E (4) | → E (4) |
| | E (4) | $\xrightarrow{1}$ F (5) | → F (5) | → F (5) | $\xrightarrow{11}$ E (4) | → E (4) | $\xrightarrow{1}$ F (5) | → F (5) | → F (5) | $\xrightarrow{1}$ G♭ (6) | $\xrightarrow{3}$ A (9) | $\xrightarrow{1}$ B♭ (10) | $\xrightarrow{8}$ F♯ (6) | $\xrightarrow{1}$ G (7) |
| | D (2) | → D (2) | $\xrightarrow{1}$ E♭ (3) | → E♭ (3) | $\xrightarrow{11}$ D (2) | → D (2) | → D (2) | $\xrightarrow{1}$ E♭ (3) | → E♭ (3) | → E♭ (3) | $\xrightarrow{11}$ D (2) | → D (2) | → D (2) | → D (2) |
| PC-int sum (Mod 12) | | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 |
| Sum Class | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 0 | 0 | 1 |

**Figure 1:** *Example of a passage with members of sc. (024) and (025) arranged in a Cube Dance.*

OctaTowers as bridge sets, due to its quality in connecting with two target sets of the graph.

The graph known as Cube Dance ([2, p. 104, Fig. 5.24]) is a "unified model of triadic voice-leading space" ([2, p. 83]) because it "includes the four hexatonic cycles and the four Weitzmann regions as contiguous subgraphs" ([2, p. 83]). The target sets are all members of sc. (037) that are placed in the voice-leading zones 1, 2, 4, 5, 7, 8, 10, and 11 and the members of sc. (048), placed in the voice-leading zones 0, 3, 6, and 9, are the pivot sets, as they connect to all sets in its two adjacent voice-leading zones. In the same way, we can consider the Power Towers ([3, p. 256, Fig. 10]) as a unified model of seventh chords voice-leading space, since it includes the three octatonic towers and the three Boretz regions as contiguous subgraphs. In this graph, all the members of sc.

(0258), placed in the odd voice-leading zones, are the target sets, while the members of sc. (0369), diminished seventh chords placed in voice-leading zones 2, 6, and 10, are the pivot sets, and the members of sc. (0358), minor seventh chords placed in voice-leading zones 0, 4, and 8, are the bridges sets.

Douthett and Steinbach have shown how the components of these graphs are embedded in one set which is associated to one of modes of limited transposition ([3, pp. 245–247]). We will refer to these sets as supersets. The hexatonic collection, sc. (014589), is the superset of each component of the HexaCycles and of each cube of Cube Dance; the octatonic collection, sc. (0134679t), is the superset of each component of the OctaCycles, each component of OctaTowers, and also of each circuit constituted by all sets between two diminished seventh chords in the Power Towers[1]. Nonatonic collection, sc. (01245689t), is the superset of each component of the Weitzmann graph and each component of the EnneaCycles.

Therefore, there are four types of sets in neo-Riemannian graphs: target sets, pivot sets, bridge sets, and supersets, each one of them playing a different role in the graphs. Since our goal in this work is to build graphs for sets other than triads and seventh chords, we shall summarize and generalize these roles.

- **Target sets** are the main sets of a graph; they are necessarily members of a set class with neither inversion or transpositional symmetry, which therefore have 24 sets; these 24 sets connect with themselves and with members of other set classes in the graph. For members of a set class to be target sets in a graph of trichords, they must be distributed in the sum classes[2] 1, 2, 4, 5, 7, 8, 10, and 11[3], they are sc. (013), (014), (016), (025), (026), (027) and (037). In tetrachord graphs, all members of the target sets must be distributed in odd sum classes. They are sc. (0124), (0126), (0135), (0137), (0146), (0148), (0157), (0236), (0247) and (0258).
- **Pivot sets** make connections with every target set placed in their adjacent voice-leading zones; they are necessarily member of a set class with inversional and transpositional symmetry, with its pitches dividing equally the octave by a single interval. Pivot sets therefore belong to set classes that have a maximum of 6 members[4]. In trichord graphs, all the members of the pivot set must be distributed in the sum classes 0, 3, 6, and 9, and in tetrachord graphs, all members of the pivot sets must be distributed in sum classes 2, 6, and 10. Sc. (048) is the only pivot set among the trichords, while sc. (0369) is the only pivot set among the tetrachords.

[1] Douthett and Steinbach have changed the original Power Tower including the French-sixth chord in voice-leading zones 0, 4, and 8 along with minor seventh chords ([3, p. 262]). In this way, the circuit constituted by all sets between two diminished seventh chords in the Power Towers becomes a tesseract. They named this version of graph as 4-Cube Trio, since it connects three tesseracts in the same way that Cube Dance connects four cubes. The octatonic collection is the superset of each tesseract of the 4-Cube Trio.

[2] According to Joseph Straus: "Two pitch sets are equivalent as members of the same sum class if their pitch integers have the same sum" ([9, p. 2]). This concept can be associated to the concept of voice-leading zones, since two sets in the same sum class are necessarily in the same voice-leading zone. In this work, we will use both concepts, as we find it more appropriate.

[3] In his article named Sum Class ([9]), Straus provides a table with all trichords that shows that they are divided into two groups, one group whose members are distributed in sum classes 1, 2, 4, 5, 7, 8, 10, and 11 - trichords that can be target sets in a graph - and other group whose members are distributed in sum classes 0, 3, 6, and 9 - trichords that can be pivot or bridges sets in a graph - ([9, p. 22]). He also provides a table with all tetrachords that shows that they are divided into three groups, one group whose members are distributed in odd sum classes - tetrachords that can be target sets in a graph - a second group whose members are distributed in sum classes 2, 4, and 6 - tetrachords that can be pivot or bridge sets in a graph - and a third group whose members are distributed in sum classes 0, 4, and 8 - tetrachords that can be bridge sets in a graph - ([9, pp. 41–44]).

[4] I exclude the aggregate of the 12 pitches because it is the only one of cardinality 12 and therefore it has no way to be the pivot in a graph.

- **Bridge sets** make connection with some, but not all target sets placed in their adjacent voice-leading zones; they can be members of a set class with inversional or transpositional symmetry, which therefore have a maximum of 12 sets, but they also can be member of a non-symmetric set class. In trichord graphs, all the members of the bridge sets must be distributed in the sum classes 0, 3, 6, and 9. They are sc. (012), (015), (024), (027) and (036). In tetrachord graphs, all members of the pivot sets must be distributed in sum classes 2, 6, and 10 or in sum classes 0, 4, and 8. They are sc. (0123), (0125), (0127), (0134), (0136), (0145), (0147), (0156), (0158), (0167), (0235), (0237), (0246), (0248), (0257), (0268), (0347), and (0358).
- **Supersets** are collections that embody all the pitches of the sets of a graph or of one component of a graph; they are necessarily member of a set class with inversional and transpositional symmetry, which therefore have a maximum of 6 sets. Here we will relate these collection to Messiaen's modes of limited transposition, thus the label WT (the whole-tone collection) will be used for his first mode, sc. (02468t), which interval cycle is a **<2>** chain; OCT (the octatonic collection) for his second mode, sc. (0134679t), which interval cycle is a **<1,2>** chain; NON (the nonatonic collection) will be used for his third mode, sc. (01245689t), which interval cycle is a **<1,1,2>** chain; MM4 will be used for his fourth mode, sc.(01236789), which interval cycle is a **<1,1,3>** chain; MM5 will be used for his fifth mode, sc. (012678), which interval cycle is a **<1,1,4>** chain; MM6 will be used for his sixth mode, sc. (0124678t), which interval cycle is a **<1,1,2,2>** chain; and MM7 will be used for his seventh mode, sc. (0123466789t), which interval cycle is a **<1,1,1,2>** chain. In addition to these collections, we will use the label HEX for the hexatonic collection, sc. (014589), which is not one of the Messiaen's modes and its cycle is a **<1,3>**im chain and AGG for the aggregated of the 12 pitches. Table 1 summarizes the main features of these sets.

## III. Axes of Contextual Inversions

Neo-Riemannian theorists use transformations labels as **P**, **L**, **R**, **N**, **S**, among others, to determine how sets connect each other. These labels are based on voice-leading work that privileges common pitches and parsimonious movements between sets. In this article we will not use these traditional labels for two reasons: 1) although some authors such as Morris ([5, pp. 186–193]) and Straus ([7, pp. 53–67]) have adapted these transformations to sets other than triads and seventh chords, those adaptations can bring some subtle difficulties, as one of these labels may be associated to more than one connection, or a single connection may be associated to more than one label for some sets[5]; 2) for constructing graphs representing the voice-leading space for all trichords and tetrachords we need to describe a greater number of connections between sets than those available with the traditional labels of Neo-Riemannian Theory, and, even if we create new labels based on the same voice-leading work principles, they would also be subject to same difficulty mentioned above.

The approach we present here to label the connections between sets is based on contextual inversions, so the labels will indicate the position of the axis on which the two sets invert themselves. Contextual inversions only occur between target sets, and this type of set class always contains 24 members divided into two different types of OPC equivalent sets: 12 members represented by normal form A and 12 members represented by normal form B[6]. We will determine

[5]For example, following Straus ([7, p. 56, Tab. 1]) **L** transforms [0,2,4] in both [t,0,2] and [2,4,6] because these connections retain i2. However, these same two connections can be labeled **R**, because it also retains i2 for this set.

[6]Following Solomon's set class table available at `http://solomonsmusic.net/pcsets.htm`, we will call normal form A those that are most packed to the left, and normal form B those that are most packed to the right.

**Table 1:** *List of the collections that can be supersets in a graph.*

| Meessiaen's Mode | PCs | Set | Collection | $T_n$ | Modes | Interval Cycle |
|---|---|---|---|---|---|---|
| Mode 1 (MM1) | 6 | (02468t) | Whole-Tone | 2 | $WT_0$ [0,2,4,6,8,t]; $WT_1$ [1,3,5,7,9,e] | <2> |
| Mode 2 (MM2) | 8 | (0134679t) | Octatonic | 2 | $OCT_{0,1}$ [0,1,3,4,6,7,9,t]; $OCT_{1,2}$ [1,2,4,5,7,8,t,e]; $OCT_{2,3}$ [2,3,5,6,8,9,t,0] | <1,2> |
| Mode 3 (MM3) | 9 | (01245689t) | Nonatonic | 4 | $NON_{0,1,2}$ [0,1,2,4,5,6,8,9,t] $NON_{1,2,3}$ [1,2,3,5,6,7,9,t,e] $NON_{2,3,4}$ [2,3,4,6,7,8,t,e,0] $NON_{3,4,5}$ [3,4,5,7,8,9,e,0,1] | <1,1,2> |
| Mode 4 (MM4) | 8 | (01236789) | - | 6 | $MM4_{0,1,2,3}$ [0,1,2,3,6,7,8,9] $MM4_{1,2,3,4}$ [1,2,3,4,7,8,9,t] $MM4_{2,3,4,5}$ [2,3,4,5,8,9,t,e] $MM4_{3,4,5,6}$ [3,4,5,6,9,t,e,0] $MM4_{4,5,6,7}$ [4,5,6,7,t,e,0,1] $MM4_{5,6,7,8}$ [5,6,7,8,e,0,1,2] | <1,1,3> |
| Mode 5 (MM5) | 6 | (012678) | - | 6 | $MM5_{0,1,2}$ [0,1,2,6,7,8] $MM4_{1,2,3}$ [1,2,3,7,8,9] $MM5_{2,3,4}$ [2,3,4,8,9,t] $MM5_{3,4,5}$ [3,4,5,9,t,e] $MM5_{4,5,6}$ [4,5,6,t,e,0] $MM5_{5,6,7}$ [5,6,7,e,0,1] | <1,1,4> |
| Mode 6 (MM6) | 8 | (0124678t) | - | 6 | $MM6_{0,1,2}$ [0,1,2,4,6,7,8,t] $MM6_{1,2,3}$ [1,2,3,5,7,8,9,e] $MM6_{2,3,4}$ [2,3,4,6,8,9,t,0] $MM6_{3,4,5}$ [3,4,5,7,9,t,e,1] $MM6_{4,5,6}$ [4,5,6,8,t,e,0,2] $MM6_{5,6,7}$ [5,6,7,9,e,0,1,3] | <1,1,4> |
| Mode 7 (MM7) | 8 | (012346789t) | - | 6 | $MM7_{0,1,2,3,4}$ [0,1,2,3,4,6,7,8,9,t] $MM7_{1,2,3,4,5}$ [1,2,3,4,5,7,8,9,t,e] $MM7_{2,3,4,5,6}$ [2,3,4,5,6,8,9,t,e,0] $MM7_{3,4,5,6,7}$ [3,4,5,6,7,9,t,e,0,1] $MM7_{4,5,6,7,8}$ [4,5,6,7,8,t,e,0,1,2] $MM7_{5,6,7,8,9}$ [5,6,7,8,9,e,0,1,2,3] | <1,1,2,2> |
| Not a Messiaen's Mode (HEX) | 6 | (014589) | Hexatonic | 4 | $HEX_{0,1}$ [0,1,4,5,8,9] $HEX_{1,2}$ [1,2,5,6,9,t] $HEX_{2,3}$ [2,3,6,7,t,e] $HEX_{3,4}$ [3,4,7,8,e,0] | <1,3> |

the label by the two opposite positions of the axis[7] relative to the first pitch of the normal form A and the last pitch of the normal form B that will always be mirrored. The axis rotates every half semitone and therefore it has 12 different positions that will be labelled with letters from **A** to **L**. We will use **A** as the label if the position of the axis is over the first pitch of normal form A and six semitones above it, or if it is over the last pitch of the normal form B and six semitones below it. **B** will be the label for the first rotation of the axis, **C** will be the label for the second rotation, and so

[7]Since we are working in a pitch class space, that traditionally is represented by the clock face, the axis always passes over two opposite points separated by 6 semitones.

on. Figure 2 shows how the 12 positions of the axis relate to the set [0,1,3] with all its possible inversions.

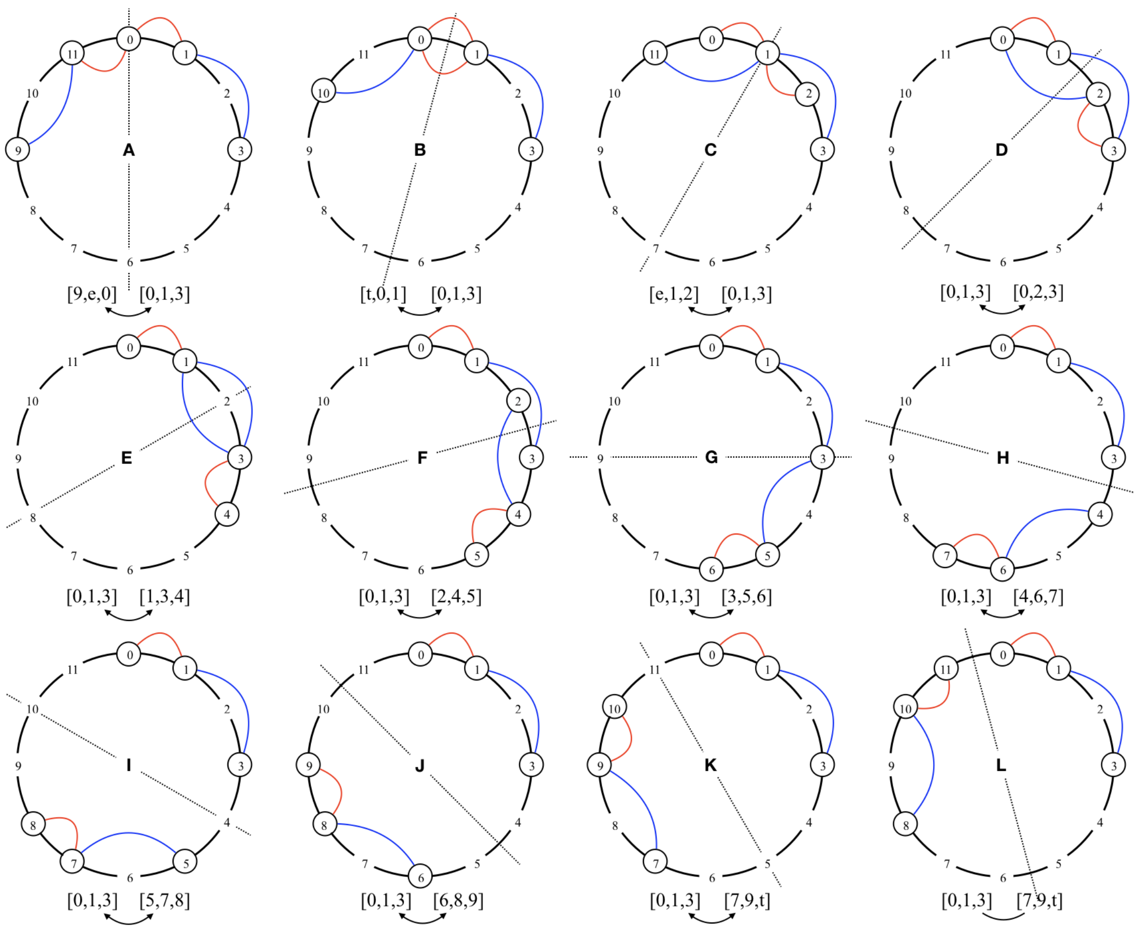



**Figure 2:** *The position of the axes for the 12 contextual inversion of set [0,1,3], all sets with pitches connected by the inside of the circle are of the normal form B.*

The difference between the axes of contextual inversion shown in Figure 2 and those used for the traditional inversion operation presented by Straus ([8, p. 61, Ex. 2.28]) is that the contextual inversion axes are not fixed in the clock face and move according to the first pitch of the normal form A or to the last pitch of the normal form B of a set[8]. Since there are 12 position of axis relating 12 pairs of sets, there will be 144 connections for each set class[9].

[8] That's why the axis **A** connects set [0,1,3] to the set [9,e,0] that is related by $I_0$, and also connects set [3,4,6] to the set [0,2,3] that is related by $I_3$.

[9] For more information about axis of contextual inversion, see [11], and for tables showing the axis that connect all the members of all set classes see [12, Vol. II, pp. 3–232].

## IV. Cycles

A cycle graph is a connected graph[10] that is regular of degree 2[11] [13, p. 17]. Cycles are a very common type of neo-Riemannian graphs: they may be a component of a larger graph which is a union of several similar cycles, such as hexatonic cycles are components of HexaCycles ([3, p. 245, Fig. 3]) or eneatonic cycles are components of the EnneaCycles ([3, p. 247, Fig. 5]), or may be a subgraph of a unified model, such as Cube Dance ([3, p. 254, Fig. 9]) or Power Towers ([3, p. 256, Fig. 10]).

All the neo-Riemannian cycles are chains of transformations using target set of the same type. Each one of the four Hexatonic cycles, for example, is a **<PL>** chain of sc. (037), but if we use the contextual inversion axis labels presented in the previous section, these cycles become <HD> chains, as Figure 3 shows.

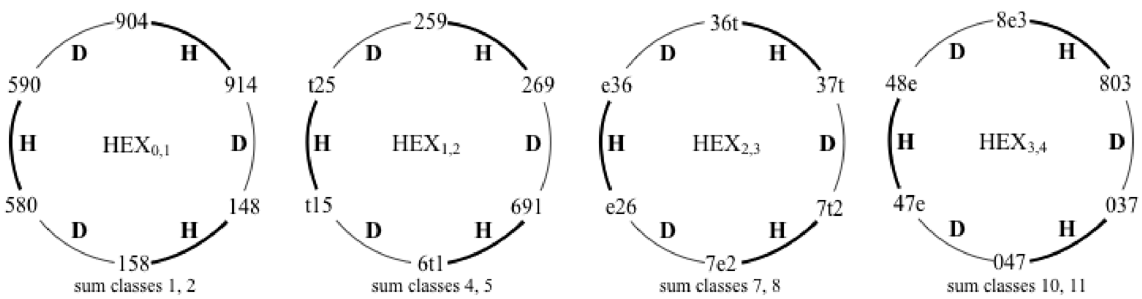



**Figure 3:** *The graph known as HexaCycles, with the four Hexatonic cycles as* ***<HD>*** *chain.*

Hexatonic cycles are subgraphs for both HexaCycles and Cube Dance graphs. This is because they have several important features that make them special cases among the trichord cycles used by the Neo-Riemannian Theory. Here are of some of these features:

- **1) All the sets belong to the same set class.** This is a common feature to all cycles used by the Neo-Riemannian Theory, because they all relate sets by a chain of contextual inversions[12].
- **2) All notes of each cycle are embedded in a symmetric superset.** The superset of each hexatonic cycle is the hexatonic collection that is listed inside them. All cycles used in the Neo-Riemannian Theory are embedded in one of the symmetric collections shown in Table 1, or in the aggregate of all PCs.
- **3) All sets of each cycle belong to two adjacent sum classes.** Sets in the cycle embedded in $HEX_{0,1}$ belong only to sum class 1 or 2; sets in the cycle embedded in HEX1,2 belong only to sum class 4 or 5; sets in the cycle embedded in $HEX_{2,3}$ belong only to sum class 7 or 8; and sets in the cycle embedded in $HEX_{3,4}$ belong only to sum class 10 or 11.
- **4) Parsimonious voice-leading[13].** Since **H** and **D** are the only position of the axis that hold fixed two notes between sets with the remaining note moving by a single semitone, it is possible to connect by parsimonious voice-leading between all the members of sc. (037) positioned at adjacent vertices in the hexatonic cycles.

[10]Connected graphs are those in which each vertex is connect to any other vertex by a path of edges.
[11]Degree is the number of connections of a vertex. If all vertices have a same degree, the graph is regular.
[12]However, it would certainly be possible to create cycles with members of two or more set classes if the criterion to relating these sets were neither transposition nor inversion.
[13]In this paper, I will consider that parsimonious voice-leading occurs only between two sets that can be transformed by moving one of its pitches by one semitone while the others remain fixed.

Only the hexatonic cycles have all these four features because just the sc. (037), among all sets of cardinality 3, can connect one of its members to two others by parsimonious voice-leading. Since parsimonious voice-leading has been a very important feature for the development of Neo-Riemannian Theory, all the trichords graphs have been limited to exclusively using members of sc. (037) as target sets. In order to build graphs with target sets other than sc. (037), it is necessary give up the voice-leading parsimony in all connections between its members. It should be noted that voice-leading parsimony is a particular case of connection between members of two adjacent sum classes, and if this kind of connection is only possible between members of sc. (037), there are other kinds of voice-leading that can connect members of trichords (013), (014), (016), (025), (026), and (027) in two adjacent sum classes. Figure 4 shows some examples of connections that keep sets in two adjacent sum classes.

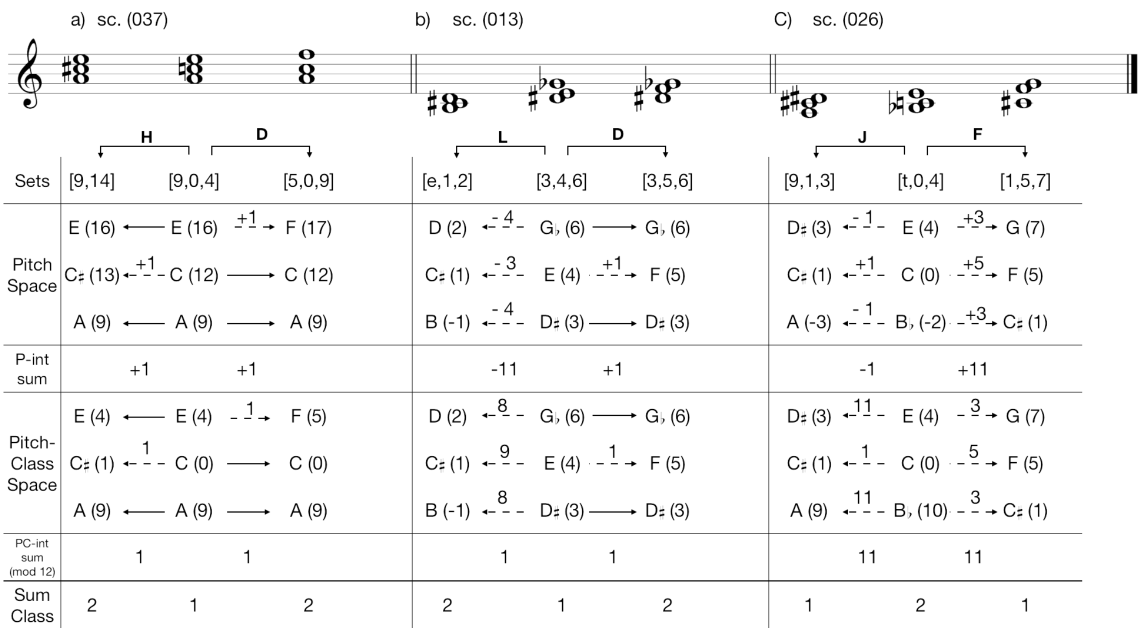



**Figure 4:** *Connections between members of sc. (037), (013) and (026) that keep sets in two adjacent sum classes.*

Figure 4 shows examples of connections between members of sc. (037), (013), and (026) that keep the sets in sum classes 1 and 2. The upper part of Figure 4 shows the music notation for the sets, while the table below shows their representation on both pitch and pitch-class space with arrows showing the voice-leading (with solid lines connecting two common pitches, and dashed lines representing the movement between two different pitches).

Straus shows that two sets of the same cardinality are in the same sum class if the sum of the directed pitch intervals (pitch space class) or direct pitch-class intervals (pitch-class space) is 0 ([9, p. 1]). In the same way, we can conclude that two sets of the same cardinality are in two adjacent sum classes if the sum of the directed pitch intervals or the direct pitch-class intervals is either 1 or 11 (mod 12). One can see in Figure 5a) how position **H** of the axis connects set [9,0,4] to set [9,1,4], and position **D** connects it to set [5,0,9]. All members of sc. (037) belonging to hexatonic cycle embedded in $HEX_{0,1}$. The parsimonious voice-leading in those connections comes from the fact that the sum of either the directed pitch intervals or the direct pitch-class intervals, is equal to 1, and hence the three sets are in sum classes 1 or 2. We can see similar features in Figure 5b), where the set [3,4,6] connects by **L** with [e,1,2] and by **D** with [3,5,6]. All the three sets are also in sum classes 1 and 2 because the sum of either the directed pitch intervals or the direct pitch-class intervals is equal to 1, but in these example only [3,4,6] and [3,5,6] are connect by the parsimony voice-leading. Figure 5c) shows that the set [t,0,4] connects by **J** with [9,1,3], and by **F** with [1,5,7] and, even neither of these connections are by parsimonious voice-leading, all the three sets are in sum classes 1 or 2 again, since the sum of either the directed pitch intervals or the direct pitch-class intervals is equal to 11. With these kind of connections it is possible to make cycles with members of trichords (013), (014), (016), (025), (026), and (027) which, as in hexatonic cycles, are divided into two adjacent sum classes. Figure 5 shows all cycles for target sets with cardinality 3. One can see that within each hexatonic cycle (Figure 5f)) is indicated the hexatonic collection which is its superset. The superset for all the remaining cycles is the aggregated of the 12 pitches.

Figure 6 shows some examples of voice-leading in cycles with sc. (014) (Figure 5b)), with sc. (026) (Figure 5e)) and with sc. (016) (Figure 5c)). Note that no matter which pitch arrangement is used for each set, the PC intervals sum of the voice-leading will always be 1 or 11, which keeps the sets in two adjacent sum classes.

It is not possible to build similar cycles with tetrachords because none of them have members in adjacent sum classes. In order to build cycles with the target sets of cardinality 4, their members must be in two adjacente odd sum classes. This will generate two types of graphs for each set class: those in which the components keep the sets in the sum classes 11/1, 3/5, and 7/9, and those in which the sets are in the sum classes 1/3, 5/7, and 9/11. Some of those graphs are divided in three components (cycles) and others are divided in six, as shown in Figure 7[14]. The supersets, except the aggregate of 12 pitches, are listed within each cycle according to Table 1.

Figure 8 shows some examples of voice-leading in tetrachords cycles with sc. (0148) (Figure 7f)), with sc. (0157) (Figure 7g)) and with sc. (0236) (Figure 7h)), in the same way that Figure 6 shows with trichords. Note that no matter which pitch arrangement is used for each set, the PC intervals sum of the voice-leading will always be 2 or 10, which keeps the sets in two adjacent even or odd sum classes.

Back to the set of cardinality 3, all the remaining trichords that are not included in the cycles of Figure 6 are either bridge or pivot, sets and have their members in sum classes 0, 3, 6, and 9, and therefore there is no way to build cycles with sets in adjacent sum classes as done before. For those set classes we are going to build cycles with members in the same sum class, using the labels for the position of contextual inversion axis for sc. (015) cycle, and transposition (T4 and T8) for sc. (012), (024), (027), and (036)[15]. In both cases the sets are connected by "pure contrary voice-leadings", in which, according to Tymoczko, "the amount of ascending motion exactly balances the amount of descending motion" ([13, p. 89]). Figure 9 shows examples of this kind of connection between members of sc. (015) and (027).

Figure 10 shows cycles for all bridge sets in which all members are connected by pure contrary voice leadings, and therefore are in the same sum class. Unlike the cycles shown in Figure 6, which the sets are related by contextual inversion, the sets in almost all of these cycles are related by transposition of 4 semitones, the exception being the cycle with sc. (015), as this is the only trichord without inverse symmetry in which its sets are distributed in the sum classes 0, 3, 6, and 9.

Figure 11 shows some examples of voice-leading in cycles with sc. (015) (Figure 10b) and with sc. (027) (Figure 10d). Note that any pitch arrangement that is used for sets results in a connection with pure contrary voice-leading, which keeps the sets in the same sum class.

[14] Some set classes have more than one chain that keep its members in two adjacent odd sum classes, like sc. (0236) and (0247).

[15] It is not possible to build this kind of cycle with members of sc. (048) since they map onto themselves both by inversion and by transposition.

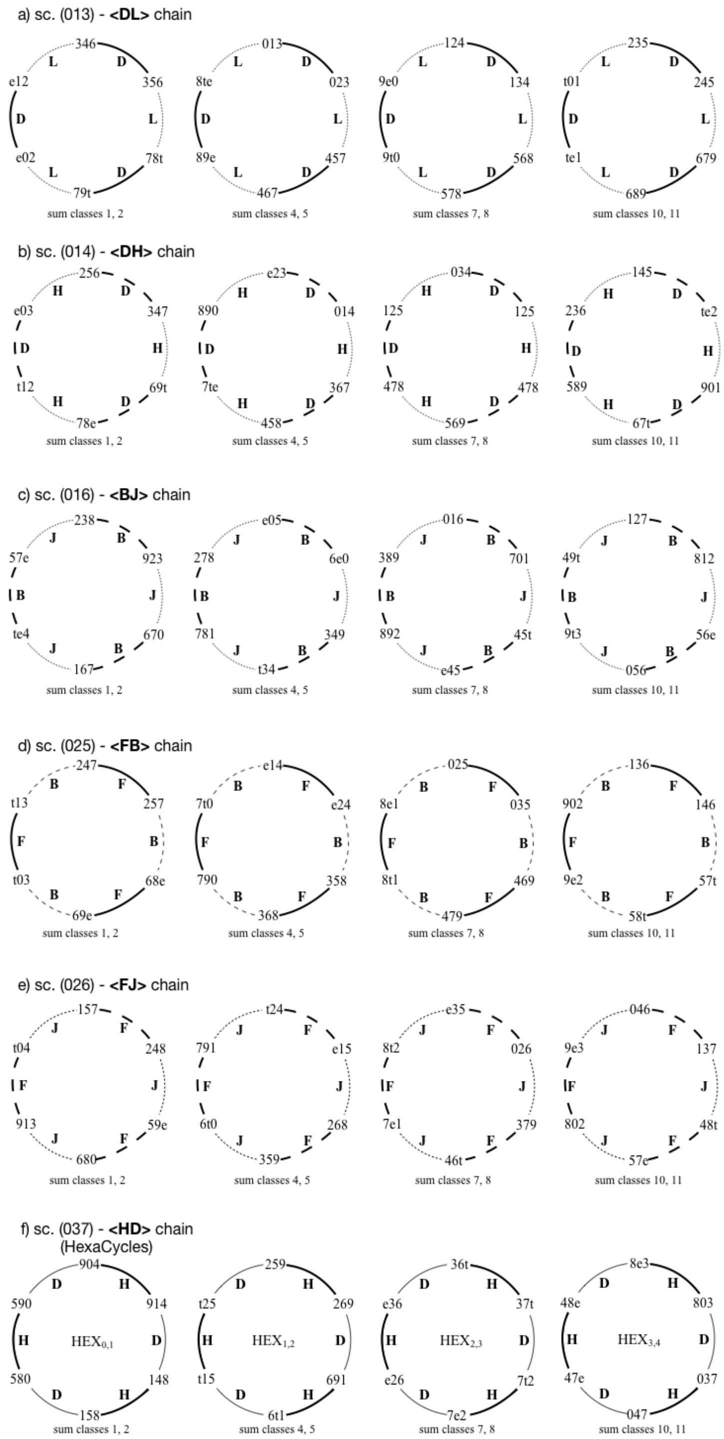
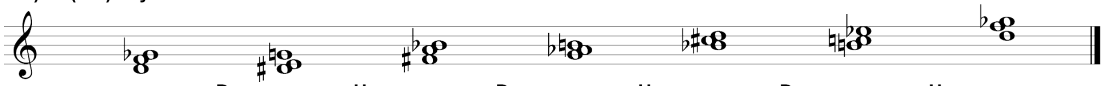


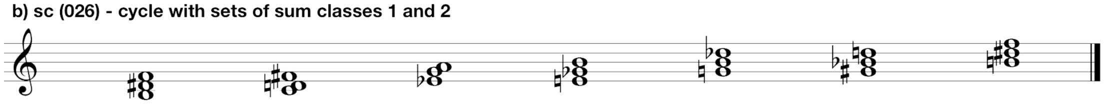

**Figure 5:** *Cycles for all target sets with cardinality 3 (solid lines represent parsimonious connections while dashed and dotted lines represent non-parsimonious connections).*

**a) sc (014) - cycle with sets of sum classes 1 and 2**



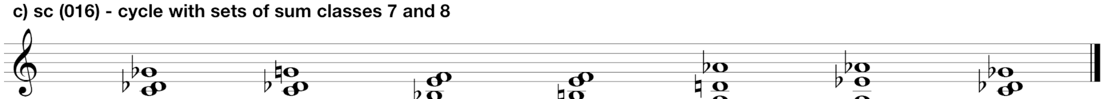
| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | D | | H | | D | | H | | D | | H | |
| Sets | | [2,5,6] | | [3,4,7] | | [6,9,t] | | [7,8,e] | | [t,1,2] | | [e,0,3] | | [2,5,6] |
| Pitch Space | | G♭ (6) | +1 | G (7) | +3 | B♭ (10) | +1 | B (11) | +3 | D (14) | +1 | E♭ (15) | +3 | G♭ (18) |
| | | F (5) | -1 | E (4) | +5 | A (9) | -1 | A♭ (8) | +5 | C♯ (13) | -1 | C (12) | +5 | F (17) |
| | | D (2) | +1 | D♯ (3) | +3 | F♯ (6) | +1 | G (7) | +3 | B♭ (10) | +1 | B (11) | +3 | D (14) |
| P-int sum | | | 1 | | 11 | | 1 | | 11 | | 1 | | 11 | |
| PC Space | | G♭ (6) | 1 | G (7) | 3 | B♭ (10) | 1 | B (11) | 3 | D (2) | 1 | E♭ (3) | 3 | G♭ (6) |
| | | F (5) | 11 | E (4) | 5 | A (9) | 11 | A♭ (8) | 5 | C♯ (1) | 11 | C (0) | 5 | F (5) |
| | | D (2) | 1 | D♯ (3) | 3 | F♯ (6) | 1 | G (7) | 3 | B♭ (10) | 1 | B (11) | 3 | D (2) |
| PC-int sum (mod 12) | | | 1 | | 11 | | 1 | | 11 | | 1 | | 11 | |
| Sum Class | | 1 | | 2 | | 1 | | 2 | | 1 | | 2 | | 1 |

**b) sc (026) - cycle with sets of sum classes 1 and 2**



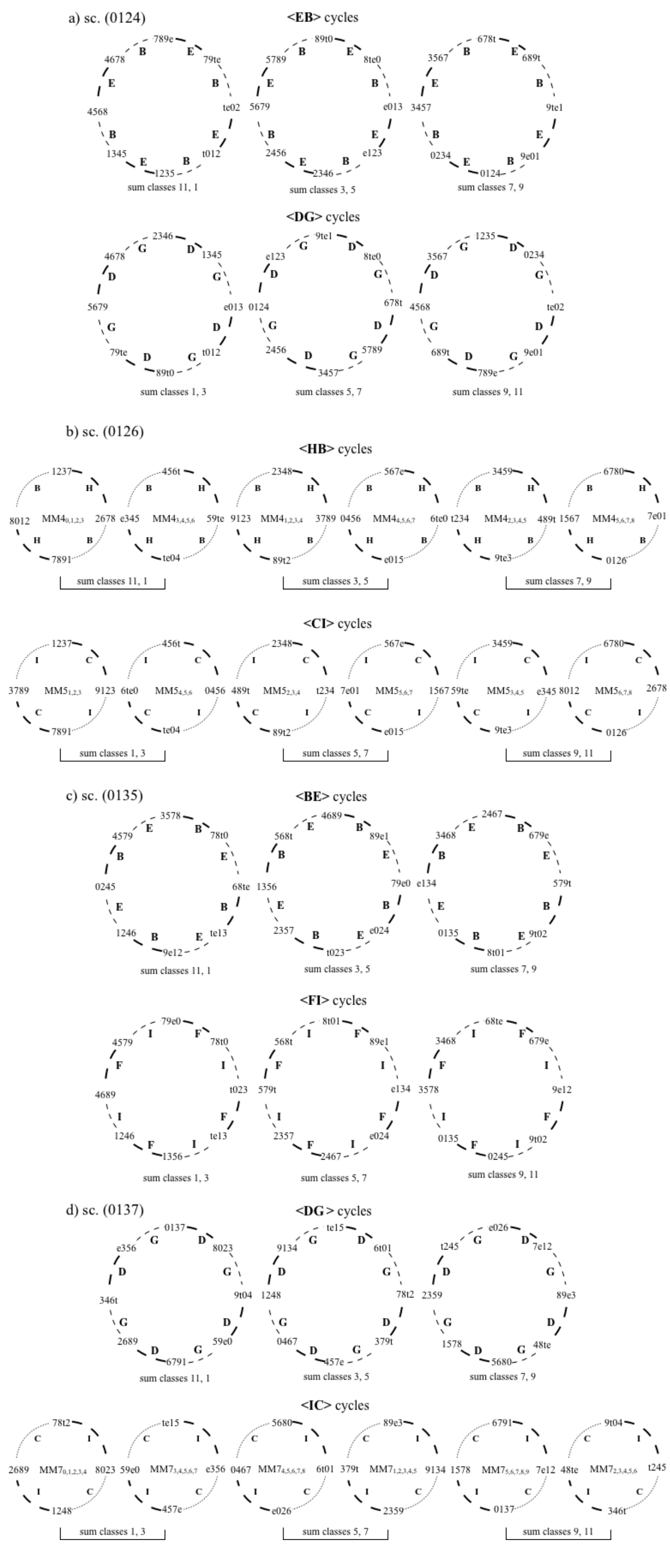
| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | F | | J | | F | | J | | F | | J | |
| Sets | | [e,3,5] | | [0,2,6] | | [3,7,9] | | [4,6,t] | | [7,e,1] | | [8,t,2] | | [e,3,5] |
| Pitch Space | | F (5) | +1 | F♯ (6) | +3 | A (9) | +1 | B♭ (10) | +3 | D♭ (13) | +1 | D (14) | +3 | F (17) |
| | | D♯ (3) | -1 | D (2) | +5 | G (7) | -1 | G♭ (6) | +5 | B (11) | -1 | B♭ (10) | +5 | D♯ (15) |
| | | B (-1) | +1 | C (0) | +3 | E♭ (3) | +1 | E (4) | +3 | G (7) | +1 | G♯ (8) | +3 | B (11) |
| P-int sum | | | 1 | | 11 | | 1 | | 11 | | 1 | | 11 | |
| PC Space | | F (5) | 1 | F♯ (6) | 3 | A (9) | 1 | B♭ (10) | 3 | D♭ (1) | 1 | D (2) | 3 | F (5) |
| | | D♯ (3) | 11 | D (2) | 5 | G (7) | 11 | G♭ (6) | 5 | B (11) | 11 | B♭ (10) | 5 | D♯ (3) |
| | | B (-1) | 1 | C (0) | 3 | E♭ (3) | 1 | E (4) | 3 | G (7) | 1 | G♯ (8) | 3 | B (11) |
| PC-int sum (mod 12) | | | 1 | | 11 | | 1 | | 11 | | 1 | | 11 | |
| Sum Class | | 7 | | 8 | | 7 | | 8 | | 7 | | 8 | | 7 |

**c) sc (016) - cycle with sets of sum classes 7 and 8**



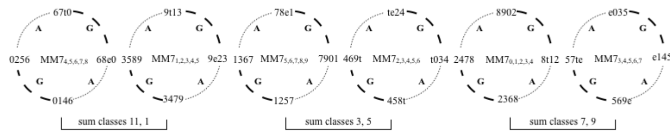
| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | B | | J | | B | | J | | B | | J | |
| Sets | | [0,1,6] | | [7,0,1] | | [4,5,t] | | [e,4,5] | | [8,9,2] | | [3,8,9] | | [0,1,6] |
| Pitch Space | | G♭ (6) | +1 | G (7) | -2 | F (5) | → | F (5) | +3 | A♭ (8) | → | A♭ (8) | -2 | G♭ (6) |
| | | D♭ (1) | → | D♭ (1) | +3 | E (4) | → | E (4) | -2 | D (2) | +1 | E♭ (3) | -2 | D♭ (1) |
| | | C (0) | → | C (0) | -2 | B♭ (-2) | +1 | B (-1) | -2 | A (-3) | → | A (-3) | +3 | C (0) |
| P-int sum | | | 1 | | -1 | | 1 | | -1 | | 1 | | -1 | |
| PC Space | | G♭ (6) | 1 | G (7) | 10 | F (5) | → | F (5) | 3 | A♭ (8) | → | A♭ (8) | 10 | G♭ (6) |
| | | D♭ (1) | → | D♭ (1) | 3 | E (4) | → | E (4) | 10 | D (2) | 1 | E♭ (3) | 10 | D♭ (1) |
| | | C (0) | → | C (0) | 10 | B♭ (10) | 1 | B (11) | 10 | A (9) | → | A (9) | 3 | C (0) |
| PC-int sum (mod 12) | | | 1 | | 11 | | 1 | | 11 | | 1 | | 11 | |
| Sum Class | | 7 | | 8 | | 7 | | 8 | | 7 | | 8 | | 7 |

**Figure 6:** *Examples of voice-leading in cycles with sc. (014), sc. (026) and sc. (016).*

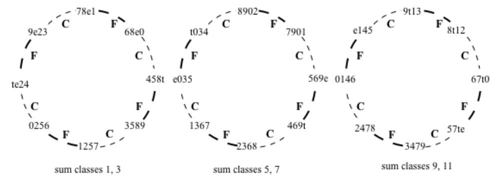

a) sc. (0124)
<EB> cycles
sum classes 11, 1
sum classes 3, 5
sum classes 7, 9
<DG> cycles
sum classes 1, 3
sum classes 5, 7
sum classes 9, 11
b) sc. (0126)
<HB> cycles
sum classes 11, 1
sum classes 3, 5
sum classes 7, 9
<CI> cycles
sum classes 1, 3
sum classes 5, 7
sum classes 9, 11
c) sc. (0135)
<BE> cycles
sum classes 11, 1
sum classes 3, 5
sum classes 7, 9
<FI> cycles
sum classes 1, 3
sum classes 5, 7
sum classes 9, 11
d) sc. (0137)
<DG> cycles
sum classes 11, 1
sum classes 3, 5
sum classes 7, 9
<IC> cycles
sum classes 1, 3
sum classes 5, 7
sum classes 9, 11

e) sc. (0146)

<GA> cycles

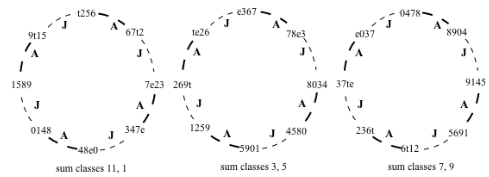



<FC> cycles

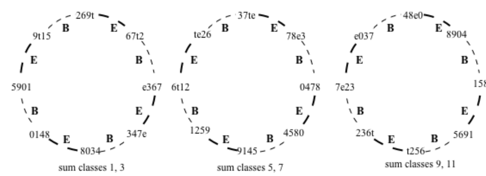



f) sc. (0148)

<AJ> cycles

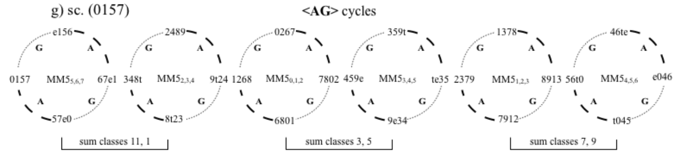



<EB> cycles

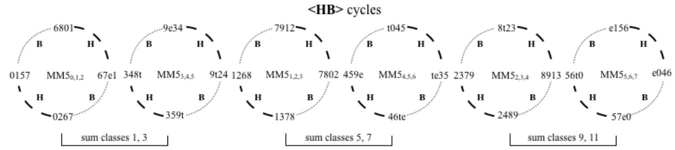



g) sc. (0157)

<AG> cycles

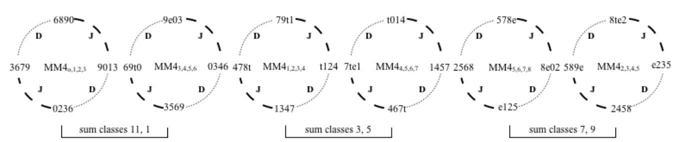



<HB> cycles

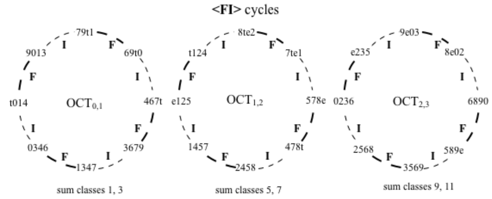



h) sc. (0236)

<JD> cycles




<FI> cycles



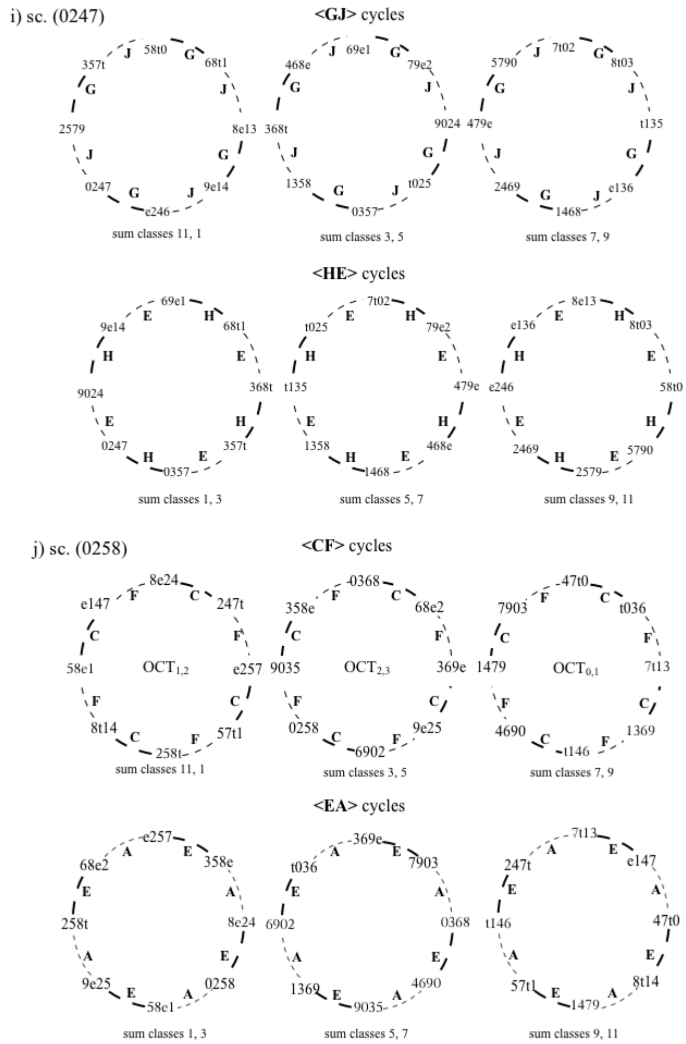



**Figure 7:** *Cycles for all target sets with cardinality 4.*

The bridge sets with cardinality 4 have its members in even sum classes and are divided in two groups: bridge sets I, which members are in sum classes 0, 4, and 8; bridge sets II, which members are in sum classes 2, 6, and 10. It is possible to build tetrachords cycles similar to those with trichords shown in Figure 10, in which all members of a target set are of the same class of sum. The sets of these cycles can be related by a chain of contextual inversions or by a chain of transpositions of 3 semitones. Table 2 provides the labels of the relation that connects bridge sets I and II with supersets of each cycle.

## V. Trees and other graphs with two different set classes

In addition to the cycles, *trees*, which are connected graphs containing no cycles ([4, p. 18]), are another type of graph used by the Neo-Riemannian Theory. The two most well known neo-Riemannian tree graphs are the Weitzmann graph and the Boretz Spiders. Cohn have designed these graphs using the chord symbols but, in order to keep the same criteria used in our previous

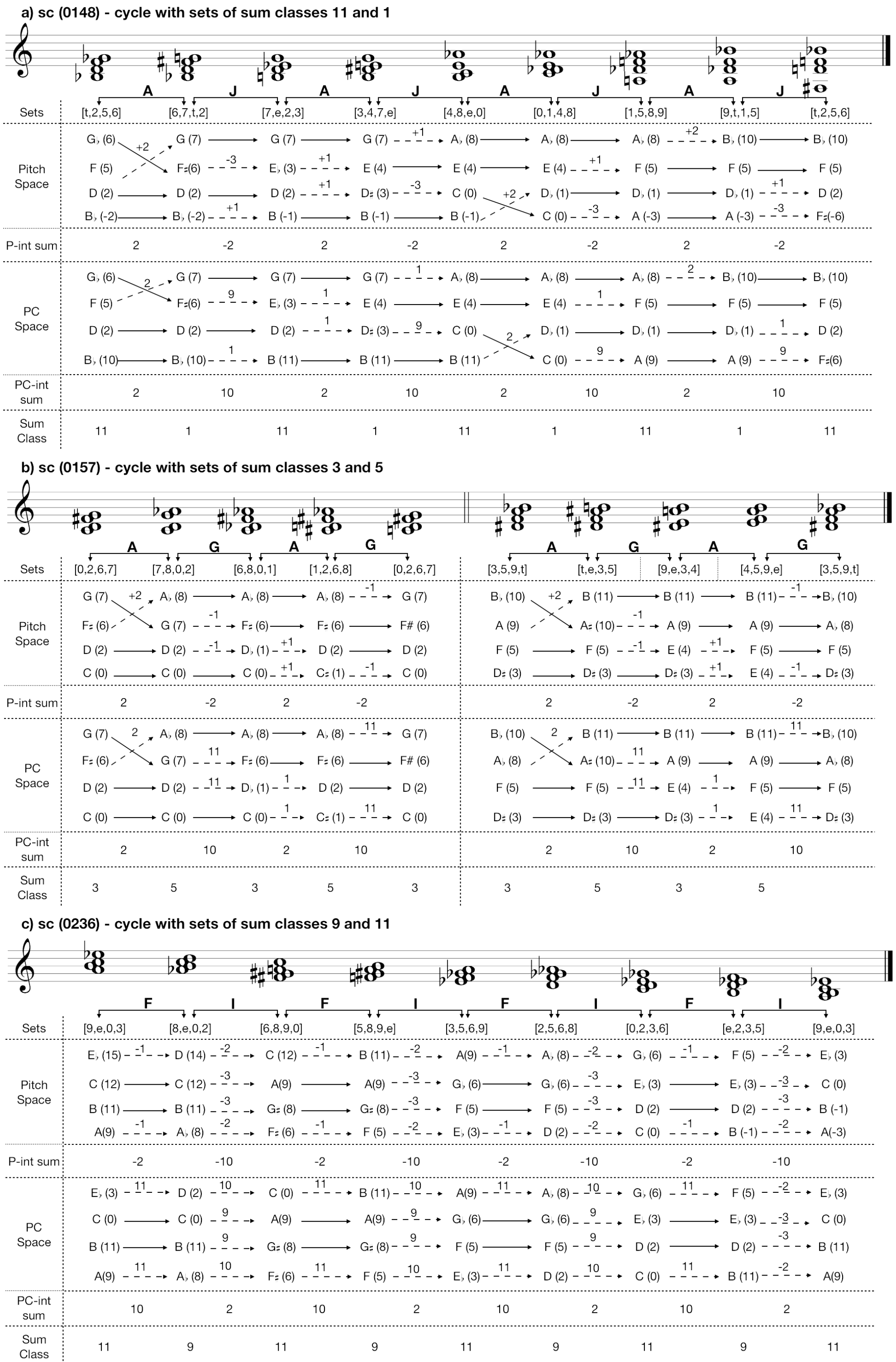
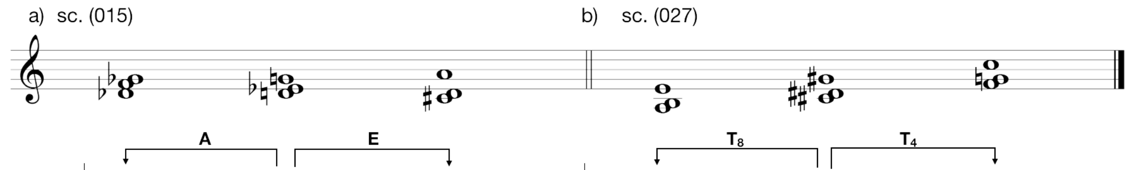



**Figure 8:** *Examples of voice-leading in cycles with sc. (0148), sc. (0157) and sc. (0236).*




| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | A | | E | | | $T_8$ | | $T_4$ | |
| Sets | [1,5,6] | | [2,3,7] | | [9,1,2] | [9,e,4] | | [1,3,8] | | [5,7,9] |
| Pitch Space | G♭ (6) | ← -1 | G (7) | +2 → | A (9) | E (4) | ← -4 | G♯ (8) | +4 → | A (9) |
| | F (5) | ← +2 | E♭ (3) | -1 → | D (2) | B (-1) | ← -4 | D♯ (3) | +4 → | G (7) |
| | D♭ (1) | ← -1 | D (2) | -1 → | C♯ (1) | A (-3) | ← -4 | C♯ (1) | +4 → | F (5) |
| P-int sum | | 0 | | 0 | | | -12 | | 12 | |
| Pitch-Class Space | G♭ (6) | ← 11 | G (7) | 2 → | A (9) | E (4) | ← 8 | G♯ (8) | 4 → | A (9) |
| | F (5) | ← 2 | E♭ (3) | 11 → | D (2) | B (11) | ← 8 | D♯ (3) | 4 → | G (7) |
| | D♭ (1) | ← 11 | D (2) | 11 → | C♯ (1) | A (9) | ← 8 | C♯ (1) | 4 → | F (5) |
| PC-int sum (mod 12) | | 0 | | 0 | | | 0 | | 0 | |
| Sum Class | 0 | | 0 | | 0 | 0 | | 0 | | 0 |

**Figure 9:** *Connections between members of sc. (015) and (027) that keep sets in the same sum class.*

graphs, they are reproduced in Figure 12 with all sets notated in the normal form. The nodes in the center of each graph is related to the pivot sets. They are connect with other sets by a gray solid line. The solid line, as before, represents a parsimonious connection between two graphs, and the gray color indicates that this connection is between members of different set classes.

Since these two graphs are subgraphs of both Cube Dance and Power Towers, and may therefore serve as a model to build other graphs, we will highlight some of their main features as we did previously with HexaCycles.

- **1) All the sets belong to two different set classes.** Both graphs have four components with a root (a node that is connect with all others) that is related to a pivot set (sc. (048) in Weitzmann graph and sc. (0369) in Boretz Spiders). All remaining nodes are related to a target sets (sc. (037) in Weitzmann graph and sc. (0258) in Boretz Spiders);
- **2) Each component is embedded in a symmetric collection.** The superset of each component in Weitzmann graph is the nonatonic collection listed below it, and the superset of each Boretz spider is the aggregated of 12 pitches;
- **3) All sets of each component belong to three adjacent sum classes.** All components of both graphs are limited to only three sum classes, with the pivot set of each component being in the sum class between the two sum classes of the target sets.
- **4) Parsimonious voice-leading between the pivot and the target sets.** Since the target sets do not connect to each other in either graph, all connections are between members of different sets of classes, which is a remarkable difference between tree graphs and cycles previously discussed. However, the target and the pivot sets in each graph were chosen because they connect parsimoniously.

We can use Weitzmann graphs and Boretz Spiders as models in order to build new graphs using different target sets, but since there are only one pivot set for cardinality 3 and one for

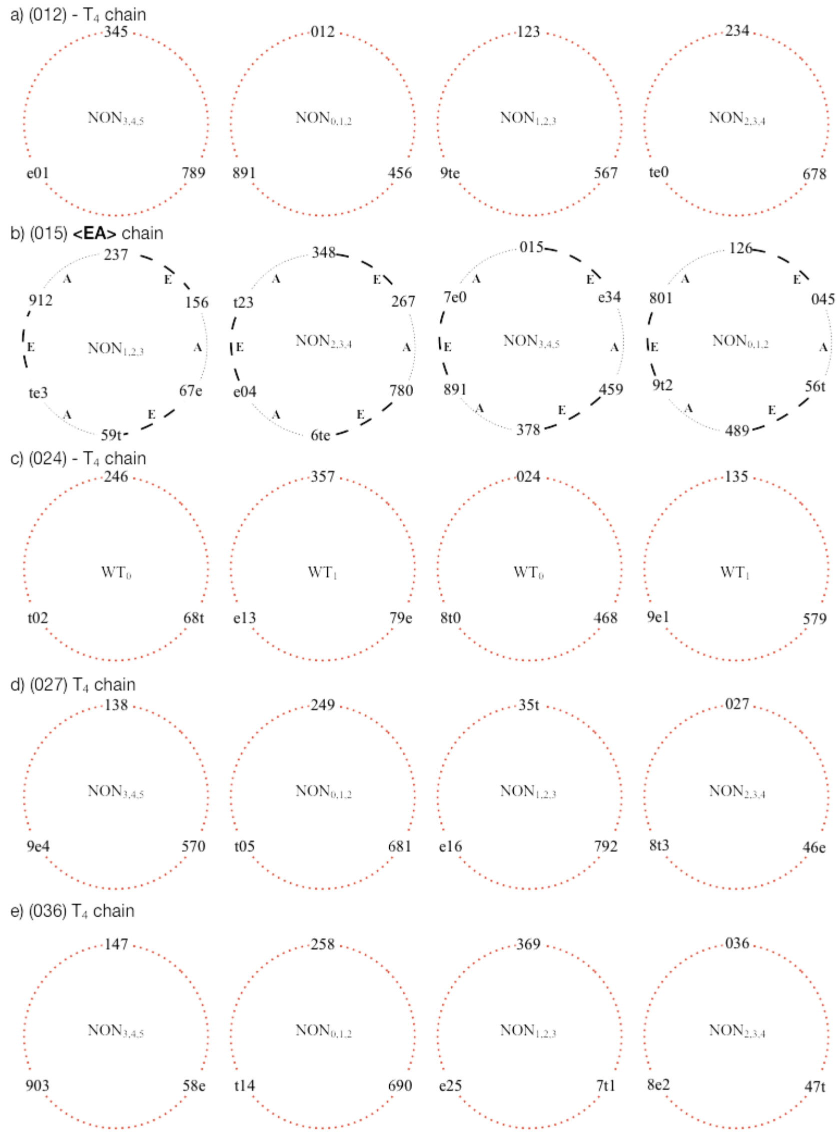



**Figure 10:** *Cycles for all pivot sets with cardinality 3 (solid black lines represent parsimonious connections, dashed and dotted black lines represent non-parsimonious connections, and red dotted lines represent T4 connections).*

cardinality 4 (sc. (048) and sc. (0369), respectively), we will have to replace them by bridge sets that can connect to these target sets by parsimonious voice-leading. As previously mentioned, the difference between the pivot and bridge sets is that the former make connections with every target set placed in their adjacent voice-leading zones, while the latter make connections with some, but not all target sets placed in their adjacent voice-leading zones. Because of this, we will replace in both trichord and tetrachord graphs the nodes of the pivot sets by cycles that we have previously built, and which connect bridge sets in the same sum class by transposition. In this way, each bridge set in the cycle will connect at least to two target sets (see the models for those graphs in Figure 13). Note that although we have used the graphs of Figure 14 as models, these cannot be considered as tree graphs because they include a cycle of bridge sets. In this way we can build

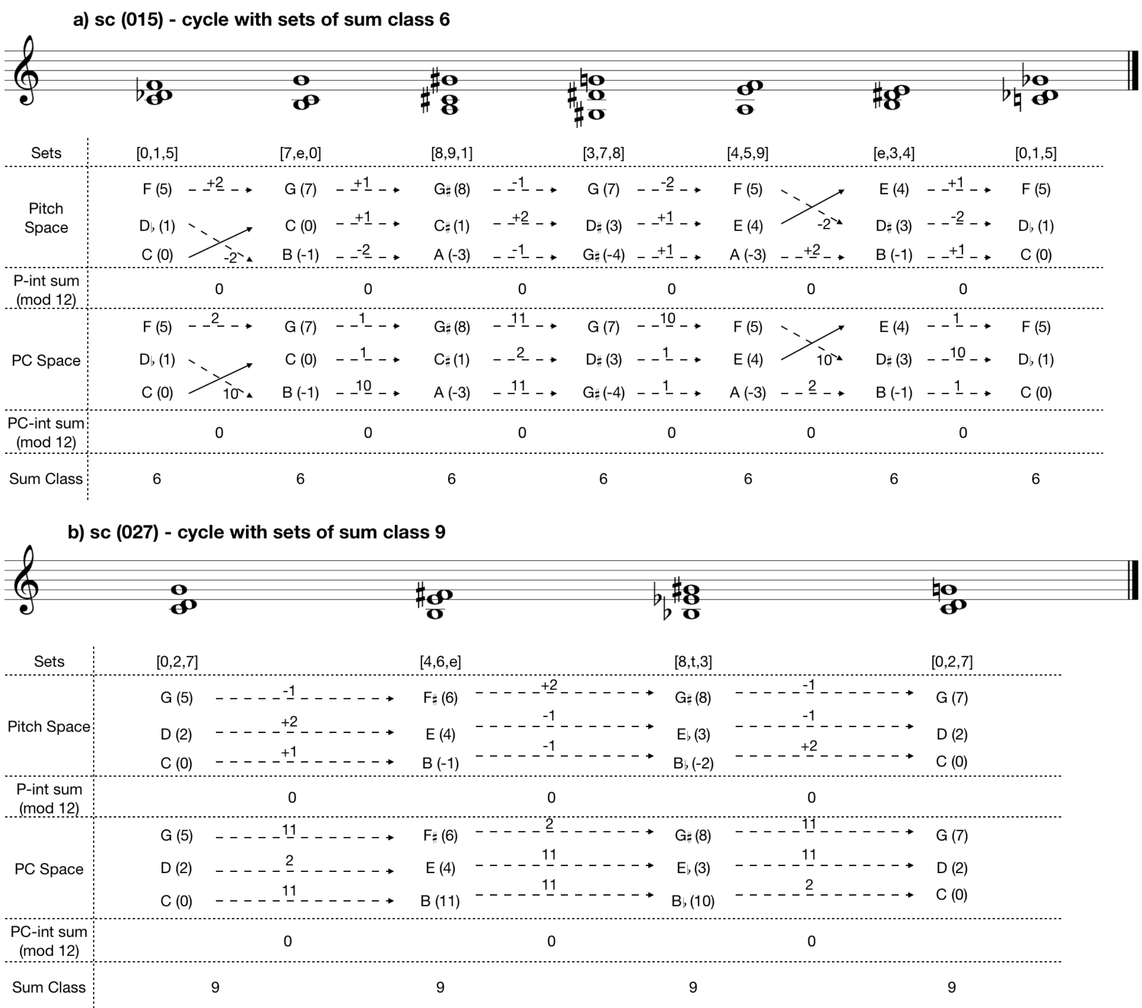



**Figure 11:** *Examples of voice-leading in cycles with sc. (015) and sc. (027).*

12 graphs that connect two sets of trichords by parsimony voice-leading. Those graphs include all set classes of cardinality 3. Each graph has four components and the sets of each one are in three adjacent sum classes listed below them. The supersets of those components are listed in the middle of the cycle, and if not, the superset is the aggregate of 12 pitches. The arrangement of the sets in all trichords graphs which sc. (015) is the bridge set is different from those with the symmetrical sc. (012), (024), (027), and (036), and therefore there are two different types of designs for the components of these graphs shown in Figure 13. Components of type 1 have a central cycle in which members of a inversional symmetric set class are connected by T4. Each bridge set is also connected to two target sets, so this type of component has 9 members. Components of type 2 have a central cycle in which members of sc. (015) are connected by contextual inversion. Each bridge set is also connected to one target set, so this type of component has 12 members.

Figure 14 shows all 12 graphs that connect two different sets of trichords. There are 8 with components of type 1 and 4 with components of type 2. I will represent the graphs using the prime form of both sets separated by a slash, the first is always the pivot or the bridge set while the second is the target set.

**Table 2:** *The table on the left provides information about tetrachord cycles with bridge sets I and, the table on the right provides information about tetrachord cycles with bridge sets II.*

| Bridge Sets I<br>Sum Classes 0, 4 and 8 | | | | Bridge Sets II<br>Sum Classes 2, 6 and 10 | | |
|---|---|---|---|---|---|---|
| Set Class | Cycle<br>(all sets in the same sum class) | Superset | | Set Class | Cycle<br>(all sets in the same sum class) | Superset |
| A (0125)<br>B (0345) | six cycles<br>**<HB>** chain | MM4 | | (0123) | three cycles<br>$T_3$ related | AGG |
| (0134) | three cycles<br>$T_3$ related | MM2 | | (0127) | three cycles<br>$T_3$ related | AGG |
| A (0147)<br>B (0367) | three cycles<br>**<AG>** chain | MM7 | | A (0136)<br>B (0356) | **<CF>** and **<FI>** chains<br>three cycles each | AGG |
| (0156) | three cycles<br>**<AG>** chain | AGG | | (0145) | three cycles<br>$T_3$ related | AGG |
| A (0237)<br>B (0457) | six cycles<br>**<JD>** chain | MM4 | | (0158) | three cycles<br>$T_3$ related | AGG |
| (0246) | three cycles<br>$T_3$ related | AGG | | (0167) | three cycles<br>$T_3$ related | MM2 |
| (0268) | three cycles<br>$T_3$ related | MM2 | | (0235) | three cycles<br>$T_3$ related | MM2 |
| (0358) | three cycles<br>$T_3$ related | MM2 | | (0248) | three cycles<br>$T_3$ related | AGG |
| | | | | (0257) | three cycles<br>$T_3$ related | AGG |
| | | | | (0347)<br>$T_3$ related | three cycles | MM2 |

Next we can describe the voice-leading between sets in two components of two different graphs. Figure 15a) is an example of voice-leading between sets of the third component (left to right) of graph 024/014 (shown in Figure 14b)), while Figure 15b) is an example of voice-leading between sets of the third component (left to right) of graph 015/026 (shown in Figure 14l)). One can see in the left side of Figure 15 the patch I choose to connect the sets on the graph and in the right side one of the possible voice-leadings for these sets. Note that I have used parsimony voice-leading for every connection between members of two different sets (red arrows in the graphs), while the members of the same set were connected by pure contrary motion (blue arrows).

There are 57 graphs that connect two different tetrachords sets by parsimony voice-leading, and each one has three components with its members in three adjacent sum classes listed below them. The components of those graphs are divided in 5 different types: components of type 1 have a central cycle whose members of a inversional and transpositional symmetric set classes are connected by $T_6$. Each one of the bridge sets is also connected to four target sets, so this type of component has 10 members. Components of type 2 have a central cycle whose members of a inversional symmetric set classes are connected by $T_3$. Each one of the bridge set is also connected to two target sets, so this type of component has 12 members. Components of type 3 have a central cycle whose members are connected by contextual inversion. Each one of the bridge set is also connected to one target set, so this type of component has 16 members arranged in two subtypes

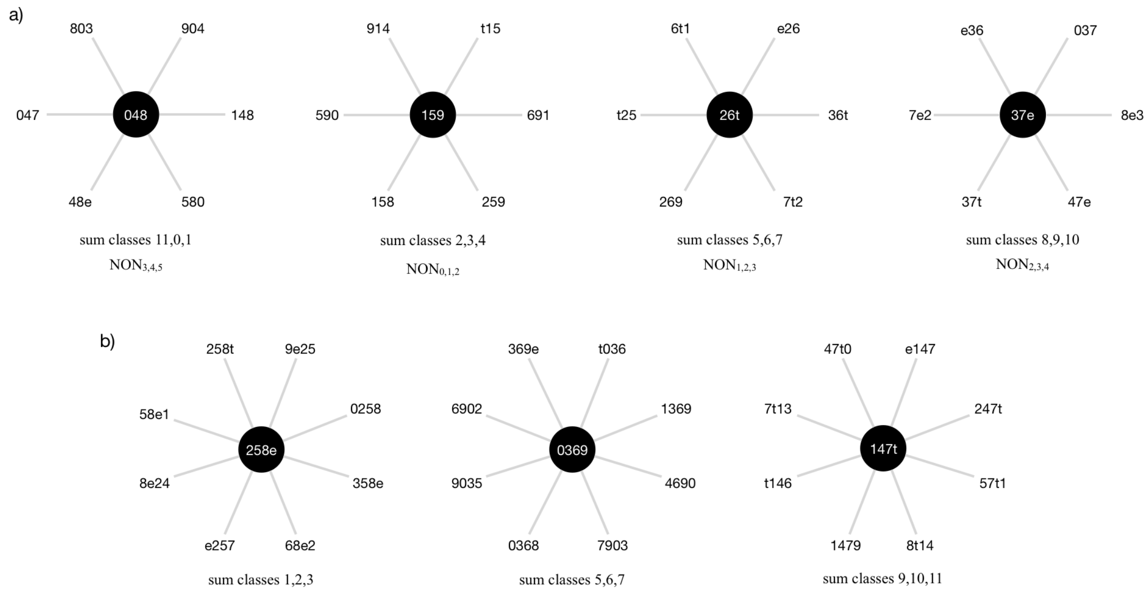



**Figure 12:** *a) Weitzmann graph; b) Boretz Spiders.*

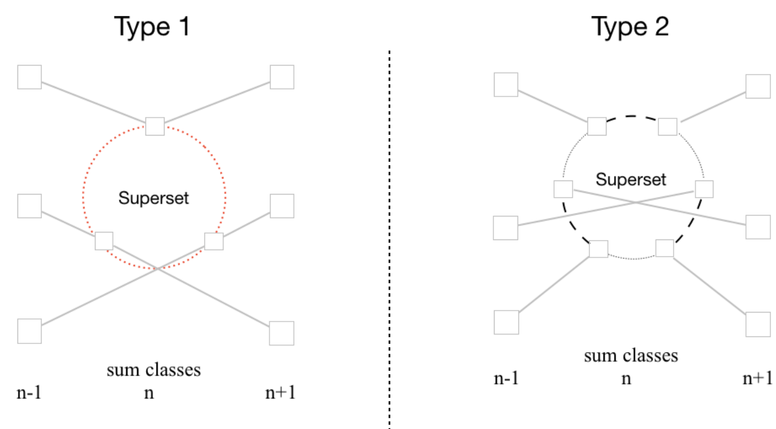
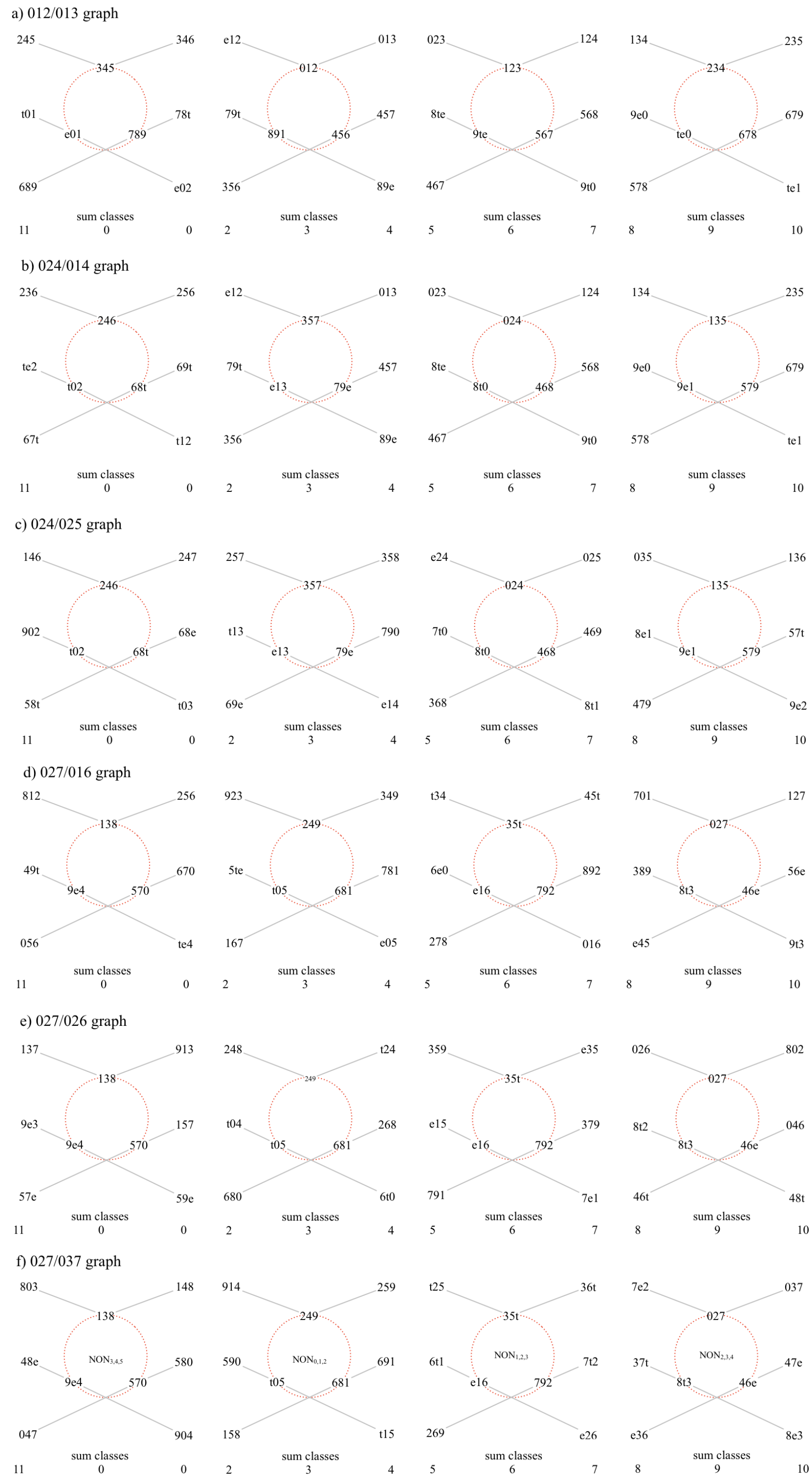



**Figure 13:** *Two types of components in graphs that connect two sets of trichords.*

of components, 3a and 3b. Each component of type 4 is divided in two cycles in which 4 members are connect by contextual inversion. Each one of the bridge set is also connected to one target set, so this type of component has 16 members arranged in two subtypes of components, 4a and 4b. There is only one graph with components of type 5, and we will analyze it separately below. Figure 16 shows the design of the components of type 1, 2, 3, and 4.

Figure 17 shows one example of graphs that connect two different tetrachords with each type and subtype of component. The graph of Figure 17a) shows three components of type 1 with members of sc. (0268) as bridge sets and members of sc. (0157) as target sets; Figure 17b) shows a graph divided in components of type 2 with members of sc. (0127) as bridge sets and members of sc. (0126) as target sets; Figure 17c) shows a graph divided in components of type 3a with members of sc. (0136) as bridge sets and members of sc. (0126) as target sets; Figure 17d) shows a graph divided in components of type 3b with members of sc. (0136) as bridge sets and members

Type 1



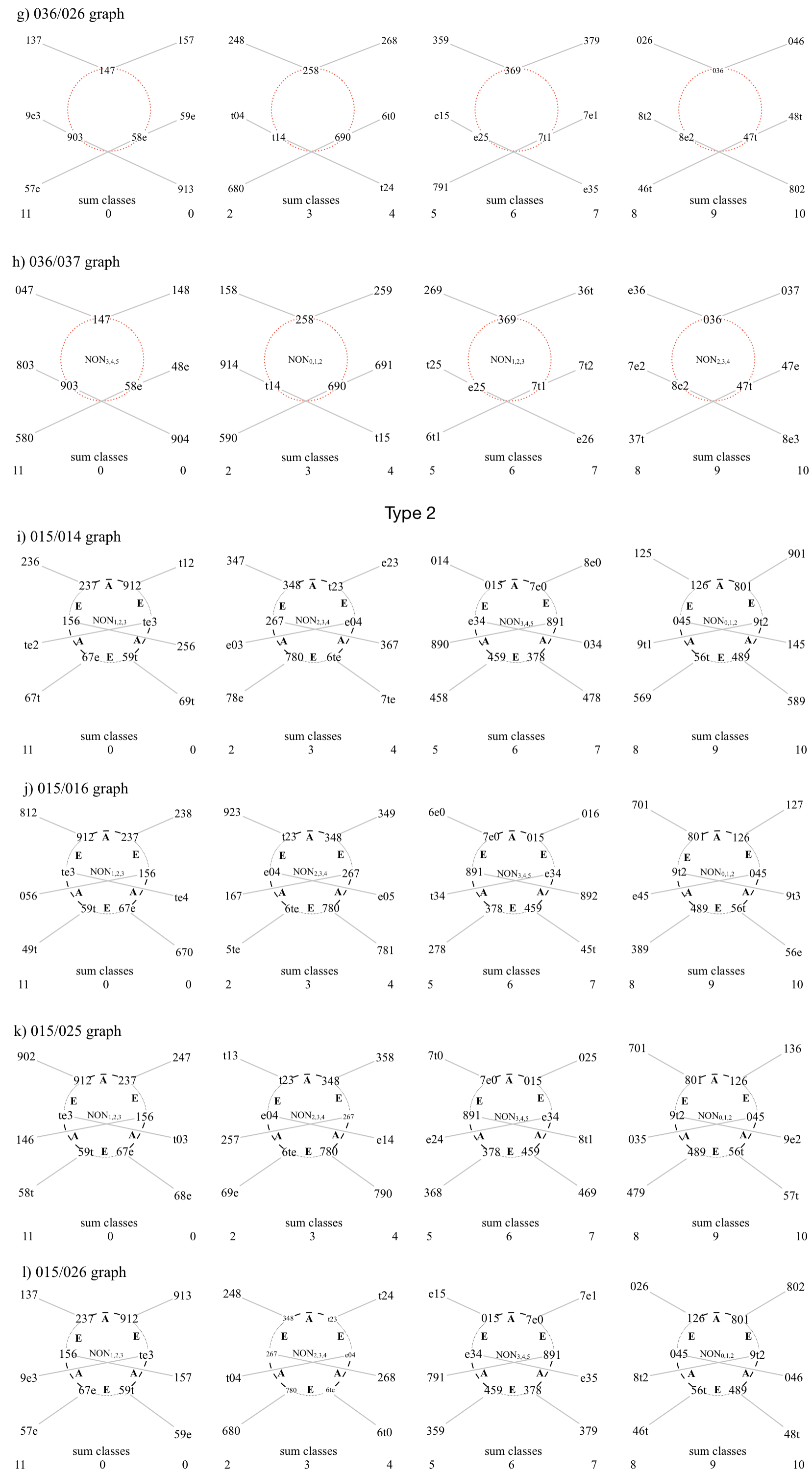



**Figure 14:** *Graphs that connect two sets of trichords by parsimony voice-leading.*

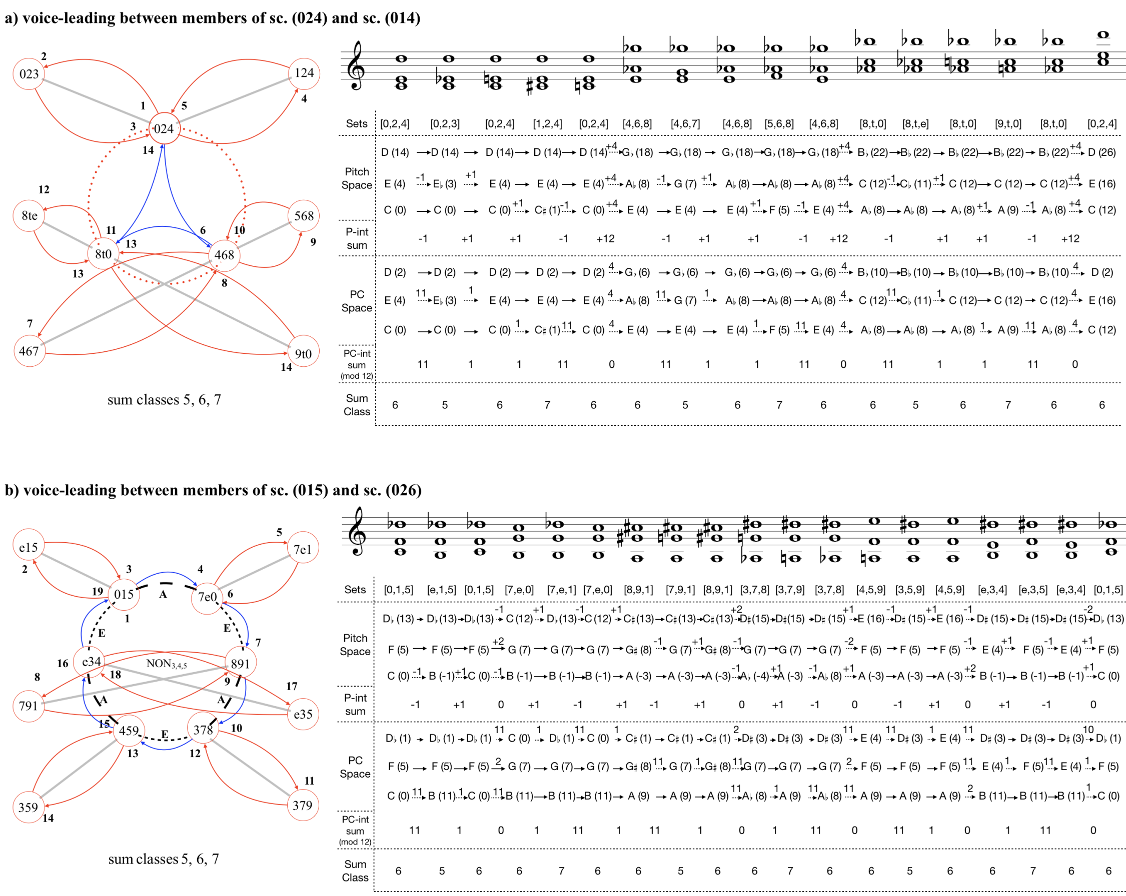



**Figure 15:** *Examples of voice-leading for sets of the 024/014 and 015/026 graphs.*

of sc. (0236) as target sets; Figure 17e) shows a graph divided in components of type 4a with members of sc. (0237) as bridge sets and members of sc. (0236) as target sets; Figure 17f) shows a graph divided in components of type 4b with members of sc. (0147) as bridge sets and members of sc. (0146) as target sets.

In Table 3 of Appendix are listed all 57 graphs that connect two different tetrachords. In it all the graphs are divided by the type of its components, which are listed in the left column. In the middle column, the set classes of the graphs are listed. The first is the bridge set, and the second is the target set on the chart. It can be seen in the right column, in which sum classes the members of the three components of each graph are distributed.

Graph 0358/0258 is the only one with components of type 5. This graph is similar to the Douthett/Steinbach's OctaTowers ([3, p. 246, Fig. 4]), but Figure 18 shows it with an alternative design and with the normal forms for each set instead the chord symbols in order to keep same appearance in all graphs. Therefore, this graph has three components with a central cycle whose members of sc. (0358) are connected by $T_3$. Each member of this set class is also connected to four members of sc. (0258), so all the components have 12 members.

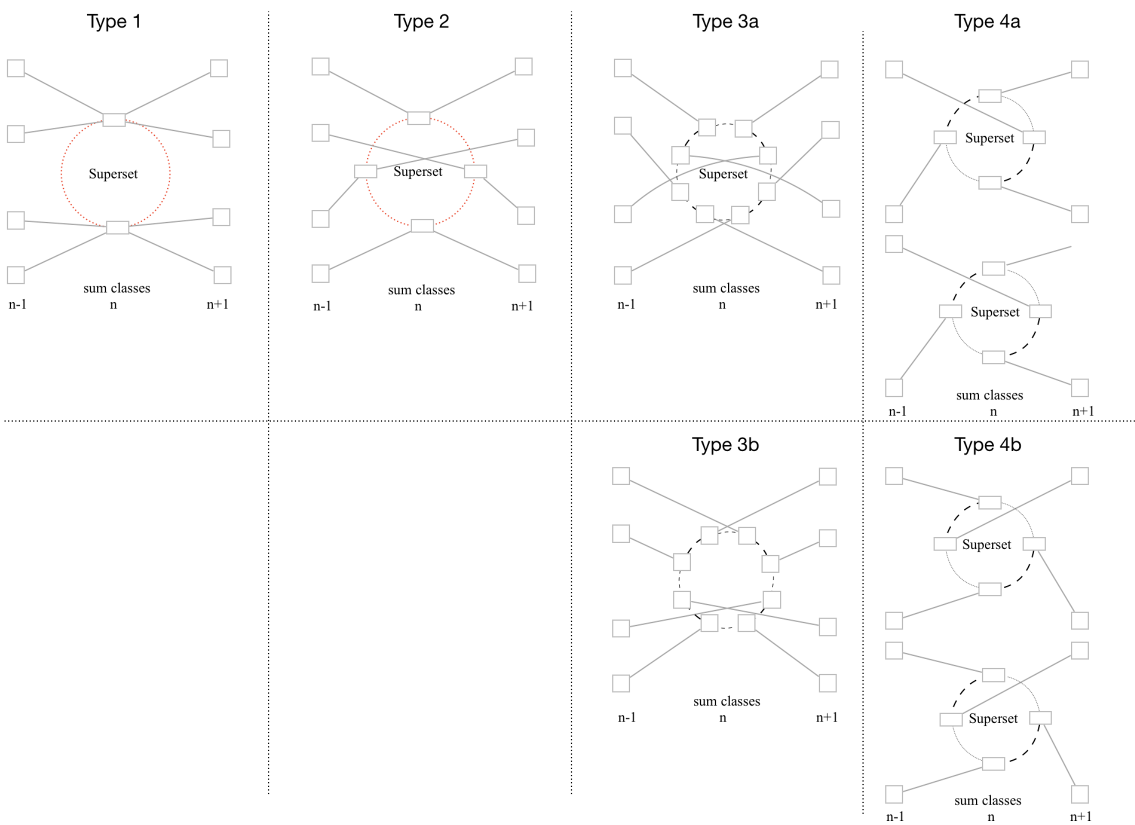



**Figure 16:** *Components of type 1,2,3, and 4 for graphs that connect two sets of tetrachords.*

## VI. Unified Models of Voice Leading Space

Richard Cohn refers to the graph known as Cube Dance ([2, p. 83, Fig. 5.24]) as a "unified model of triadic voice-leading space" ([2, p. 83]) since it "includes the four hexatonic cycles and the four Weitzmann regions as contiguous subgraphs" ([2, p. 85, Fig. 5.24]). I will refer to this graph as 048/037 Cube Dance, since members of sc. (048) are the pivot sets and members of sc. (037) are the target sets in it. Figure 19 shows 048/037 Cube Dance with some changes to the original graph: the sets are represented by its normal forms instead of the chord symbols; the black lines represent connections between members of the same set class and the gray lines represent connections between members of different set classes; thickest black lines represent connections using the contextual inversion axis **H**, while the thinner black lines represent connections using the contextual inversion axis **D**.

Next we list the main features of the original Cube Dance, so that these can be used to build other graphs as done above with the cycles and with tree graphs.

- **1) All the sets in the graph belong to two different set classes.** As observed by Cohn, Cube Dance has the hexatonic cycles and the Weitzmann regions as contiguous subgraphs. These subgraphs are arranged in order to make four cubes with members of sc. (048) in the intersections between them, and members of sc. (037) in the remaining vertices.
- **2) Sets are placed in voice-leading zones.** The entire graph is placed inside a clock face. The sum of pitches modulo 12 of each set is equal to the number next to it. These numbers

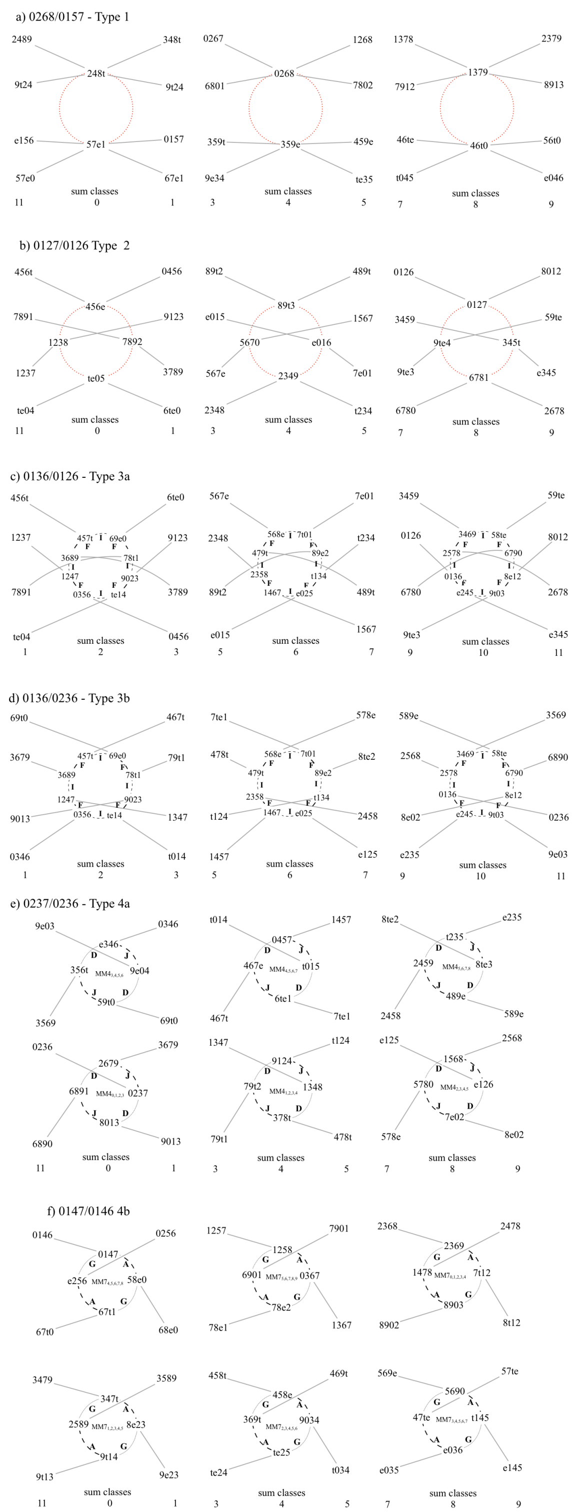


**Figure 17:** *Examples of graphs that connect two sets of tetrachords.*

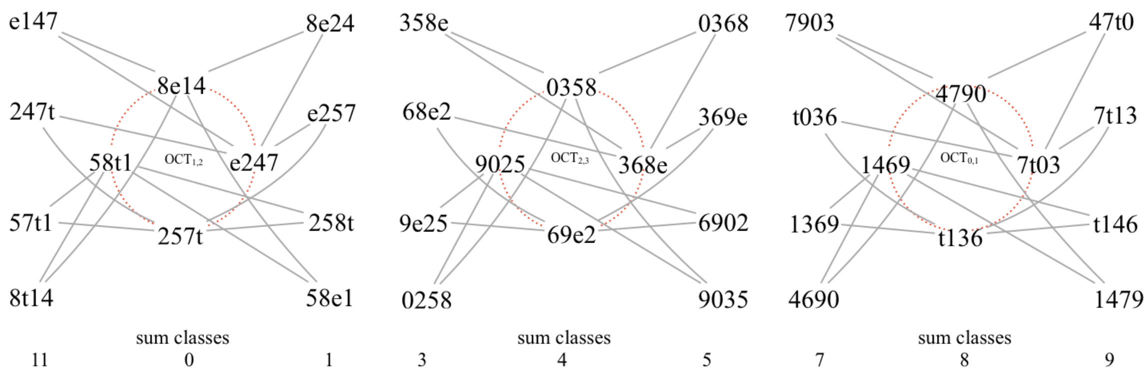



**Figure 18:** *The 0358/0258 graph is similar to Douthett/Steinbach's OctaTowers.*

are the voice-leading zones. In order words, the sets in Cube Dance are organized in adjacent sum classes.

- **3) Each cube has a symmetrical superset.** All the sets in each cube are embedded in a particular hexatonic collection. The sets in northeast cube are in $HEX_{0,1}$; sets in southeast cube are $HEX_{1,2}$; sets in southwest cube are $HEX_{2,3}$; and sets in northwest cube are $HEX_{3,4}$.
- **4) Parsimonious voice-leading between all connected sets.** Any connection between two sets in Cube Dance is parsimonious, regardless of whether the sets belong to the same set class or not. So it is possible to make a path that connects several chords with a voice-leading work that uses only displacements of one semitone.

We can build Cube Dances using trichords other than members of sc. (048) and (037). They will share the first three features of the original Cube Dance listed before[16]. As seen in cycles of trichords shown in Figure 5, the only set class in which members can relate parsimoniously by contextual inversion to two other members is sc. (037). So in order to build Cube Dances with different trichords one must admit that the voice-leading between zones 1-2, 4-5, 7-8, and 10-11 may not necessarily be parsimonious, although it connects members of adjacent sum classes.

Figure 20 shows the 024/014 Cube Dance, a graph that connects all members of these two set classes with similar patterns of voice-leading throughout their vertices. In this new Cube Dance all the vertices where were the pivot sets (in zones 0, 3, 6, and 9) in the original Cube Dance were replaced by the four cycles with sc. (024) shown in Figure 10c), so the members of this set class are the bridge sets in the graph. In the zones 1-2, 4-5, 7-8, and 10-11, the hexatonic cycle were replaced by the four cycles with sc. (014) shown in Figure 5b), and the members of this set class are the target set in the graph. In other words, the 024/014 Cube Dance includes the 024/014 graph shown in Figure 14b) and the cycle of **<DH>** chain with sc. (014) as contiguous subgraphs. The red and the black lines in the graph represent connections between members of same set class, red lines connect them by transposition and black lines by contextual inversion. Gray lines connect members of different set classes. Solid lines represent connection with parsimonious voice-leading, while dashed and dotted lines represent connections using the contextual inversions axes **D** and **H**, respectively. The graph of Figure 20 gives an example of connection with this sets represented by the path with red arrows and circles. The voice leading between these sets are shown in Figure 21. Certainly there are countless other possible ways to connect these sets that can be traced in this Cube Dance.

Figure 22 shows the 027/016 Cube Dance. It includes the 027/016 graph, shown in Figure 14d),

[16] About the third feature of the list: the supersets of each cube in all graphs is the aggregate of the 12 pitches.

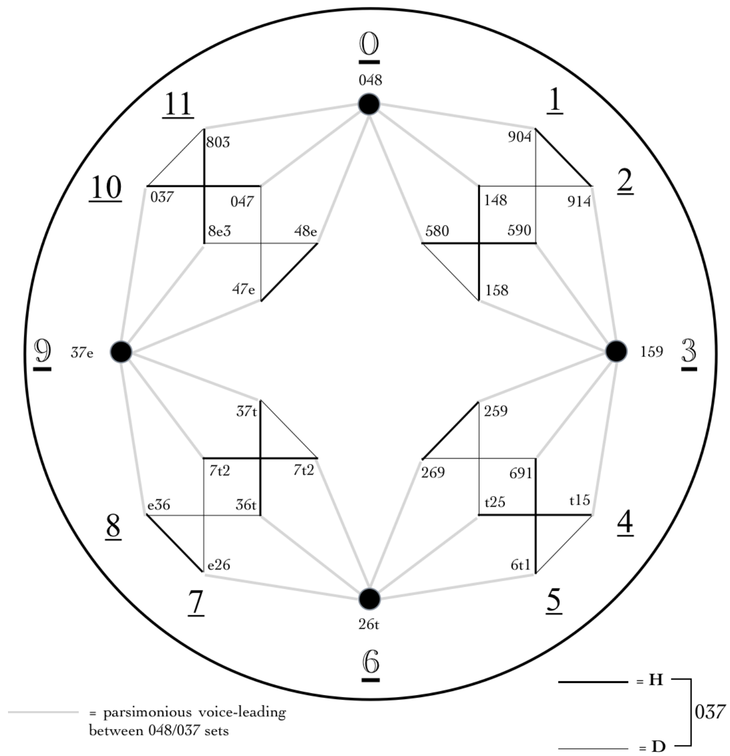



**Figure 19:** *048/037 Cube Dance.*

and the cycle of **<BJ>** chain with sc. (016) as contiguous subgraphs. The pattern of the connection lines is the same used in Figure 20, but the black solid and dotted lines represent connections using the contextual inversions axes **B** and **J**, respectively. The path drawn over this graph is an example of a connection that explores the parsimonious voice leading in this graph, as shows Figure 23.

Figure 24 shows the 015/025 Cube Dance. This graph has a different design from the previous ones, since members of sc. (015) are the bridge sets and, as previously observed, this set class is the only one in which its members are distributed in the voice-leading zones 0, 3, 6, and 9 and is not symmetrical. Thus, the vertices in the intersections between the four cubes in the original graph were replaced by the cycles of **<DH>** chain with sc. (015) shown in Figure 11b), and this is what changes the model of the graph, since cycles with 6 instead of 3 sets, are placed in those positions. That is, the Cube Dances of Figures 20 and 22 included graphs of type 1 as subgraphs between the voice-leading zones 11-0-1, 2-3-4, 5-6-7, and 8-9-10, while the 015/025 Cube Dance includes graphs of type 2 as subgraphs in these same positions (the types of graphs that connect two sets of trichords are shown in Figure 13). Another important difference between this graph and the previous ones is that in this Cube Dance there are no $T_4$ direct connections, and all the members of the same set class are connected by contextual inversion. It should be noted that members of sc. (015) are connect with only one target set in adjacent voice-leading zone, so to cross between target sets in two different cubes[17] it is necessary to connect with two members of

[17] By replacing the intersecting vertices of the original Cube Dance with cycles, the graph is deformed and the solids between the voice-leading zones 0-1-2-3, 3-4-5-6, 6-7-8-9, and 9-10-11-0 are no longer exactly cubes. However, I shall still use the term cube, considering that the intersection between these solids is a unit, even if it is represented by a cycle. In the same spirit, I shall still refer to these graphs as Cube Dance.

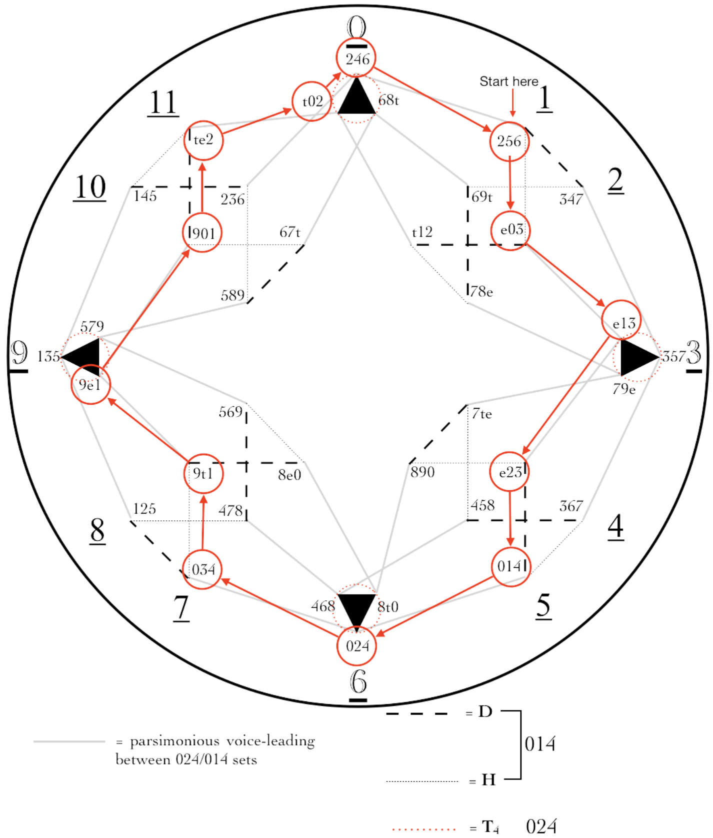
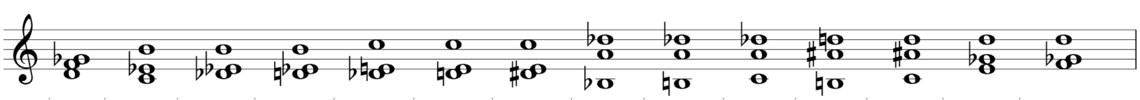



**Figure 20:** *024/014 Cube Dance.*



| Sets | [2,5,6] | [e,0,3] | [e,1,3] | [e,2,3] | [0,1,4] | [0,2,4] | [0,3,4] | [9,t,1] | [9,e,1] | [9,0,1] | [t,e,2] | [t,0,2] | [2,4,6] | [2,5,6] |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Pitch Space | G♭ (6) | $\overset{+5}{\dashrightarrow}$ B (11) | → B (11) | → B (11) | $\overset{+1}{\dashrightarrow}$ C (12) | → C (12) | → C (12) | $\overset{+1}{\dashrightarrow}$ D♭ (13) | → D♭ (13) | → D♭ (13) | $\overset{+1}{\dashrightarrow}$ D (14) | → D (14) | → D (14) | → D (14) |
| | F (5) | $\overset{-2}{\dashrightarrow}$ E♭ (3) | → E♭ (3) | → E♭ (3) | $\overset{+1}{\dashrightarrow}$ E (4) | → E (4) | → E (4) | $\overset{+5}{\dashrightarrow}$ A (9) | → A (9) | → A (9) | $\overset{+1}{\dashrightarrow}$ A♯ (10) | → A♯ (10) | $\overset{-4}{\dashrightarrow}$ G♭ (6) | → G♭ (6) |
| | D (2) | $\overset{-2}{\dashrightarrow}$ C (0) | $\overset{+1}{\dashrightarrow}$ D♭ (1) | $\overset{+1}{\dashrightarrow}$ D (2) | $\overset{-1}{\dashrightarrow}$ D♭ (1) | $\overset{+1}{\dashrightarrow}$ D (2) | $\overset{+1}{\dashrightarrow}$ D♯ (3) | $\overset{-5}{\dashrightarrow}$ B♭ (-2) | $\overset{+1}{\dashrightarrow}$ B (-1) | $\overset{+1}{\dashrightarrow}$ C (0) | $\overset{-1}{\dashrightarrow}$ B (-1) | $\overset{+1}{\dashrightarrow}$ C (0) | $\overset{+4}{\dashrightarrow}$ E (4) | $\overset{+1}{\dashrightarrow}$ F (5) |
| P-int Sum | | +1 | +1 | +1 | +1 | +1 | +1 | +1 | +1 | +1 | +1 | +1 | 0 | +1 |
| PC Space | G♭ (6) | $\overset{5}{\dashrightarrow}$ B (11) | → B (11) | → B (11) | $\overset{1}{\dashrightarrow}$ C (0) | → C (0) | → C (0) | $\overset{1}{\dashrightarrow}$ D♭ (1) | → D♭ (1) | → D♭ (1) | $\overset{1}{\dashrightarrow}$ D (2) | → D (2) | → D (2) | → D (2) |
| | F (5) | $\overset{10}{\dashrightarrow}$ E♭ (3) | → E♭ (3) | → E♭ (3) | $\overset{1}{\dashrightarrow}$ E (4) | → E (4) | → E (4) | $\overset{5}{\dashrightarrow}$ A (9) | → A (9) | → A (9) | $\overset{1}{\dashrightarrow}$ A♯ (10) | → A♯ (10) | $\overset{8}{\dashrightarrow}$ G♭ (6) | → G♭ (6) |
| | D (2) | $\overset{10}{\dashrightarrow}$ C (0) | $\overset{1}{\dashrightarrow}$ D♭ (1) | $\overset{1}{\dashrightarrow}$ D (2) | $\overset{11}{\dashrightarrow}$ D♭ (1) | $\overset{1}{\dashrightarrow}$ D (2) | $\overset{1}{\dashrightarrow}$ D♯ (3) | $\overset{7}{\dashrightarrow}$ B♭ (10) | $\overset{1}{\dashrightarrow}$ B (11) | $\overset{1}{\dashrightarrow}$ C (0) | $\overset{11}{\dashrightarrow}$ B (11) | $\overset{1}{\dashrightarrow}$ C (0) | $\overset{4}{\dashrightarrow}$ E (4) | $\overset{1}{\dashrightarrow}$ F (5) |
| PC-int sum (Mod 12) | | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 |
| Sum Class | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 0 | 0 | 1 |

**Figure 21:** *An example of voice-leading for the path shown over the 024/014 Cube Dance in Figure 20.*

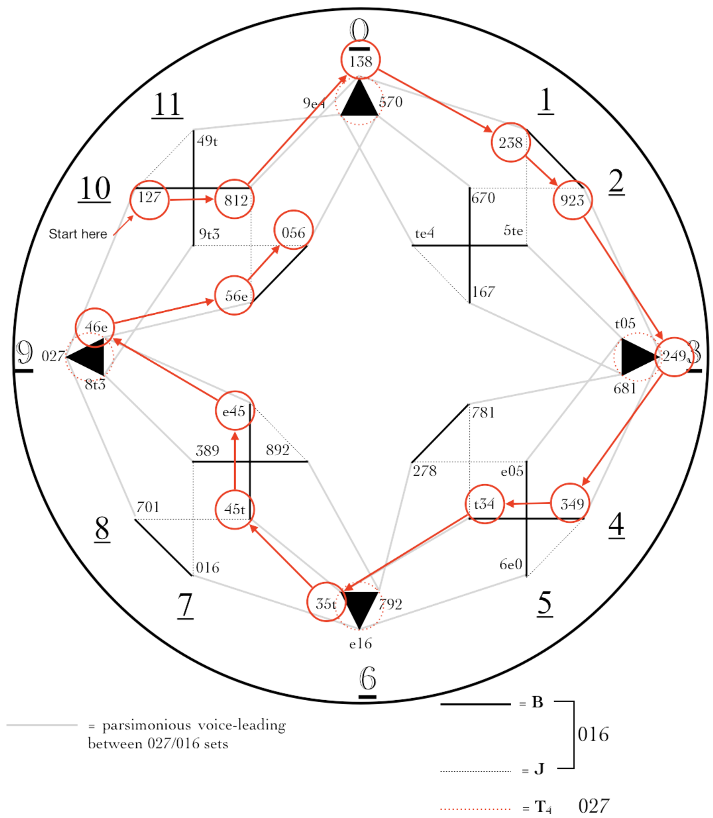
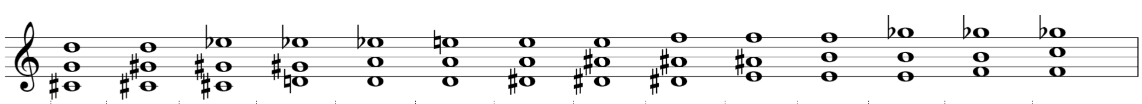



**Figure 22:** *027/016 Cube Dance.*



| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Sets | [1,2,7] | [8,1,2] | [1,3,8] | [2,3,8] | [9,2,3] | [2,4,9] | [3,4,9] | [t,3,4] | [3,5,t] | [4,5,t] | [e,4,5] | [4,6,e] | [5,6,e] | [0,5,6] |
| Pitch Space | D (14) | D (14) | E♭ (15) | E♭ (15) | E♭ (15) | E (16) | E (16) | E (16) | F (17) | F (17) | F (17) | G♭ (18) | G♭ (18) | G♭ (18) |
| | G (7) | G♯ (8) | G♯ (8) | G♯ (8) | A (9) | A (9) | A (9) | A♯ (10) | A♯ (10) | A♯ (10) | B (11) | B (11) | B (11) | C (12) |
| | C♯ (1) | C♯ (1) | C♯ (1) | D (2) | D (2) | D (2) | D♯ (3) | D♯ (3) | D♯ (3) | E (4) | E (4) | E (4) | F (5) | F (5) |
| P-int Sum | +1 | +1 | +1 | +1 | +1 | +1 | +1 | +1 | +1 | +1 | +1 | +1 | +1 | |
| PC Space | D (2) | D (2) | E♭ (3) | E♭ (3) | E♭ (3) | E (4) | E (4) | E (4) | F (5) | F (5) | F (5) | G♭ (6) | G♭ (6) | G♭ (6) |
| | G (7) | G♯ (8) | G♯ (8) | G♯ (8) | A (9) | A (9) | A (9) | A♯ (10) | A♯ (10) | A♯ (10) | B (11) | B (11) | B (11) | C (0) |
| | C♯ (1) | C♯ (1) | C♯ (1) | D (2) | D (2) | D (2) | D♯ (3) | D♯ (3) | D♯ (3) | E (4) | E (4) | E (4) | F (5) | F (5) |
| PC-int sum (Mod 12) | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | |
| Sum Class | 10 | 11 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |

**Figure 23:** *An example of voice-leading for the path shown over the 027/016 Cube Dance in Figure 22.*

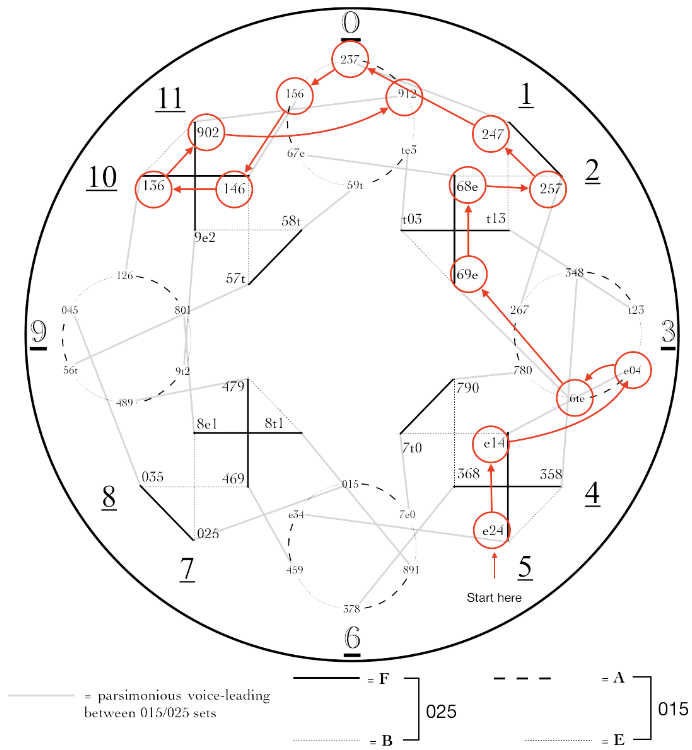



**Figure 24:** *015/025 Cube Dance.*

sc. (015). The path drawn over this Cube Dance gives an example of how the voice-leading can be with these sets. Figure 25 shows the details of this example.

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Sets | [e,2,4] | [e,1,4] | [e,0,4] | [6,t,e] | [6,9,e] | [6,8,e] | [2,5,7] | [2,4,7] | [2,3,7] | [1,5,6] | [1,4,6] | [1,3,6] | [9,0,2] | [9,e,2] |
| Pitch Space | D (14) | -1 ⇢ C♯ (11) | -1 ⇢ C (12) | -2 ⇢ A♯ (10) | -1 ⇢ A (9) | -1 ⇢ G♯ (8) | -1 ⇢ G (7) | → G (7) | → G (7) | +1 ⇢ G♭ (6) | → G♭ (6) | → G♭ (6) | +3 ⇢ A (9) | → A (9) |
| | E (4) | → E (4) | → E (4) | +2 ⇢ F♯ (6) | → F♯ (6) | → F♯ (6) | -1 ⇢ F (5) | -1 ⇢ E (4) | +1 ⇢ E♭ (3) | ⇢ F (5) | -1 ⇢ E (4) | -1 ⇢ E♭ (3) | -1 ⇢ D (2) | → D (2) |
| | B (-1) | → B (-1) | → B (-1) | → B (-1) | → B (-1) | → B (-1) | +3 ⇢ D (2) | → D (2) | → D (2) | -1 ⇢ C♯ (1) | → C♯ (1) | → C♯ (1) | -1 ⇢ C (0) | -1 ⇢ B (-1) |
| P-int Sum | | -1 | -1 | 0 | -1 | -1 | +1 | -1 | +1 | 0 | -1 | -1 | +1 | -1 |
| PC Space | D (2) | 11 ⇢ C♯ (1) | 11 ⇢ C (0) | 10 → A♯ (10) | 11 ⇢ A (9) | 11 ⇢ G♯ (8) | 11 ⇢ G (7) | → G (7) | → G (7) | 11 ⇢ G♭ (6) | → G♭ (6) | → G♭ (6) | 3 ⇢ A (9) | → A (9) |
| | E (4) | → E (4) | → E (4) | 2 ⇢ F♯ (6) | → F♯ (6) | → F♯ (6) | 11 ⇢ F (5) | 11 ⇢ E (4) | 11 ⇢ E♭ (3) | 2 ⇢ F (5) | 11 ⇢ E (4) | 11 ⇢ E♭ (3) | 11 ⇢ D (2) | → D (2) |
| | B (11) | → B (11) | → B (11) | → B (11) | → B (11) | → B (11) | 3 ⇢ D (2) | → D (2) | → D (2) | 11 ⇢ C♯ (1) | → C♯ (1) | → C♯ (1) | 11 ⇢ C (0) | 1 ⇢ B (11) |
| PC-int sum (Mod 12) | | 11 | 11 | 0 | 11 | 11 | 1 | 11 | 11 | 0 | 11 | 11 | 1 | 11 |
| Sum Class | 5 | 4 | 3 | 3 | 2 | 1 | 2 | 1 | 0 | 0 | 11 | 11 | 10 | 11 |

**Figure 25:** *An example of voice-leading for the path shown over the 015/025 Cube Dance in Figure 24.*

It is possible to build fourteen Cube Dances combining trichord cycles shown in Figure 5 and graphs that connect two trichords shown Figure 14[18]. In Table 4 of the Appendix, a list of all these Cube Dances with its subgraphs and supersets is given.

We can call the Douthett/Steinbach's Power Towers ([3, p. 256, Fig. 10]) a unified model too, since it includes the Boretz Regions and the OctaTowers as subgraphs, and the authors themselves state that "Power Towers is the seventh chord analog to Cube Dance" ([3, p. 255]). I will refer to it as 0369/0358/0258 Power Towers, since members of sc. (0369) in voice-leading zones 2, 6, and 10 are the pivot sets, members of sc. (0358) in voice-leading zones 0, 4, and 8 are the bridge sets, and members of sc. (0258) in the odd voice-leading zones are the target set in this graph. Figure 26 shows the 0369/0358/0258 Power Towers[19] with some changes to the original graph: as done before with the Cube Dances, the sets are represented by their normal forms instead the chord symbols; the OctaTowers are represented as the graph shown in Figure 18; the dotted red lines represent connection between members of same set class related by $T_3$; the solid gray lines represent connections between members of different set class by parsimony voice-leading (there are no black lines because members of same set class are not connect by contextual inversion in this graph).

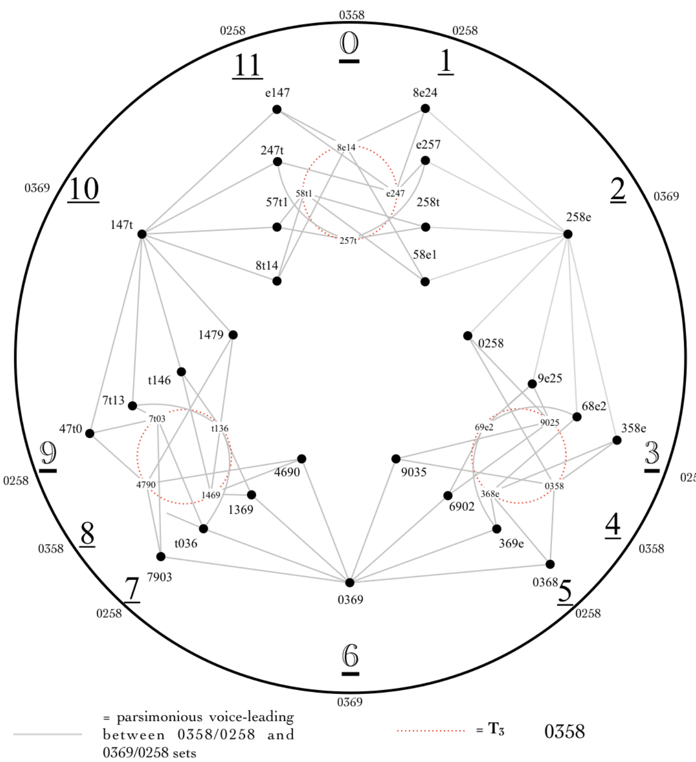



**Figure 26:** *The 0369/0358/0258 Power Towers.*

We now list the main features of the original Power Dance, as done for the original Cube Dance

[18] In all fourteen Cube Dances we keep the parsimony voice-leading between the bridges and the target sets. However, it would be possible to build Cube Dances without this kind of the connection, which would considerably increase the number of possible graphs.

[19] In Cohn's Book there is a version of this graph that also includes the members of sc. (0268) in the same voice-leading zones of members of sc. (0358) which he calls "4-Cube Trio" ([2, p. 158, Fig. 7.16]). However, for our purpose in this article it will better to limit the graph to three different set classes of the Power Towers.

before:

- **1) All the sets in the graph belong to three different set classes.** Power Tower includes set (0258) as target set in all odd voice-leading zones. Since there is no member of this set class in even voice-leading zones, it is necessary to include members of two different sets of sets to fill these gaps. Because of this, members of sc. (0369) are placed in voice-leading zones 2, 6, and 10 as pivot sets and members of sc. (0358) are placed in voice-leading zones 0, 4, and 8 as pivot sets.
- **2) Sets are placed in voice-leading zones.** As in Cube Dance, all the sets are organized in adjacent sum classes.
- **3) Symmetrical superset.** All sets in voice-leading zone 10 to 2 are embedded in $OCT_{1,2}$, all sets in voice-leading zone 2 to 6 are embedded in $OCT_{2,3}$ and all sets in voice-leading zones 6 to 10 are embedded in $OCT_{0,1}$.
- **4) Parsimonious voice-leading between all connected sets.** As in Cube Dance, any connection between two sets is parsimonious.

Next we can build Power Towers with different sets in the same way we did with the Cube Dances, and these new graphs will also share the first three features of previous list. Figure 27 shows the 0167/0147/0157 Power Towers. It includes graphs 0147/0157 (voice-leading zones 11-0-1, 3-4-5, and 7-8-9) and 0167/0157 (voice-leading zones 1-2-3, 5-6-7, and 9-10-11) as contiguous subgraphs. In this graph the red lines connect two members of sc. (0167) related by $T_6$ and the black dashed and dotted lines represent connections between members of sc. (0147) using the contextual inversions axes **A** and **G**, respectively. The path drawn over this Power Towers shows an example of a pattern of voice-leading that combine parsimony and pure contrary motion with these three sets of trichords. The details of the voice-leading of this path is shown in Figure 28.

Figure 29 shows the 0136/0246/0135 Power Towers which includes graphs 0246/0135 (voice-leading zones 11-0-1, 3-4-5, and 7-8-9) and 0136/0135 (voice-leading zones 1-2-3, 5-6-7, and 9-10-11) as contiguous subgraphs. Red lines connect two members of sc. (0246) related by $T_3$ and the black dashed and dotted lines represent connections between members of sc. (0136) using the contextual inversions axes **D** and **F**, respectively. The path drawn over this Power Towers shows another example of a pattern of voice-leading that combine parsimony and pure contrary motion, and the details of this voice-leading is shown in Figure 30.

It is possible to build 87 Power Towers combining the tetrachord graphs of Table 3 in the Appendix, where we also give a list of all these Power Towers with its subgraphs and supersets in Table 5.

## VII. Conclusion

The visual advantages that are offered by neo-Riemannian graphs are of unquestionable importance both to determine how sets that are not embedded in a scale or collection connect in a given passage, and to highlight certain types of voice-leading. Graphs have performed these two tasks efficiently and have been widely used, especially in the analyzes, in the last decades. However, the fact that most neo-Riemannian graphs include only triads or seventh chords limited the scope of these analyses to a specific type of repertoire primarily comprised of works composed in the 19th century, chord progressions that Cohn call "pantriadic" ([2, p. 34]).

In this paper we have proposed ways to construct several types of graphs that include any set class of trichord and tetrachord. If on the one hand this approach had to renounce the requirement that all the connections between the sets included in the graphs are parsimonious, on the other

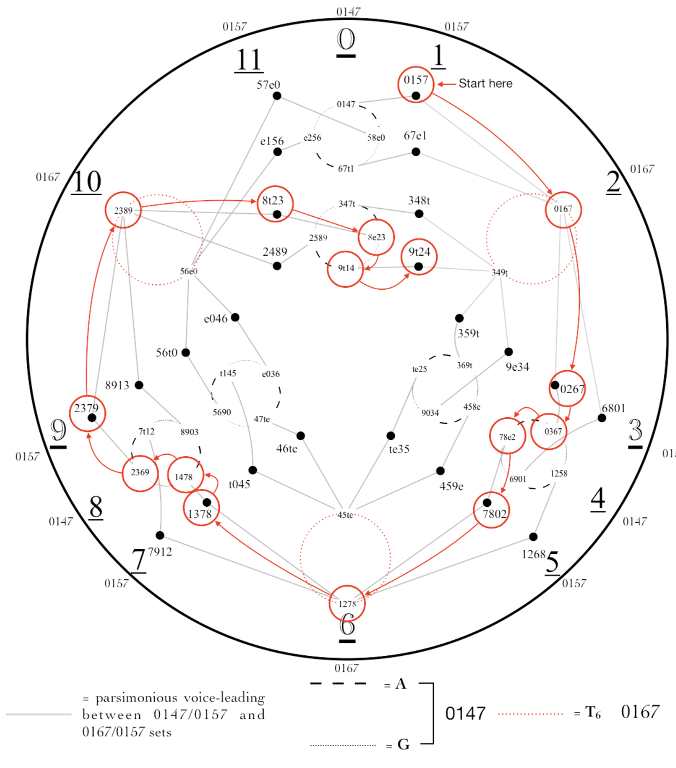



**Figure 27:** *0167/0147/0157 Power Towers.*

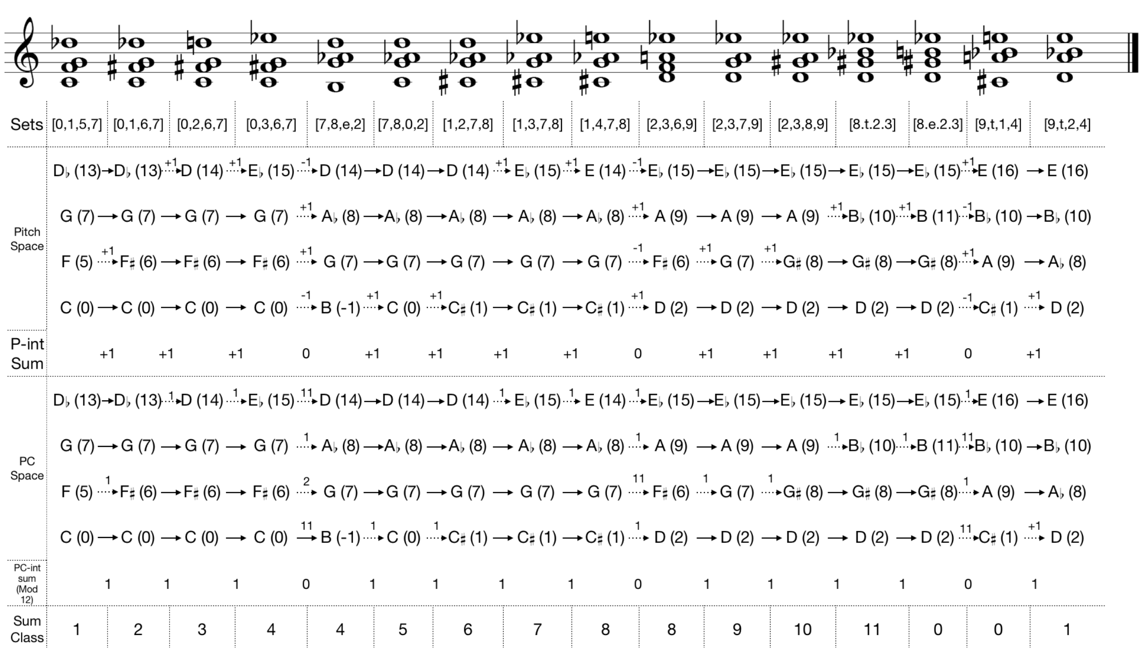



**Figure 28:** *An example of voice-leading for the path shown over the 0167/0147/0157 Power Towers in Figure 27.*

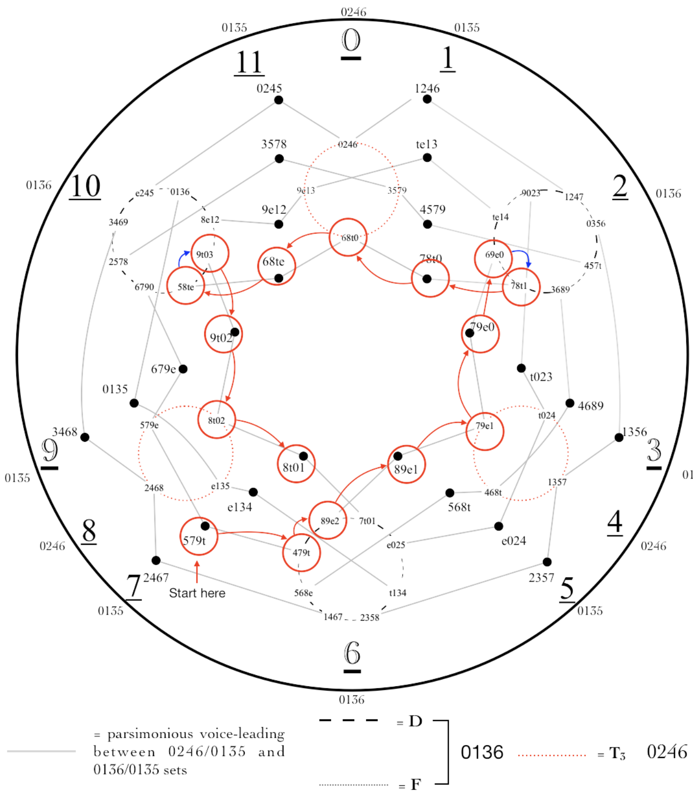



**Figure 29:** *The 0136/0246/0135 Power Towers.*

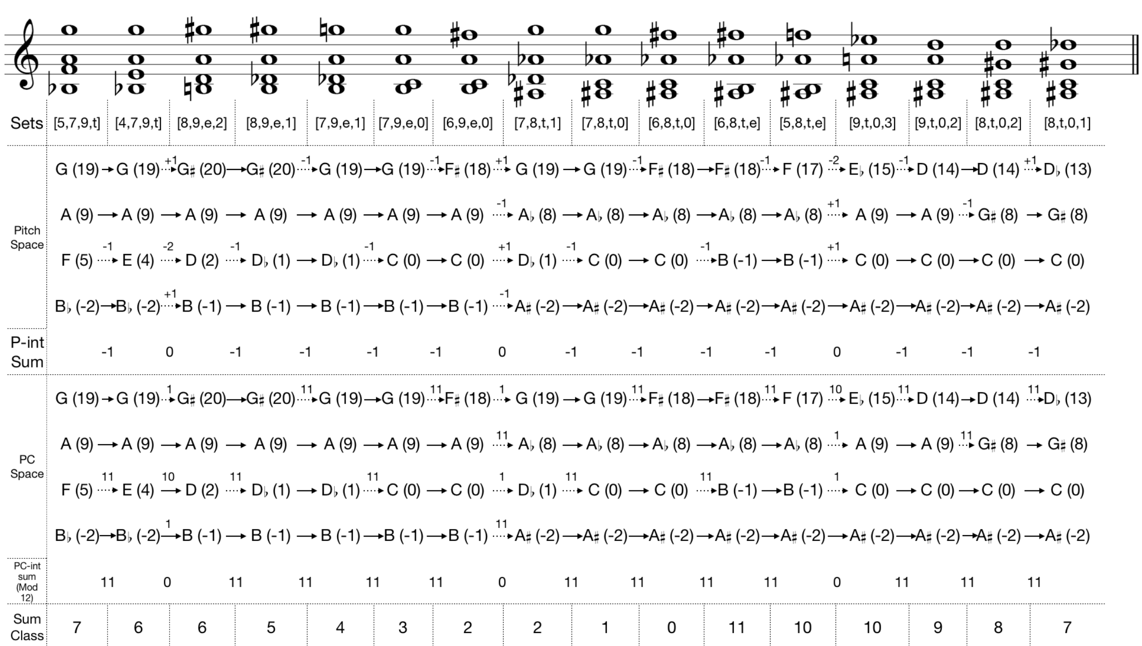



**Figure 30:** *An example of voice-leading for the path shown over the 0136/0246/0135 Power Towers in Figure 29.*

hand in the new graphs the sets are connected in adjacent voice-leading zones in the same way as in traditional graphs, and this maintains the leading voice consistency in the passages of the musical examples created with them. It is hoped that these new graphs can contribute both to analyses of music composed after the 19th century and to pre-compositional work of new music.

This approach can be easily expanded in many different ways, such as for graphs that would include sets of other cardinalities (dyads, pentachords, hexachords, etc.), graphs in which the pivot/bridge and target sets do not connect parsimoniously, graphs that include more than three set classes in their vertices, graphs in mod. 7, mod. 8, etc., among others.

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# A. Appendix

**Table 3:** *List of all graphs that connect two sets of tetrachords.*

| Type of Components | Bridges/Targets | Sum Classes | Superset |
|---|---|---|---|
| TYPE 1 | 0167/0157 | 1,2,3-5,6,7-9,10,11 | AGG |
| | 0268/0157 | 11,0,1-3,4,5-7,8,9 | AGG |
| | 0268/0258 | 11,0,1-3,4,5-7,8,9 | MM2 |
| TYPE 2 | 0123/0124 | 1,2,3-5,6,7-9,10,11 | AGG |
| | 0127/0126 | 11,0,1-3,4,5-7,8,9 | AGG |
| | 0127/0137 | 1,2,3-5,6,7-9,10,11 | AGG |
| | 0134/0124 | 11,0,1-3,4,5-7,8,9 | AGG |
| | 0134/0135 | 11,0,1-3,4,5-7,8,9 | AGG |
| | 0145/0135 | 1,2,3-5,6,7-9,10,11 | AGG |
| | 0145/0146 | 1,2,3-5,6,7-9,10,11 | AGG |
| | 0156/0146 | 11,0,1-3,4,5-7,8,9 | AGG |
| | 0156/0157 | 11,0,1-3,4,5-7,8,9 | AGG |
| | 0158/0148 | 1,2,3-5,6,7-9,10,11 | AGG |
| | 0158/0157 | 1,2,3-5,6,7-9,10,11 | AGG |
| | 0158/0258 | 1,2,3-5,6,7-9,10,11 | AGG |
| | 0235/0124 | 1,2,3-5,6,7-9,10,11 | AGG |
| | 0235/0135 | 1,2,3-5,6,7-9,10,11 | AGG |
| | 0235/0236 | 1,2,3-5,6,7-9,10,11 | MM2 |
| | 0246/0135 | 11,0,1-3,4,5-7,8,9 | AGG |
| | 0246/0146 | 11,0,1-3,4,5-7,8,9 | AGG |
| | 0246/0236 | 11,0,1-3,4,5-7,8,9 | AGG |
| | 0246/0247 | 11,0,1-3,4,5-7,8,9 | AGG |
| | 0248/0137 | 1,2,3-5,6,7-9,10,11 | AGG |
| | 0248/0148 | 1,2,3-5,6,7-9,10,11 | AGG |
| | 0248/0247 | 1,2,3-5,6,7-9,10,11 | AGG |
| | 0248/0258 | 1,2,3-5,6,7-9,10,11 | AGG |
| | 0257/0146 | 1,2,3-5,6,7-9,10,11 | AGG |
| | 0257/0157 | 1,2,3-5,6,7-9,10,11 | AGG |
| | 0257/0247 | 1,2,3-5,6,7-9,10,11 | AGG |
| | 0257/0258 | 1,2,3-5,6,7-9,10,11 | AGG |
| | 0347/0148 | 1,2,3-5,6,7-9,10,11 | AGG |
| | 0347/0236 | 1,2,3-5,6,7-9,10,11 | MM2 |
| | 0347/0247 | 1,2,3-5,6,7-9,10,11 | AGG |
| | 0358/0247 | 11,0,1-3,4,5-7,8,9 | AGG |
| | 0358/0148 | 11,0,1-3,4,5-7,8,9 | AGG |

| **Table 3**: Continuation | | | |
|---|---|---|---|
| Type of Components | Bridges/Targets | Sum Classes | Superset |
| TYPE 3a | 0136/0126 | 1,2,3-5,6,7-9,10,11 | AGG |
| | 0136/0135 | 1,2,3-5,6,7-9,10,11 | AGG |
| | 0136/0247 | 1,2,3-5,6,7-9,10,11 | AGG |
| TYPE 3b | 0136/0137 | 1,2,3-5,6,7-9,10,11 | AGG |
| | 0136/0146 | 1,2,3-5,6,7-9,10,11 | AGG |
| | 0136/0236 | 1,2,3-5,6,7-9,10,11 | AGG |
| TYPE 4a | 0125/0124 | 11,0,1-3,4,5-7,8,9 | AGG |
| | 0125/0236 | 11,0,1-3,4,5-7,8,9 | AGG |
| | 0147/0148 | 11,0,1-3,4,5-7,8,9 | MM7 |
| | 0147/0157 | 11,0,1-3,4,5-7,8,9 | MM7 |
| | 0147/0247 | 11,0,1-3,4,5-7,8,9 | MM7 |
| | 0237/0236 | 11,0,1-3,4,5-7,8,9 | MM4 |
| | 0237/0247 | 11,0,1-3,4,5-7,8,9 | AGG |
| TYPE 4b | 0125/0126 | 11,0,1-3,4,5-7,8,9 | AGG |
| | 0125/0135 | 11,0,1-3,4,5-7,8,9 | AGG |
| | 0147/0137 | 11,0,1-3,4,5-7,8,9 | MM4 |
| | 0147/0146 | 11,0,1-3,4,5-7,8,9 | MM7 |
| | 0147/0258 | 11,0,1-3,4,5-7,8,9 | MM7 |
| | 0237/0126 | 11,0,1-3,4,5-7,8,9 | AGG |
| | 0237/0137 | 11,0,1-3,4,5-7,8,9 | MM4 |
| | 0237/0148 | 11,0,1-3,4,5-7,8,9 | AGG |
| Type 5 | 0358/0258 | 11,0,1-3,4,5-7,8,9 | mm2 |

**Table 4:** *List of fourteen Cube Dances.*

| Cube Dance (Intersection/target) | Subgraph 1 (Sum 1/2, 4/5, 7/8 and 10/11) | Superset | Subgraph 2 (Sum 11/0/1, 2/3/4, 5/6/7 and 8/9/10) | Superset |
|---|---|---|---|---|
| 012/013 | four **<DL>** chains of 013 sets | AGG | four graphs combining sets 012 and 013 (type 1) | AGG |
| 015/014 | four **<DH>** chains of 014 sets | AGG | four graphs combining sets 015 and 014 (type 1) | AGG |
| 024/014 | four **<DH>** chains of 024 sets | AGG | four graphs combining sets 014 and 014 (type 1) | AGG |
| 015/016 | four **<BJ>** chains of 016 sets | AGG | four graphs combining sets 015 and 016 (type 2) | AGG |
| 027/016 | four **<BJ>** chains of 016 sets | AGG | four graphs combining sets 027 and 016 (type 1) | AGG |
| 015/025 | four **<FB>** chains of 025 sets | AGG | four graphs combining sets 015 and 025 (type 2) | AGG |
| 024/025 | four **<FB>** chains of 025 sets | AGG | four graphs combining sets 024 and 025 (type 1) | AGG |
| 036/026 | four **<FJ>** chains of 026 sets | AGG | four graphs combining sets 036 and 026 (type 1) | AGG |
| 015/026 | four **<FJ>** chains of 026 sets | AGG | four graphs combining sets 015 and 026 (type 2) | AGG |
| 027/026 | four **<FJ>** chains of 026 sets | AGG | four graphs combining sets 027 and 026 (type 1) | AGG |
| 036/026 | four **<FJ>** chains of 026 sets | AGG | four graphs combining sets 036 and 026 (type 1) | AGG |
| 027/037 | four **<HD>** chains of 037 sets | HEX | four graphs combining sets 027 and 037 (type 1) | MM3 |
| 036/037 | four **<HD>** chains of 037 sets | HEX | four graphs combining sets 036 and 037 (type 1) | MM3 |
| 048/037 | four **<HD>** chains of 037 | HEX | four Weitzmann Regions | MM3 |

**Table 5:** *List of 87 Power Towers.*

| Power Towers | Subgraph 1 (Sum 1/2/3, 5/6/7, and 9/10/11) | Superset | Subgraph 2 (Sum 11/0/1, 3/4/5, and 7/8/9/) | Superset |
|---|---|---|---|---|
| 0123-0125-0124 | graph 0123/0124 | AGG | graph 0125/0124 | AGG |
| 0235-0125-0124 | graph 0235/0124 | AGG | graph 0125/0124 | AGG |
| 0123-0134-0124 | graph 0123/0124 | AGG | graph 0134/0124 | AGG |
| 0235-0134-0124 | graph 0235/0124 | AGG | graph 0134/0124 | AGG |
| 0127-0125-0126 | graph 0127/0126 | AGG | graph 0125/0126 | AGG |
| 0136-0125-0126 | graph 0136/0126 | AGG | graph 0125/0126 | AGG |
| 0237-0127-0126 | graph 0127/0126 | AGG | graph 0237/0126 | AGG |
| 0136-0237-0126 | graph 0136/0126 | AGG | graph 0237/0126 | AGG |
| 0136-0125-0135 | graph 0136/0135 | AGG | graph 0125/0135 | AGG |
| 0145-0125-0135 | graph 0145/0135 | AGG | graph 0125/0135 | AGG |
| 0235-0125-0135 | graph 0235/0135 | AGG | graph 0125/0135 | AGG |
| 0136-0134-0135 | graph 0136/0135 | AGG | graph 0134/0135 | AGG |
| 0145-0134-0135 | graph 0145/0135 | AGG | graph 0134/0135 | AGG |
| 0235-0134-0135 | graph 0235/0135 | AGG | graph 0134/0135 | AGG |
| 0136-0246-0135 | graph 0136/0135 | AGG | graph 0246/0135 | AGG |
| 0145-0246-0135 | graph 0145/0135 | AGG | graph 0246/0135 | AGG |
| 0235-0246-0135 | graph 0235/0135 | AGG | graph 0246/0135 | AGG |
| 0127-0147-0137 | graph 0127/0137 | AGG | graph 0147/0137 | MM4 |
| 0136-0147-0137 | graph 0136/0137 | AGG | graph 0147/0137 | MM4 |
| 0248-0147-0137 | graph 0248/0137 | AGG | graph 0147/0137 | MM4 |
| 0127-0237-0137 | graph 0127/0137 | AGG | graph 0237/0137 | MM4 |
| 0136-0237-0137 | graph 0136/0137 | AGG | graph 0237/0137 | MM4 |
| 0248-0237-0137 | graph 0248/0137 | AGG | graph 0237/0137 | MM4 |
| 0136-0147-0146 | graph 0136/0146 | AGG | graph 0147/0146 | MM7 |
| 0145-0147-0146 | graph 0145/0146 | AGG | graph 0147/0146 | MM7 |
| 0257-0147-0146 | graph 0257/0146 | AGG | graph 0147/0146 | MM7 |
| 0136-0156-0146 | graph 0136/0146 | AGG | graph 0156/0146 | AGG |
| 0145-0156-0146 | graph 0145/0146 | AGG | graph 0156/0146 | AGG |
| 0257-0156-0146 | graph 0257/0146 | AGG | graph 0156/0146 | AGG |
| 0136-0246-0146 | graph 0136/0146 | AGG | graph 0246/0146 | AGG |
| 0145-0246-0146 | graph 0145/0146 | AGG | graph 0246/0146 | AGG |
| 0257-0246-0146 | graph 0257/0146 | AGG | graph 0246/0146 | AGG |
| 0158-0147-0148 | graph 0158/0148 | AGG | graph 0147/0148 | MM7 |
| 0248-0147-0148 | graph 0248/0148 | AGG | graph 0147/0148 | MM7 |
| 0158-0147-0157 | graph 0158/0157 | AGG | graph 0147/0157 | MM7 |
| 0167-0147-0157 | graph 0167/0157 | AGG | graph 0147/0157 | MM7 |
| 0257-0147-0157 | graph 0257/0157 | AGG | graph 0147/0157 | AGG |
| 0158-0156-0157 | graph 0158/0157 | AGG | graph 0156/0157 | AGG |
| 0167-0156-0157 | graph 0167/0157 | AGG | graph 0156/0157 | AGG |
| 0257-0156-0157 | graph 0257/0157 | AGG | graph 0156/0157 | AGG |
| 0158-0268-0157 | graph 0158/0157 | AGG | graph 0268/0157 | AGG |
| 0167-0268-0157 | graph 0167/0157 | AGG | graph 0268/0157 | AGG |
| 0257-0268-0157 | graph 0257/0157 | AGG | graph 0268/0157 | AGG |
| 0136-0125-0236 | graph 0136/0236 | AGG | graph 0125/0236 | AGG |

| **Table 5**: Continuation | | | | |
|---|---|---|---|---|
| Power Towers | Subgraph 1 (Sum 1/2/3, 5/6/7, and 9/10/11) | Superset | Subgraph 2 (Sum 11/0/1, 3/4/5, and 7/8/9/) | Superset |
| 0235-0125-0236 | graph 0235/0236 | MM2 | graph 0125/0236 | AGG |
| 0347-0125-0236 | graph 0347/0236 | AGG | graph 0125/0236 | AGG |
| 0136-0237-0236 | graph 0136/0236 | AGG | graph 0237/0236 | MM4 |
| 0235-0237-0236 | graph 0235/0236 | MM2 | graph 0237/0236 | MM4 |
| 0347-0237-0236 | graph 0347/0236 | AGG | graph 0237/0236 | MM4 |
| 0136-0246-0236 | graph 0136/0236 | AGG | graph 0246/0236 | AGG |
| 0235-0246-0236 | graph 0235/0236 | MM2 | graph 0246/0236 | AGG |
| 0347-0246-0236 | graph 0347/0236 | AGG | graph 0246/0236 | AGG |
| 0235-0237-0236 | graph 0235/0236 | MM2 | graph 0237/0236 | MM4 |
| 0347-0237-0236 | graph 0347/0236 | AGG | graph 0237/0236 | MM4 |
| 0136-0246-0236 | graph 0136/0236 | AGG | graph 0246/0236 | AGG |
| 0235-0246-0236 | graph 0235/0236 | MM2 | graph 0246/0236 | AGG |
| 0347-0246-0236 | graph 0347/0236 | AGG | graph 0246/0236 | AGG |
| 0136-0147-0247 | graph 0136/0247 | AGG | graph 0147/0247 | MM7 |
| 0248-0147-0247 | graph 0248/0247 | AGG | graph 0147/0247 | MM7 |
| 0347-0147-0247 | graph 0347/0247 | AGG | graph 0147/0247 | MM7 |
| 0136-0237-0247 | graph 0136/0247 | AGG | graph 0237/0247 | AGG |
| 0248-0237-0247 | graph 0248/0247 | AGG | graph 0237/0247 | AGG |
| 0347-0237-0247 | graph 0347/0247 | AGG | graph 0237/0247 | AGG |
| 0136-0246-0247 | graph 0136/0247 | AGG | graph 0246/0247 | AGG |
| 0248-0246-0247 | graph 0248/0247 | AGG | graph 0246/0247 | AGG |
| 0347-0246-0247 | graph 0347/0247 | AGG | graph 0246/0247 | AGG |
| 0158-0147-0258 | graph 0158/0258 | AGG | graph 0147/0258 | AGG |
| 0248-0147-0258 | graph 0248/0258 | AGG | graph 0147/0258 | MM7 |
| 0257-0147-0258 | graph 0257/0258 | AGG | graph 0147/0258 | MM7 |
| 0369-0147-0258 | graph 0369/0258 | AGG | graph 0147/0258 | MM7 |
| 0158-0268-0258 | graph 0158/0258 | AGG | graph 0268/0258 | MM2 |
| 0248-0268-0258 | graph 0248/0258 | AGG | graph 0268/0258 | MM2 |
| 0257-0268-0258 | graph 0257/0258 | AGG | graph 0268/0258 | MM2 |
| 0369-0268-0258 | graph 0369/0258 | AGG | graph 0268/0258 | MM2 |
| 0158-0358-0258 | graph 0158/0258 | AGG | graph 0358/0258 | MM2 |
| 0248-0358-0258 | graph 0248/0258 | AGG | graph 0358/0258 | MM2 |

# Tonal Progressions Identification Through Kripke Semantics


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***Abstract:*** *The automatic identification of tonal chord sequences has already been addressed through several formalisms. We return to this problem for didactic reasons, as we seek a formal solution that lends itself to automatic explanation of the way in which tonal sequences are identified. There is a search for a correspondence between the formal steps, which lead to the solution, and how a human agent does it to solve the problem him/herself. Given a sequence of chords, the task is to answer whether or not it constitutes a tonal progression, and how and why. It is an interesting problem because its formal solution, once easily automated, can give birth to educational software of real value in the case of young musicians whose access to harmony teachers is scarce or even null. This formalism applied to a large test body allows empirical proof of the fundamental idea that we can describe the tonal sequences by chaining together a minimal collection of basic tonal sequences. Students who do not have access to a harmony teacher will benefit from this harmonic analysis companion.*



***Keywords****: Tonality identification. Modal semantics. Model checking. Automatic harmonic reasoning.*


## I. Introduction

The classification of a chord sequence[1] as a tonal sequence is a fundamental task in the study of harmony and is one of the pillars of musical analysis. This is sufficient reason for it to be formalized. The difficulty of such a task can take on great proportions if long pieces are analyzed completely. This is a sufficient reason for it to be automated. In this study, sequences are defined as any concatenation of chords. Here are two situations of identification of a harmonic progression:

1) To identify that [Em - $A^7$ - Dm - G - C] is a tonal progression, in the key of C major, we must explain the role played by the non-diatonic chord, $A^7$.


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[1]The term *sequence*, as defined in this work, is not related to the term *melodic sequence*, which consists of the repetition of motifs or phrases at different levels of pitch, maintaining the same interval pattern (modulatory sequence) or the same contour (diatonic sequence); neither is it related to the medieval sequence, which consists of "the most important type of addition to the official Catholic liturgical song" ([6, p. 739]). For a more in-depth examination of the term *sequence*, as it is traditionally used in the Western musical context, see [6, pp. 739–741].

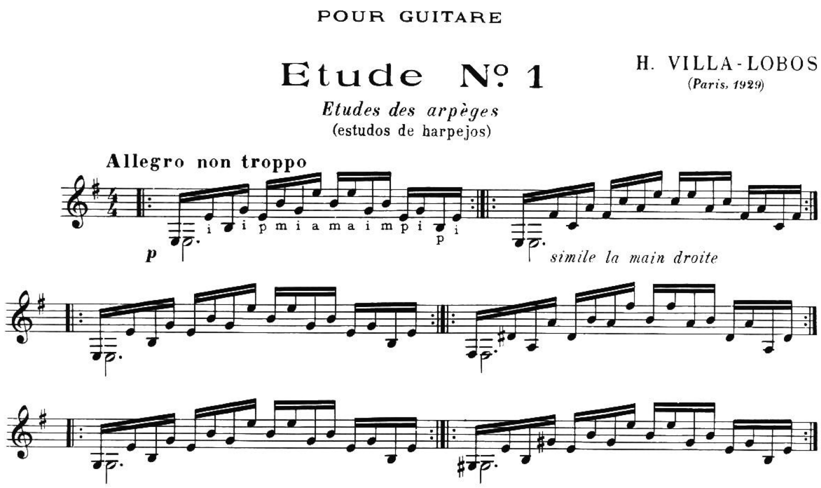


**Figure 1:** *The first six measures of Villa-Lobos' Étude No. 1, for guitar.*

2) Another type of difficulty is found in this excerpt from Étude No. 1, by Villa-Lobos (1953) ([5, p. 1]) shown in Figure 1, whose chord progression is: [Em - F#°/E - Em - B$^7$/F# - Em/G]. With the exception of the second chord, the others make up a traditional diatonic sequence in the key of E minor. The point here then becomes to explain the "tonal presence" of this chord.

## II. The problem

It is worth remembering that the same sequence of chords can be constructed in more than one way, and therefore, it has more than one explanation. The problem faced here is not only if a sequence is tonal, but also why it is so. Moreover, the explanation must be formal: we are looking for a formal system to explain how the solution of a theoretical musical issue is resolved. In addition, as this question has several answers, the system must be able to find all of them.

### i. Our contribution in a glance

We borrow the concept of *basic tonal sequence* from traditional harmony and, from there, we formalize the concept of *tonal sequence*. We achieve this through a recursive function inspired by Kripke Semantics ([8, pp. 83–94], [1, pp. viii–xi]) and Model Checking ([3, pp. 1–28]). Some consequences directly associated with these concepts are:

- We have developed an unprecedented formalism for identifying tonal progressions.
- Because it is a formalism similar to propositional logic in the syntactic part, and similar to Kripke's structure[2] in the semantic part, it is immediately convertible into an algorithm.
- We demonstrate that the proposed formalism is correct and complete in relation to the AHO formalism for detecting tonal harmonies based on context-free grammar ([7, p. 87–88]).

[2] A sophisticated example of applying Kripke structures in Game Theory can be found in [9, pp. 60–182].

# III. Our proposal

To expose our idea, let us return to situation 1) in Section I (identification of a harmonic progression), and note that the chords in common with the two sub-sequences merge into one. To explain the role played by the non-diatonic chord $A^7$ in the sequence [Em - $A^7$ - Dm - G - C], we imagine that it is, indeed, two sequences [Em - $A^7$ - Dm] and [Dm - G - C]; and that both are chained together motivated by the presence of the common chord Dm. Note that the chords in common with the two subsequences merged together. This phenomenon occurs in natural language, in phonetics, and is called *crase*. In situation 2) in Section I, there is an ambiguity with respect to the second chord, which can be also interpreted as an Am6/E. These two types of solution are included in the formalism that we propose because what we are modelling are not sequences of chords but, rather, sequences of tonal functions.

# IV. The proposed solution

In this work, we developed a formal system that performs the tonal harmonic analysis of musical pieces. The analysis is twofold: the conclusion about tonalness and the explanation how it was formed. This system produces a harmonic analysis from the chord sequence of a musical piece, identifying the key of the piece, its musical cadences, and the role of each chord in the cadences.

# V. Formalization

## i. Background

Model Check ([2, pp. 49—58]) is a procedure in which a semantic structure serves as a model representing a system. In addition, there is a specification that you would like the system to respect, any property that this system could have. Moreover, you have an automatic device called "Model Check Tool", an automatic tool that responds positively if that system has that property.

From a technical point of view, it answers whether that formal model respects the specification which is also written in a formal language. Model Check gives an answer "yes" if that model satisfies that specification.

Here we subvert this idea. The specifications, which are expressions of the formal language, for us, represent an object. That object will have a certain property. For us the model continues to represent the system, in the same way. However, instead of testing whether the system has a property, we are going to test whether that property is actually recognized by the system.

For us the model is the ruler and the specification is up to us. In this work here, in particular, the model represents the tonal system while the formal specification represents sequences of candidate chords for tonal harmonic progressions.

## ii. Language

The syntax of our formal language consists of an alphabet formed by chord symbols considered as atomic constituents on which we abstract the details and consider them correctly written, and by a single rule of formation, the concatenation of these symbols.

## iii. Semantics

We established three basic (primary) tonal sequences ([4, pp. 19–233]), formed as follows:

**Definition 1** (Basic tonal sequences)**.** Basic tonal sequences show one of these formats:

- **Basic sequence 1**: *The dominant function, followed by the tonic function*;
- **Basic sequence 2**: *The subdominant function, followed by the tonic function*;
- **Basic sequence 3**: *The subdominant function, followed by the dominant function, followed by the tonic function*.

*Example* 1 (Basic tonal sequences)*.* For example, in the C major key, we have:

- **Basic sequence 1**: [G - C] or [B° - C].
- **Basic sequence 2**: [F - C] or [Dm - C].
- **Basic sequence 3**: [F - G - C] or [Dm - $G^7$ - C].

## iv. Formation rules

Three different ways of grouping sequences are possible: by *juxtaposition*, by *elision*, and by *crase*. In the *juxtaposition*, two strings are joined only by concatenation; in the *elision* and in the *crase* there is a collapse between the sequences: one of the two sequences loses a component. More formally:

**Definition 2** (Chaining rules)**.** Let $s = \langle s_1, s_2, ..., s_n \rangle$ and $r = \langle r_1, r_2, ..., r_m \rangle$ be sequences.

- The *juxtaposition* between them produces the sequence $\langle s_1, s_2, ..., s_n, r_1, r_2, ..., r_m \rangle$;
- The *crase* between them produces the sequence $\langle s_1, s_2, ..., s_n, r_2, ..., r_m \rangle$, if $s_n = r_1$;
- The *elision* between them produces the sequence $r = \langle s_1, s_2, ..., s_{(n-1)}, r_1, r_2, ..., r_m \rangle$ if $s_n \equiv r_1$.

Note that the crase requires equality between the extreme elements of the strings while the elision requires only equivalence. These two concepts are established by extension in formal language. For example, the equivalence between two points in the sequence can be defined as follows: $P \equiv Q$ when, for all label function $L \in \mathbb{L}$ and all paths $\pi$ in a given structure $K$, there is

$$P \in L(\pi_i) \text{ if and only if } Q \in L(\pi_i). \tag{1}$$

In musical terms, if in the context of a key ($L$), both chords (points $P$ and $Q$ in a path) perform the same tonal function.

*Example* 2 (Chaining rules)*.* We have that:

- **Juxtaposition**: [G - C] and [B° - C] produces the sequence [G - C - B - B° - C].
- **Crase**: [Em - $A^7$ - Dm] and [Dm - G - C] produces the sequence [Em - $A^7$ - Dm - G - C].
- **Elision**: [Em - $A^7$ - F6] and [Dm - G - C] produces the sequence [Em - $A^7$ - Dm - G - C].

## v. Structure and Representation

To model the idea of sequence, the states represent the elements with which the sequences are built. Relations between states represent the concatenation between the elements. The displacement between states obeying the dictates of accessibility relations, shapes the sequences.

To formally check out if they are indeed tonal sequences, one imagines each point of the sequence as a state[3]. Those states, which form sequences, are interconnected by accessibility relationships, and the *path*[4] between these states is used as a representation of linear sequences. The states are characterized by a *label function* that denotes their meaning. The following Kripke framework provides the necessary formalism.

[3]Each state is formally a node of a graph and represents a tonal function.
[4]The concept of *path* will be defined later in this work (Definition 4).

**Definition 3** (Kripke Structure)**.** A *Kripke structure*, $K = \langle S, S_0, S_F, R, L \rangle$, is defined by:

- A non-empty set $S$ of states;
- A set $S_0 \subseteq S$ of initial states;
- A set $S_F \subseteq S$ of final states;
- A total relation $R \subseteq S \times S$, i.e., for every state $s \in S$ there is a state $s' \in S$, such that $\langle s, s' \rangle \in R$.
- A collection $\mathbb{L}$ of label functions $L_i : S \to \Pi$, where $\Pi$ is the set of all chords. The label function assigns to each state a chord symbol.

Example 3 shows an instance of Definition 3 with only 3 label functions:

*Example* 3. [Kripke Structure] Example of a Kripke structure $K = \langle S, S_0, S_F, R, L \rangle$, where

- $S = \{s_f, s_g, s_c\}$.
- $S_0 = \{s_f, s_g\}$.
- $S_F = \{s_c\}$.
- $\langle s_f, s_c \rangle \in R$, $\langle s_g, s_c \rangle \in R$, and $\langle s_f, s_g \rangle \in R$.
- $\mathbb{L} = \{L_1, L_2, L_3\}$ where

$$\begin{aligned} &L_1(s_c) = \{\mathrm{C}\}, && L_1(s_f) = \{\mathrm{F}\}, && L_1(s_g) = \{\mathrm{G}\}, \\ &L_2(s_c) = \{\mathrm{Am}\}, && L_2(s_f) = \{\mathrm{Bm}\}, && L_2(s_g) = \{\mathrm{E}^7\}, \\ &L_3(s_c) = \{\mathrm{Cm}\}, && L_3(s_f) = \{\mathrm{D}^7_{5\flat}\}, && L_3(s_g) = \{\mathrm{G}^7, \mathrm{B}^\circ\}. \end{aligned} \tag{2}$$

In Example 3, $\Pi = \{\mathrm{C, F, G, Am, Bm, E^7, Cm, D^7_{5\flat}, G^7, B^\circ}\}$ is a set of chord symbols; the $L_1$ function associates the $s_c$ state with a single chord symbol, the C symbol, and this means that in the key represented by the $L_1$ function, all other chord symbols are prohibited in this state, only $L_1$ is accepted. Note that the $L_3$ function associates with the state, $s_g$, both chord symbols, $\mathrm{G}^7$ and $\mathrm{B}^\circ$.

Figure 2 shows the $K^*$ structure with nodes labeled by the $L_1$ label function.

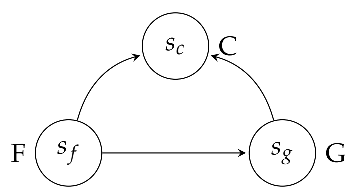


**Figure 2:** *Structure $K^*$ labeled by $L_1$.*

The labeling of the $K^*$ structure by the label functions $L_2$ and $L_3$ is illustrated in the Figures 3 and 4.

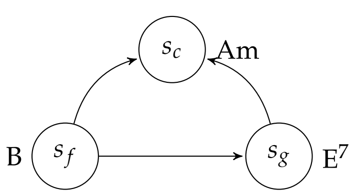


**Figure 3:** *$K^*$ structure labeled by $L_2$.*

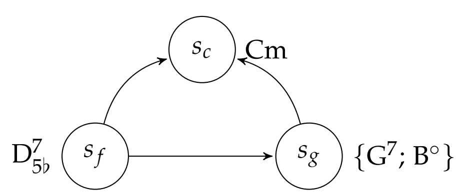


**Figure 4:** *$K^*$ structure labeled by $L_3$.*

Note that the multiple labelling in $K^*$ allows an interpretation of "worlds of structures" ,where each world represents a possible key. The accessibility relationship between states determine possible displacements over a Kripke structure. A series of two-by-two accessible states defines a *path* over the structure:

**Definition 4** (Path)**.** A *path* in a Kripke structure from a state $s \in S$ is defined by:

- A sequence $\pi = s_0, s_1, ..., s_n$ such that $s = s_0$ and $\forall k \in \{0, 1, ..., n\}$ , $\langle s_k, s_{k+1} \rangle \in R$;
- A path starting in the $s$ state is said to be an $s$-path;
- The state $s_k$ of path $\pi$ is notated by $\pi_k$;
- The prefix $s_0, s_1, ..., s_k$ of $\pi$ is notated $\pi_{0,k}$;
- The suffix $s_k, s_{k+1}, ..., s_n$ of $\pi$ is notated $\pi_{k,n}$.

*Example* 4. [Paths in a Kripke Structure] In the structure of Example 3, from the state $s_f$ we have the following paths: $\pi_A = [s_f, s_g, s_c]$ and $\pi_B = [s_f, s_c]$; from the $s_c$ state we have the path: $\pi_C = [s_g, s_c]$. A succession of arrows represents a path in the graph.

Let $\Pi$ be a set of chord symbols and $P$ a chord symbol in $\Pi$. Let $K = \langle S, S_0, S_F, R, L \rangle$ be a Kripke structure and $\pi$ a path of the $K$ structure. Finally, let $\phi$ be a formal expression. We define now when $K$ *satisfies* $\phi$ *on the path* $\pi$, denoted by $K \models_\pi \phi$:

**Definition 5** (Satisfaction)**.** Let $K = \langle S, S_0, S_F, R, L \rangle$ be a Kripke structure, $\pi$ and $\pi'$ be paths in $K$, $L$ and $L'$ be label functions in $K$ and $\phi$ a formal expression. We say that $K$ satisfies $\phi$ when:

$$K, L \models_\pi P \iff P \in L(\pi_0) \tag{3}$$

$$K, L \models_\pi (P\phi) \iff K, L \models_\pi P \quad \text{and} \quad (K, L \models \pi_{1,n}\phi \quad \text{or} \quad K, L \models_{\pi'} \phi) \tag{4}$$

$$K, L \models_\pi (P\phi) \iff K, L \models_{\pi_0} P \quad \text{and} \quad K, L' \models_{\pi'_0} P \quad \text{and} \quad K, L' \models_{\pi'_{1,n}} \phi \tag{5}$$

An expression is said to be satisfied if it is read entirely, until the last of its chord symbols without resulting in error[5].

We assume a function of labels for each possible key, assigning the chords that perform the tonic function to the $s_c$ state, those who fulfills the function of dominant to the state $s_g$, and those who fulfill the function of subdominant to the state $s_f$.

## VI. Examples of identifying tonal progressions

The sequence of triads will be interpreted in reverse, $s_0$ being the last triad of harmony. The following are examples of satisfaction of sequences whose formation took place in different ways. In them, satisfaction occurs according to the definition 5 in view of the structure of example 3, labeled, each time, with the proper label function.

In the following examples, we will use the next label functions that will be numbered to facilitate exposure

$$L_1(s_c) = \{\text{C}, \text{Am}\}, L(s_f) = \{\text{F}, \text{Dm}\} \text{ and } L(s_g) = \{\text{G}^7, \text{B}^\circ\} \tag{6}$$

$$L_5(s_c) = \{\text{F}, \text{Dm}\}, L(s_f) = \{\text{B}^\flat, \text{Gm}\} \text{ and } L(s_g) = \{\text{C}^7, \text{E}^\circ\} \tag{7}$$

$$L_{10}(s_c) = \{\text{A}, \text{F}\sharp\text{m}\}, L(s_f) = \{\text{D}, \text{Bm}\} \text{ and } L(s_g) = \{\text{E}^7, \text{G}\sharp^\circ\} \tag{8}$$

We are going to study three chord sequences: [B$^\flat$ - C - F], [G - C - F - G - C] and [G - C - B$^\flat$ - C - F - G - C]. The third example exposes the basic procedure for identifying modulations ([4, ppd. 169-280]).

[5] We say that $K$ is a model for $\phi$.

Let $\phi$ be a chord sequence, $K = \langle S, S_0, S_F, R, L\rangle$ be a Kripke structure and $\pi = [s_f, s_g, c_c]$ be a path in $K$.

*Example* 5 ([B$^\flat$ - C - F] – Perfect cadence in the key of F major). The sequence of triads will be read in reverse: [F - C - B$^\flat$]. We want to show that $K \models_\pi$ F(CB$^\flat$). To Show that $K \models_\pi$ FCB$^\flat$, we have, by Equation 4 in Definition 5:

$$K \models_\pi \mathrm{FCB}^\flat \iff \mathrm{F} \in L(\pi_0) \text{ and } K \models_{\pi_{1,2}} \mathrm{CB}^\flat. \tag{9}$$

Taking $L = L_5$, and $\pi = \pi_A$ we have F $\in L(\pi_{A_0})$. Let's show that $K \models_{\pi_{1,n}}$ CB$^\flat$, by Equation 4 in Definition 5:

$$K \models_{\pi_{1,n}} \mathrm{CB}^\flat \iff \mathrm{C} \in L_5(\pi_1) \text{ and } K \models_{\pi_{2,n}} \mathrm{B}^\flat. \tag{10}$$

Taking again $L = L_5$, we have C $\in L_5(\pi_{A_1})$. Let's show that $K \models_{\pi_{2,2}}$ B$^\flat$, by Equation 3 in Definition 5:

$$K \models_{\pi_{2,2}} \mathrm{B}^\flat \iff \mathrm{B}^\flat \in L(\pi_2). \tag{11}$$

Taking again $L = L_5$, we have B$^\flat \in L_5(\pi_2)$.

There is a "B flat major" interpretation for $L$, i.e., $L_5$ which, together with the "*perfect cadence*" path $\pi_A$, recognize the sentence FCB$^\flat$ as a tonal progression. The recursive dynamics of Definition 5 takes that interpretation and recognize the sequence FCB$^\flat$ as a tonal progression.

*Example* 6 ([(GC)F]GC)] – Dominant cadence followed by a perfect cadence in the key of F major). The sequence of triads will be read in reverse: CGFCG. We want to show that $K \models_\pi$ C (G [F (C G)]). To show that $K \models_\pi$ C (G [F (C G)]), we have, by Equation 4 in Definition 5:

$$K \models_\pi \mathrm{C\ (G\ [F\ (C\ G)])} \iff C \in L(\pi_0) \text{ and } K \models_{\pi_{1,n}} \mathrm{G[F(CG)]}. \tag{12}$$

Taking $L = L_1$ and $\pi = \pi_A$, we have C $\in L(\pi_{A_0})$. Let's show that $K \models_{\pi_{1,n}}$ G[F (C G)], by Equation 4 in Definition 5:

$$K \models_{\pi_{1,n}} \mathrm{G[F(CG)]} \iff \mathrm{G} \in L_1(\pi_1) \text{ and } K \models_{\pi_{2,n}} \mathrm{[F(CG)]}. \tag{13}$$

Taking again $L = L_5$, we have G $\in L_1(\pi_{A_1})$. Let's show that $K \models_{\pi_{2,n}}$ [F(CG)], by Equation 4 in Definition 5:

$$K \models_{\pi_{1,n}} \mathrm{[F(CG)]} \iff \mathrm{F} \in L_1(\pi_2) \text{ and } K, L^* \models_{\overline{\pi}} \mathrm{CG}. \tag{14}$$

Taking again $L = L_1$, we have F $\in L_1(\pi_{A_2})$. Let's show that $K, L^* \models_{\overline{\pi}}$ CG, by Equation 4 in Definition 5:

$$K, L^* \models_{\overline{\pi}} \mathrm{CG} \iff \mathrm{C} \in L^*(\overline{\pi}_0) \text{ and } K, L^* \models_{\overline{\pi}_{1,n}} \mathrm{G}. \tag{15}$$

Taking again $L^* = L_1$, and $\overline{\pi} = \pi_C$, we have C $\in L_1(\pi_{C_0})$. Let's show that $K, L^* \models_{\overline{\pi}_{1,n}}$ G, by Equation 3 in Definition 5:

$$K, L^* \models_{\overline{\pi}_{1,n}} \mathrm{G} \iff \mathrm{G} \in L^*(\overline{\pi}_1). \tag{16}$$

Taking again $L^* = L_1$ and $\overline{\pi} = \pi_C$, we have $G \in L^*(\overline{\pi}_1)$.

There are a "B flat major" interpretation for $L$, i.e., $L_5$ which, together with the "*perfect cadence*" path $\pi_A$, recognize the sentence FGC as a tonal progression. There is also another "B flat major" interpretation for $L$, i.e., $L_1$ which, together with the *plagal cadence* path $\pi_C$, recognize the sentence GC as a tonal progression. The recursive dynamics of Definition 5 take those two interpretations and recognize the entire sequence CGFCG as a tonal progression.

*Example* 7 ([CG$^7$DmA$^7$E$^7$Bm] – Progression with diatonic and non-diatonics chords in C major). The sequence of triads will be read in reverse: we want to show that $K, L \models_\pi$ C, G$^7$, Dm, A$^7$, E$^7$, Bm. By Equation 4 in Definition 5 this happens if, and only if, we have:

$$K, L \models_\pi \mathrm{C\ G^7Dm(A^7E^7Bm)} \iff \mathrm{C} \in L(\pi_0) \text{ and } K, L \models_{\pi_{1.n}} \mathrm{G^7Dm(A^7E^7Bm)}. \tag{17}$$

Taking $L = L_1$ and $\pi = \pi_A$, we have $C \in L_1(\pi_{A_0})$. Let's show that $K, L \models_{\pi_{1.n}} \mathrm{G^7Dm(A^7\{E^7Bm\})}$, by Equation 4 in Definition 5:

$$K, L \models_{\pi_{1.n}} \mathrm{G^7Dm(A^7E^7Bm)} \iff \mathrm{G^7} \in L(\pi_{1,n}) \text{ and } K, L \models_{\pi_{2,n}} \mathrm{Dm(A^7E^7Bm)}. \tag{18}$$

Taking $L = L_1$ and $\pi = \pi_A$, we have $\mathrm{G} \in L_1(\pi_{A_1})$. Let's show that $K, L \models_{\pi_{2,n}} \mathrm{Dm(A^7\{E^7Bm\})}$, by Equation 5 if Definition 5:

$$K, L \models_{\pi_{1.n}} \mathrm{Dm(A^7E^7Bm)} \iff \mathrm{Dm} \in L(\pi_{2,n}) \text{ and } \mathrm{Dm} \in L'(\pi'_0) \text{ and } K, L' \models_{\pi_{1.n}} \mathrm{A^7E^7Bm}. \tag{19}$$

Taking again $L = L_1$, and $\pi = \pi_A$, we have $\mathrm{Dm} \in L_1(\pi_{A_2})$. Taking $L' = L_5$ and $\pi' = \pi_A$, we have $\mathrm{Dm} \in L_5(\pi'_{A_0})$. Let's show that $K, L' \models_{\pi'_{1.n}} \mathrm{A^7E^7Bm}$, by Equation 5 in 5:

$$K, L' \models_{\pi'_{1.n}} \mathrm{A^7E^7Bm} \text{ iff } \mathrm{A^7} \in L'(\pi'_1) \text{ and } \mathrm{A^7} \in L''(\pi^*_0) \text{ and } K, L'' \models_{\pi^*_{1,n}} \mathrm{E^7Bm}. \tag{20}$$

Taking again $L' = L_5$ and $\pi' = \pi_A$, we have $\mathrm{A^7} \in L_5(\pi_{A_1})$. Taking again $L'' = L_{10}$, and $\pi* = \pi_A$, we have $\mathrm{A^7} \in L_{10}(\pi^*_{A_0})$. Let's show that $K, L'' \models_{\pi^*_{1,n}} \mathrm{E^7Bm}$, by Equation 4 in Definition 5:

$$K, L'' \models_{\pi^*_{1,n}} \mathrm{E^7Bm} \iff \mathrm{E^7} \in L''(\pi^*_{1,n}) \text{ and } K, L'' \models_{\pi^*_{2,n}} \mathrm{Bm}. \tag{21}$$

Taking again $L'' = L_{10}$, and $\pi^* = \pi_A$, we have $\mathrm{E^7} \in L_{10}(\pi^*_{A_1})$. Let's show that $K, L'' \models_{\pi^*_{2,n}} \mathrm{Bm}$, by Equation 3 in Definition 5:

$$K, L'' \models_{\pi^*_{2,n}} \mathrm{Bm} \iff \mathrm{Bm} \in L''(\pi^*_2). \tag{22}$$

Taking $L'' = L_{10}$ and $\pi^* = \pi_A$, we have $\mathrm{Bm} \in L''(\pi^*_2)$.

In this Example we take the "C major" label function $L_1$, the "D minor" label function $L_5$ and the "A major" label function $L_{10}$ (see equations 6, 7, and 8). We also take the paths $\pi$, $\pi'$, and = $\pi^*$ all equal to $\pi_A$ ("*perfect cadences*").

The recursive dynamics of Definition 5 take those interpretations and recognize the entire sequence CGFCB$^\flat$CG as a tonal progression.

## VII. Conclusions and further work

We prove the correctness and completeness between this system and AHO (context-free) grammar developed in [7, p. 32–77]. A body of 100 works between classical music and Brazilian popular music was used as a test. The formalization employed naturally provides the basis for immediate computational implementation. The methodology employed in this work proved satisfactory in identifying tonal harmonic sequences, showing that a minimal structure can synthesize the idea of tonal harmonic progression.

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# A Conceptual Note on Gesture Theory

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***Abstract:*** *This paper is a conceptual counterpart of the technical developments of gesture theory. We base all our discussion on Saint-Victor's definition of gesture. First, we unfold it to a philosophical reflection that could establish a dialogue between semiotic and pre-semiotic approaches to musical gestures, thanks to Peirce's ideas. Then we explain how the philosophical definition becomes a mathematical one, and provide reflections on important concepts involved in gesture theory, some possible relations to other branches of mathematics and mathematical music theory, and several open and closed questions. We include a non-mathematical discussion related to the environment of this subject and a glossary of specialized terms.*

***Keywords***: *Gestures. Mathematical Music Theory. Category Theory. Geometry. Philosophy.*

## To the reader

The following lines try to explain some concepts in gesture theory by means of general ideas, motivations, and musical examples. We do not pretend that the reader understands all mathematical details. Regarding the technical concepts that we consider as essential to understand the big picture, we include a glossary, to which the reader will be directed after clicking on a red term, like *topological gesture*. However, it is desirable that the reader has a minimum acquaintance with category theory [18, pp. 10-23]. On the other hand, we end each example with the symbol ♣ for organization.

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## I. Some motivations and antecedents

The first version of *The Topos of Music* [28] was an enterprise that achieved a framework for musicology based on topos theory, including a theory of performance and a very complete account of the mathematical structures present in music. Soon after this accomplishment, Mazzola became aware that his own activity as a free jazz pianist had little to do with the structures and procedures described in his monograph; see [23, Chapter 24]. *Gestures*, rather than formulas, were the essence of his performance. Certainly, free jazz improvisation is mainly determined by the movements of the body's limbs, that is, by a *dancing of the body*, the classical structures of Western music being secondary and auxiliary. Therefore, a rigorous reflection on gestures was necessary, and not only in the case of musical improvisation, but in music in general, since all its power and intensity relies in its realization in bodily terms, even in the Western classical tradition.

Indeed, the task of studying gestures in music is not new and has received considerable attention from other musicologists in recent times, though following different approaches to formulate a suitable conceptualization of musical gestures. In fact, these studies are far from a unified treatment of the concept of gesture. On the one hand, some authors [13, Chapter 1] emphasize the need of content, meaning, and signification in the definition of gesture arguing that the understanding of a gesture as a mere movement of the body can lead to an ambiguity regarding whether this movement expresses a musical intention or not, like in the case of the gesture of a listener in response to music [13, § 1.2.4]. On the other hand, studies like [10] have an inclination for more flexible definitions of gesture, in which the notion of meaning or signification need not be included:

> "We consider that the word *gesture* (or the French equivalent *geste*) necessarily makes reference to a human being and to its body behaviour—whether they be useful or not, significant or meaningless, expressive or inexpressive, conscious or not, intentional or automatic/reflex, completely controlled or not, applied or not to a physical object, effective or ineffective, or suggested." [10, p. 73]

However, most of definitions seem to have a common point: a gesture is the movement of the body (plus several nuances).

## II. Saint-Victor's definition and its pragmaticist development

The point of departure for all, both formal and informal, developments discussed in this paper, is the philosophical definition of gesture given by Hugues de Saint-Victor in the chapter XII of [11]: "Gesture is the movement and configuration of the body's limbs, towards an action and having a modality."[1]

In Section III, we comment in detail how it is possible to translate this definition into mathematical terms, but before we explore its pragmaticist essence, as pointed out in [37]. More precisely, we will apply Peirce's phaneroscopy and see how a semeiotic framework, which is not presupposed in the definition but a consequence of certain particularities of Western music, naturally emerges after triadic elaboration in the phaneron. However, this does not mean that all gradations of the definition in the phaneron are semiotic.

[1] Our translation of the original quote: "*Gestus est motus et figuratio membrorum corporis, ad omnen agendi et habendi modum.*"

## i. Peirce's phaneroscopy

According to Peirce, *phaneroscopy*, or *phenomenology*, is the study or description of the *phaneron* defined as the complete collective that is present to the mind; see [36, Chapter 3] and [33, p. 74]. Phaneroscopy includes the study of the cenopythagorean categories, the *three modes of being*, or the tints occurring in phenomena. We use the synthesis of the three categories made in [36, Chapter 3], which relates these to keywords as follows:

> *Firstness*: immediacy, first impression, freshness, sensation, unary predicate, monad, chance, possibility.
> *Secondness*: action-reaction, effect, resistance, alterity, binary relation.
> *Thirdness*: mediation, order, law, continuity, knowledge, ternary relation, triad, generality, necessity.

In what follows, we regard these categories as the three fundamental *modes* from which we progressively stratify Saint-Victor's definition, and use them by recursion.

## ii. Movement and aim

The fundamental observation of Fernando Zalamea, in the introduction to [37], that Saint-Victor's definition of gesture as "the movement and configuration of the body's limbs with an aim" is fully pragmaticist in Peirce's sense, can initially be interpreted as the fact that it is related to thirdness. If a gesture is the *movement* and configuration of the body with an *aim*, then it is by definition a mediation between two states of the body; a former state or beginning, and a second state or aim. In turn, the movement implies *continuity* and it is an essential manifestation of thirdness, so *a gesture is marked by thirdness*.

As the beginning of a gesture, a first state of the body is firstness. As the end of a gesture, the aim is secondness. The aim necessarily refers to a first state of the body, so it has a dyadic character. According to [33, p. 76]: "The beginning is first, the end second, the middle third. The end is second, the means third. The thread of life is a third; the fate that snips it, its second." The modalities of this thirdness will lead to semeiotics.

## iii. Modality

Now we add to our discussion a new term from Saint-Victor's definition, namely *modality*, which is a new trace of pragmaticism. We may relate it to the three categories of Peirce again to give modal sub-determinations of gestures.

For simplicity, let us consider the instance of piano performance. There is a certain tension between the elasticity of the performer's body and the stability of the instrument, which is mediated by the gesture of the interpreter. Three initial modes of the gesture regarding the piano can be considered.

*1. A raw movement, potential but not reactive.* Not all movements of the performer produce an effective touch of the piano, and hence sounds. Precisely, we refer to those auxiliary gestures in a state of firstness that suggest moods, directions, waves, or continuities, and that are spontaneous and immediate. They are a bodily envelope for sound. Here we recall the famous hums and envelopes of Glenn Gould and Keith Jarrett.

*2. The movement that effectively acts on the instrument and produces sounds.* This is the active-reactive movement of the body. Besides touch, or the active character of the movement, and a first reaction, or the feel of the instrument, there is the main reaction of that touch of the instrument: sound. When the body touches the instrument, it vibrates and this vibration, when propagates

$$\begin{tikzcd} \text{body} \arrow[r, shift left=2, "touch"] & \text{instrument} \arrow[l, squiggly, "sound"] \arrow[l, shift left=2, "feel"] \end{tikzcd}$$

**Figure 1:** *Diagram describing the active-reactive movement of the body.*

through the thickness of air and space, is the sound itself. We picture this situation in the diagram of Figure 1. Here, we remember the violent chords from Bartók's *Allegro Barbaro* or Scriabin's Study Op. 8 No. 12.

*3. Coordination-performance: the movement mediates between the touch of the instrument and the sounds that are produced.* A first mediating instance is *hearing*, or the reactive action of the sound on the body with a certain degree of consciousness.[2] The body is embedded in sound and is enveloped by it, but it reverberates inside the body, makes it vibrate, and, in particular, it affects the brain. Hearing and feeling the sound and the instrument are first instances of interaction and communication in the performance, and allow the performers to take their own decisions about how their movement should become so as to have the appropriate modality when coordinating touch and sound. These modalities are the particularities of each performer; their modes of being regarding the instrument. This opens up a new opportunity to apply the three Peircean modes.

## iv. Natural emergence of semeiotics

In a vague way, three further subdeterminations of coordination-performance can be distinguished, according to the Peircean categories. In this case, modality is introduced in the way that the sound is produced: a spontaneous one, one given by the opposition between forces, or one given by a law or design.

*3.1. Free improvisation.* Probably related to a strong back-and-forth between conscious and subconscious states of the body-mind, free improvisation is spontaneous and is hardly based on previous references or on limitations of musical resources (space, techniques, instrumentation). This is fully exemplified by Mazzola's vision of his activity as a free jazz pianist as one related to immediate gestures rather than mediating formulas. As in the mathematical notion of *free object* [17, Section III.1], in free improvisation the structure is not totally absent, but is skeletal. According to [22, p. 7], free jazz musicians generate their music partaking in a dynamic and sophisticated game, whose rules are incessantly generated and/or recycled, not prescribed, and negotiating what they are going to play. The skeletal configuration, related to these rules, allows a high capacity of transformation and internal movement (free objects have special properties of projection on the objects of the same category), which tends to *multiplicity* and *freshness*.

*3.2. Thematic improvisation.* Related to the dialectics *facts/concepts* and *visible/non-visible*, it is the improvisation that consciously uses a determined system of musical techniques or concepts. If we are located within jazz standards and academic traditions of music, we can define improvisation as the live (while playing) creation with a wide knowledge of musical techniques (harmony, counterpoint, composition).

*3.3. Mediated interpretation.* Perhaps the most important mediation in music performance is

[2]Following Merleau-Ponty's late ideas [30], this definition is insufficient since perception cannot be reduced to the action of the perceived on the perceiver. Moreover, the introduction of consciousness is also problematic since its definition is far from being clear. However, this first approximation is useful for relating musical gestures and Peirce's ideas.

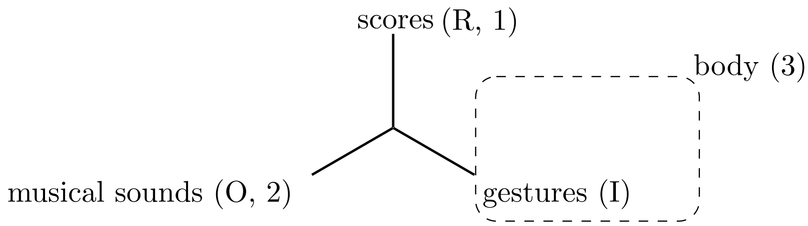


**Figure 2:** *The triad of Western performance as a Peirce's sign.*

vision, whether actual or as an image in the mind. Interpretation, in traditional sense, takes place according to the design of a score. At this point of considerable triadic elaboration, the fundamental sign of the following discussion emerges in a natural way. The sign is Mazzola's triad *sounds/scores/gestures* of Western musical performance, which we study in the following section.

## v. Semeiotics and Mazzola's triad of Western musical performance

According to Peirce, knowledge occurs through signs.[3] A *sign*, in Peirce's sense, consists of an *object* O (or second 2), a *representamen* R (or first 1), and a *quasi-mind* (or third 3), such that the representamen replaces the object for the quasi-mind, in which the representamen is transformed into the *interpretant* I of the sign. The quasi-mind need not be related to a human mind; rather it corresponds to the interpretation context where the *semeiosis*, that is, the transformation of the representamen into the interpretant, occurs. Similarly, the terms O, R, and I are quite flexible: they can be things, concepts or even signs, giving place to *endless semeiosis*, which refers to the infinite iteration of signs.

The sign *sounds/scores/gestures* of Western musical performance, introduced in Section iv, can initially be schematized by means of Figure 2. There, O is *musical sounds*, R is *scores*, and we regard the *body* as the quasi-mind (3) where the scores are transformed into *gestures* (I).

The exhibition of the sign is only the point of departure for the process of endless semeiosis. Certainly, formulas and gestures are different representations of sounds,[4] and more dramatically, the terms O, R, and I are transmutations of each other: scores produce gestures in the body, which produce new sounds by means of the interaction with an instrument, which in turn produce scores in an appropriate (quasi-)mind (think of the process of transcription of a musical piece, or some process of codification of sound), and so on, in an endless spiral. Within this logic of endless semeiosis, there are more possibilities. First, the musical sounds considered can also be representations of previous concepts; in fact, musical ideas can be related to previous ones or even to non-musical ideas like those of poetry or some kind of text. Also, scores can be the interpretant of previous ideas that represent musical sounds; for example, motives, or the subjects on which fugues are developed.

[3]According to one of the reviewers: "This is exactly what Mazzola's gesture theory questions, and this is also congruent with the approach of important French diagrammatic philosophers, such as Alunni and Châtelet." However, we do not see a marked opposition of these philosophers with the more general Peirce's phaneroscopy (Section i), since both knowledge and gesture are thirdness, and hence are strongly related. Thus, in the same way as we located gestures (Section ii) and musical semeiotics (Section iv, Paragraph 3.3) in the phaneron, we might pursue French gestural ideas (also including Merleau-Ponty, Cavaillès, and Desanti).

[4]Although the main point of musical gesture theory is that gestures generate sounds, they can also be representations of sounds, like a dancer's gestures in response to music.

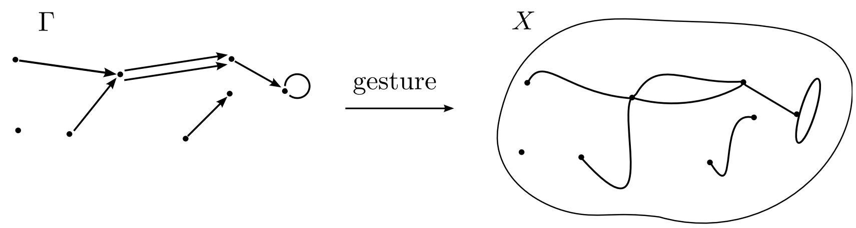


**Figure 3:** *A gesture in a topological space $X$ shaped by a digraph $\Gamma$.*

## III. The hypergestural approach

Saint-Victor's definition[5] is also the inspiration for mathematical gesture theory, as discussed in what follows. In this section, we present Mazzola's initial mathematical insights.

### i. Configuration, body, and space

Based on Saint-Victor's definition, Mazzola and Andreatta give the first mathematical definition[6] of gesture, namely that of *topological gesture*, as a diagram of curves in a topological space $X$; see Figure 3. The shape of the gesture is given by a digraph $\Gamma$, which consists of arrows and vertices suitably connected, and the diagram incarnates the arrows as continuous paths in $X$ and the vertices as points in the space, preserving the configuration of $\Gamma$.

In terms of Saint-Victor's definition, the digraph corresponds to the *configuration* of the body's limbs and the topological space corresponds to the *space*-time where the movement of the *body* occurs. The absence of a load of significance is deliberate in this definition and corresponds to Mazzola's presemiotic approach. A gesture has an existence in its own right, regardless of whether or not it conveys any meaning, as in the cases of dancing or free improvisation. However, as we observed before (Section iv, Paragraph 3.3), the mediation of scores in performance leads to semeiotics.

**Example 1.** We can define gestures in musically meaningful spaces. Let us interpret the Euclidean space $\mathbb{R}^2$ as the *score space*, where a pair $(t, p)$ denotes a sound event with pitch $p$ that occurs at the time $t$. We choose the unities according to the situation. In the following discussion we use the quarter duration as time unity and identify the subset $\mathbb{Z}$ of $\mathbb{R}$ with the diatonic scale indicated by the key involved. For instance, the pair $(1/2, 0)$ denotes the pitch A4 occurring after an eight duration.

Consider the first phrase of Mozart's Piano Sonata[7] K. 331. The melodic contour can be regarded as a gesture in $\mathbb{R}^2$, see Figure 4. As we explain in Examples 4 and 5, the melodic contour plays an important role in Mozart's variations of this phrase since they can be regarded as the result of transforming the original gesture with homotopies and sheaves. ♣

[5] This definition was used by Jean-Claude Schmitt in [35] before the mathematical definition in Section i was formulated by Mazzola and Andreatta.

[6] This definition appeared for the first time in [27]. Other detailed presentations can be found in [3, Section 1.1] and [4].

[7] The authors prepared the excerpts from [32] (theme and variation 1) and the original manuscript (variation 4) at `https://mozart.oszk.hu/`.

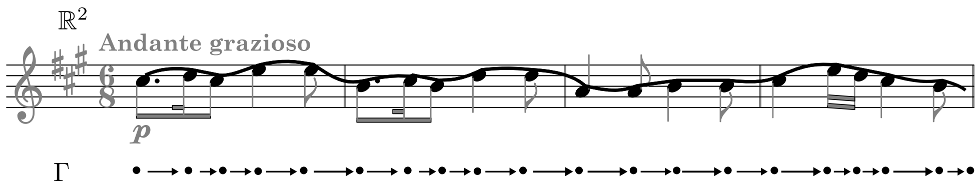


**Figure 4:** *The first phrase in Mozart's K. 331 as a gesture.*

## ii. Hypergestures

Nevertheless, it is not clear why this definition models the human body. For example, the contour gesture in Figure 4 is bi-dimensional. Mazzola's solution to this problem consists in constructing *hypergestures*, which are just gestures of gestures. If we want a gesture of gestures, then our definition requires the construction of a *space of gestures* replacing $X$. This construction is one of the central *concepts* of gesture theory. In fact, we can equip the set of all $\Gamma$-gestures in $X$ with a natural topology, in the sense that it is a categorical construction with a universal property. Thus, once we have the space of gestures $\Gamma@X$, given another digraph $\Gamma'$, we can consider $\Gamma'$-gestures in $\Gamma@X$ and the space of hypergestures $\Gamma'@\Gamma@X$. This construction leads to models of the human body.[8]

# IV. Gestures and the notion of space

Traditionally, we model space-time with the Euclidean space $\mathbb{R}^n$, for instance, in classical mechanics. However, it is worth to ask whether space and body, which is embedded in space, are decomposable in terms of points, like $\mathbb{R}^n$ or any topological space.

## i. Locales

*Locales* are generalized topological spaces[9] whose primitive notion is that of open, regardless of whether or not the latter has points. This is more congruent with the fact that we tend to experience space and time in terms of neighborhoods and intervals, and that our bodies are more the result of the synergy of organs and limbs rather than aggregates of atoms.

Since locales need not have points, which are possible incarnations of the vertices of a digraph, it is not reasonable to define an individual gesture in a locale. Instead, we consider *locales of gestures* that are analogous to spaces of gestures, and can be computed via the categorical constructions of limits and exponentials. There are non-trivial locales of gestures with no points at all, as shown in the following example.

**Example 2.** In contrast to the idealized model in Example 1, a musical sound cannot exist in a time instant —instead it is a vibration that needs an interval of time to propagate. Consider the topology on $\mathbb{R}$ whose basic opens are all half-closed intervals of the forms $[a, b)$ and denote the associated topological space by $H$, which is Hausdorff. We interpret these intervals as those time spans where a sound usually exists.

[8] Drawings of approximations to the human body by means of hypergestures are in [27, p. 33]. We do not include similar figures here since, in Section V, we present an alternative and more natural model of the human body based on simplicial sets.

[9] In fact, the wide class of sober spaces, which includes Hausdorff and have the property that we do not loose information by replacing them by their lattices of opens, is embedded in the category of locales.

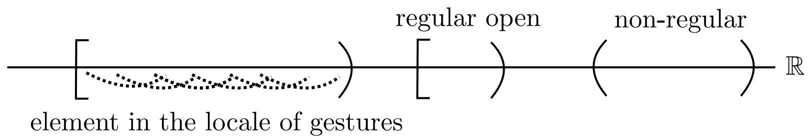


**Figure 5:** *An element of the locale of all opens in H whose closures coincide with their interiors, also called regular opens (center), which can interpreted as time spans with onset where sound can exist. A non-regular open (right-hand side). An element in a typical locale of gestures (paths), which consists of an indecomposable family of gestures, depicted as pointed curves (left-hand side).*

We consider the locale $L$ of all opens in $H$ coinciding with the interior of their closures, where the join and meet is given by the interior of the closure of the union and intersection in $H$, respectively. In this case, half-closed intervals are in $L$, whereas open intervals of the form $(a, b)$, which have no onset, are not—see Figure 5. Also, this locale is an example of locale with no points at all[10] and can be regarded as a non-atomic model of time reconstructed from the existence of sound.

On the other hand, we can consider the exponential locale of paths $L^{\mathcal{O}(I)}$, which is just the locale of gestures on $L$ for the arrow digraph $\bullet \to \bullet$. It could be interpreted as a locale of gestures on time, induced by gestures on space-time. This locale has no points again,[11] but has many elements representing indecomposable families of gestures. Indeed, the locale $L^{\mathcal{O}(I)}$ is generated [16, p. 320] by the symbols of the form $W(u, a)$, where $u \in \mathcal{O}(I)$ and $a \in L$, which represent families of paths, akin to the generators of a compact-open topology. For example, $W(I, a)$ represents the family of paths whose image is contained in $a$; see Figure 5. ♣

The fact that locales need not have points and their limits and exponentials are difficult to compute could be unattractive. Certainly, we are used to thinking analytically, in terms of points, but it is a profound question whether locales are more appropriate to model space-time and the human body than topological spaces.

## ii. Grothendieck topoi

Nevertheless, locales are a bridge between topological spaces and the broader notion of space, namely Grothendieck topoi, which is central in mathematics. *Grothendieck topologies* are defined in categories and their primitive notion is that of covering, instead of those of point or neighborhood. In a sort of yoga, Grothendieck replaces a site (category with a Grothendieck topology) by its category of *sheaves*, the latter called *Grothendieck topos*, which has all limits and colimits, in much the same way as a topology is suitably closed under intersections and unions. We have accessed the continuous realm from the discrete in an astonishing way.[12]

The potential of recovering the substantial movement behind functions and, more generally, morphisms, was one of the essential features of gestures stressed by Mazzola in his first papers on gestures [27, 24]. For example, the linear transformation of the plane associated with a rotation matrix only takes into account the beginning and end but not the intermediate movement, which is a gesture. This gave rise to the question whether there is an analogue of the Yoneda embedding that allows to represent a given category in terms of gestures. *The Yoneda embedding* naturally

[10] See [18, p. 489]. An alternative proof can follow the lines of [3, p. 20].

[11] In fact, there is a homeomorphism [3, Proposition 2.4.2] $pt(\Gamma @ L) \cong \Gamma @ pt(L)$ and $pt(L) = \emptyset$, so $pt(\Gamma @ L) = \emptyset$ whenever $\Gamma$ has at least a vertex, where $pt$ assigns to each locale its space of points.

[12] A complementary discussion on the emergence of Grothendieck topologies and their relation to *sheaves* can be found in Section VI.

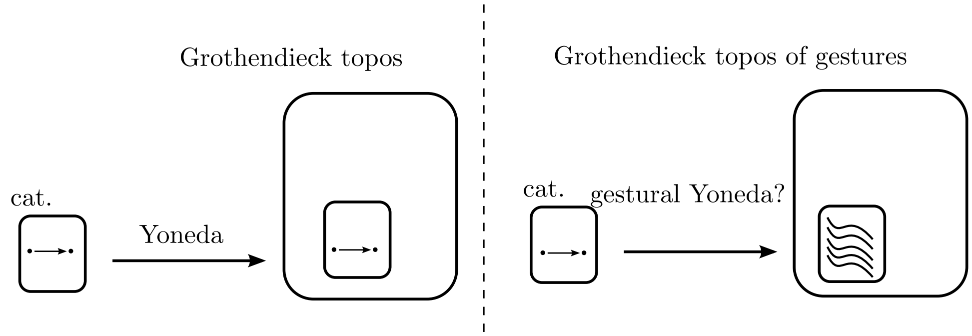


**Figure 6:** *The Yoneda embedding in a Grothendieck topos of presheaves (left-hand side) and a possible gestural Yoneda embedding (right-hand side).*

represents a given category as a full subcategory of its associated category of presheaves, the latter being a particular example of Grothendieck topos, since presheaves are sheaves for a certain trivial Grothendieck topology. This means that the Yoneda embedding is already a representation in a generalized space, so a possible gestural embedding could be refined thereof; see Figure 6.

For this reason, some authors embarked on the enterprise of defining gestures on Grothendieck topoi [3, Chapter 4]. Once again, there can be no points, so, as locales taught us, the definition of *a gesture* is not a reasonable approach. Instead, we define a Grothendieck topos of gestures, but this definition has an important difference. The natural structure of the collection of all Grothendieck topoi is not that of a category, but it is a *2-category*,[13] which also takes into account the natural transformations between geometric morphisms (the analogues of continuous maps) of Grothendieck topoi. Besides, the notion of limit, which is basic for constructing an object of gestures, is not appropriate in this context, and rather we talk about *bilimits*. Consequently, we define the *Grothendieck topos of gestures* as an appropriate bilimit. This renders the theory of gestures on Grothendieck topoi very challenging in technical terms. For example, for profound reasons, we can only ensure the existence of the Grothendieck topos of gestures if the digraph involved is finite [3, Remark 4.2.3].

Up to now, we do not know whether this notion effectively helps solve the problem of the gestural Yoneda embedding. However, the consideration of this level of generality was essential in the development of a theory of gestures [3]. First, it led to consider the intermediate case of locales, which is halfway between Grothendick topoi and topological spaces and concentrates the spatiality of Grothendieck topoi. Second, it was useful for formulating a comprehensive definition of gestures on objects of categories, by means of limits, and on objects of 2-categories, by means of bilimits, without using points.

Next, we discuss Mazzola's approach to the gestural Yoneda embedding problem.

## iii. Topological categories

Another way of introducing topologies in categories corresponds to *topological categories*. A topological category is a category such that their sets of objects and morphisms are topological

[13] Unlike other concepts in the Glossary, providing complete definitions of 2-category [17, Section XII.3], bilimit, and exponential in 2-categories, which would take tens of pages, will not improve the intuition of the non-specialists on the subject. It is enough to bear in mind the analogies topological space/Grothendieck topos, continuous function/geometric morphism, limit/bilimit, and topological space of gestures/Grothendieck topos of gestures. We only include the discussion to motivate the concepts and invite the reader to review [3, Section 4.2].

spaces and all operations (domain, codomain, identity, composition) involved in the category definition are continuous. Thus, topological categories mix point-set topology and categories.

**Example 3.** The score space in Example 1 can be regarded as a topological category. The objects are the elements $x \in \mathbb{R}^2$, and the unique morphism from $x$ to $y$ is the triple $(x, y, y-x)$, where $y-x$ is the vector difference. The third component $y-x$ is the *interval* between the sound events $x$ and $y$. For example, the third component $(1/2, -2)$ denotes a descending third that occurs between two sound events separated by an eight time interval. The composition of two such interval morphisms is given by the formula in Equation 1, and it corresponds to addition in the intervallic components:

$$(y, z, z-y) \circ (x, y, y-x) = (x, z, z-x), \tag{1}$$

The topological space of objects is $\mathbb{R}^2$ with the usual topology, and the space of morphisms is a subspace of $\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$ with the product topology. This category is actually a *groupoid* since its morphisms are invertible, and mix the algebraic intervallic and topological structures of the score space. ♣

The paper [24] proposed the existence of a bicategory of gestures, based on topological categories, as a possible universe for the Yoneda embedding. However, the proposed embedding was not full. We comment another proposed solution by Mazzola in Section VI.

Based on the original sketches [24], the complete definition of gestures on topological categories is in [1] and [6], together with some additional developments of the theory. Regarding the definition, the last paper studies the explicit topological category structure of the topological category of gestures, including constructive examples—the explicit presentation of the morphisms and topologies on morphisms and objects of the topological category of gestures were absent in [24, Section 2.2]. On the other hand, the *exponential presentation* problem of a given topological category of gestures, which relates it to a category of functors with domain the *category of continuous paths of a digraph*, was initially addressed, but not solved, in [6]. This includes a complete description of the topological structure of the mentioned category of continuous paths, which is a non-trivial *object* that mediates between the discretion of categorical diagrams (linked to transformational theory) and the continuity of gestures (linked to the performer's body). In the following section we discuss how abstract gestures are a *conceptual* unification between these discrete and continuous worlds of mathematical music theory.

## V. Abstract gestures and the simplicial approach

Though the immediate manifestation of gesturality that we perceive is linked to the human body and space, in the following lines we try to explain why the notion of gesturality is not spatial in essence. Mathematical details of this section are in [3, Chapter 3] and [7].

*Diagrams* in categories, which in a vague way can be considered as the diagrams that the mathematicians draw in the blackboard in courses of abstract algebra or homological algebra, are perhaps the most important non-spatial counterpart of bodily gestures. In turn, from a more musical point of view, diagrams of musical transformations, for example, affine transformations of musical modules, give rise to *transformational theory*, which studies the natural symmetries and correspondences that occur in music, and has important applications in musical analysis [26].

The notion of *abstract gestures* just establishes a dialogue between bodily gestures and diagrams in a natural and powerful way. Let us justify this strong assertion.

Both in topological gestures and diagrams in categories there is a common pattern. There is a skeleton or shape that represents an abstract configuration, which incarnates in a particular

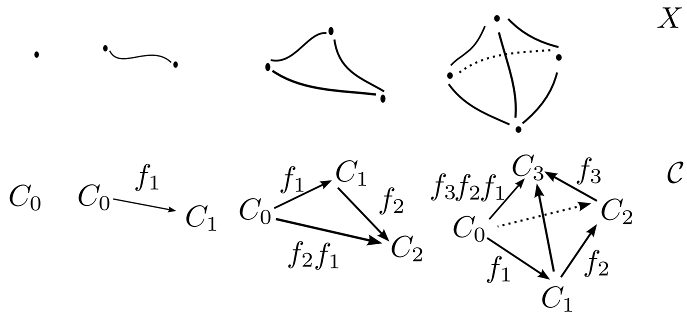


**Figure 7:** *A point, a path, a triangle, and a tetrahedron in a space $X$, which are singular $n$-simplices for $n = 0, 1, 2, 3$ (first row). An object, a morphism, a commutative triangle, and a commutative tetrahedron in a category $\mathcal{C}$, which are $n$-simplices of the nerve for $n = 0, 1, 2, 3$ (second row).*

context. In the case of topological gestures, the skeleta incarnate as paths and points in a space and, in the case of diagrams, skeleta become morphisms and objects in a category. Moreover, the existence of hypergestures and the need of a suitable model of the movement and configuration of the human body suggest that not only points and paths are important, but objects of higher dimensions are.

Poincaré's idea of studying a space by *triangulation*, leading to the concept of homology in a further stage of development (Section VII), is the base of our approach. Certainly, the *singular complex* of a space $X$ codifies all triangular objects of all dimensions that can occur in $X$ and their main relations (common faces, collapses). More formally, it is the presheaf $\mathbf{Top}(\Delta^{(-)}, X)$ on the simplicial category $\Delta$, where $\Delta^{(-)}$ is the standard simplex functor, which assigns to each natural number $n$ all continuous maps $\Delta^n \longrightarrow X$ from the standard $n$-simplex to $X$, called *singular $n$-simplices*; see Figure 7. The singular complex is a combinatorial object whose abstraction leads to the idea of simplicial set, which is an abstract configuration of simplices (resembling singular ones) suitably pasted. Formally, a simplicial set is a presheaf on $\Delta$. With these formal tools, we define a $\Sigma$-*gesture in $X$* as a natural transformation of simplicial sets from a simplicial set $\Sigma$, representing the abstract configuration of the gesture, to $\mathbf{Top}(\Delta^{(-)}, X)$, representing the possible incarnations in the space $X$. The natural transformation condition ensures that the relations that exist between abstract simplices in $\Sigma$ are preserved by their incarnations in $X$.

Following this definition, given a digraph $\Gamma$, it has a natural extension to a simplicial set $\Sigma$, whose associated notion of gesture is essentially that given by Mazzola since there is a natural bijection[14] between the set of $\Gamma$-gestures (topological gestures) in $X$ and that of $\Sigma$-gestures in $X$, so the simplicial definition generalizes Mazzola's one. However, our definition immediately models the human body by taking a simplicial set $\Sigma$ that triangulates it. Figure 8 (left-hand side) reproduces a sculpture $\Sigma$ by Xavier Veilhan, which models the human body by means of abstract tetrahedra. A suitable $\Sigma$-shaped gesture in $\mathbb{R}^3$ is a human body, as shown in Figure 8.

From the categorical point of view, Grothendieck translated Poincaré's idea into the concept of *nerve of a category*. It codifies all possible strings of composable morphisms, their compositions (faces), and their extensions by using identities (degeneracies). More formally, the nerve of a category $\mathcal{C}$ is the presheaf $N(\mathcal{C})$ on $\Delta$ that assigns to each natural number $n$ all strings of composable morphisms in $\mathcal{C}$ of the form described in Equation 2:

$$C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} \dots C_{n-1} \xrightarrow{f_n} C_n. \tag{2}$$

[14]Moreover, there is a natural homeomorphism [3, Proposition 3.5.2] between the space of gestures (digraph case) $\Gamma @ X$ and the space of gestures for simplicial sets $S \pitchfork \mathbf{Top}(\Delta^{(-)}, X)$ defined below.

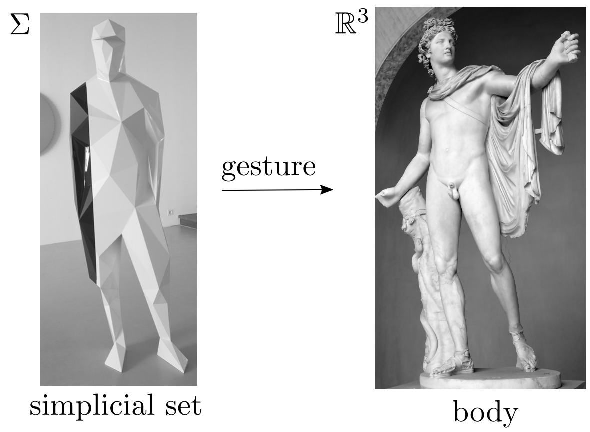


**Figure 8:** *A human body, represented by means of the Apollo Belvedere (right-hand side), can be regarded as a gesture (natural transformation) shaped by a Veilhan's sculpture (left-hand side).*

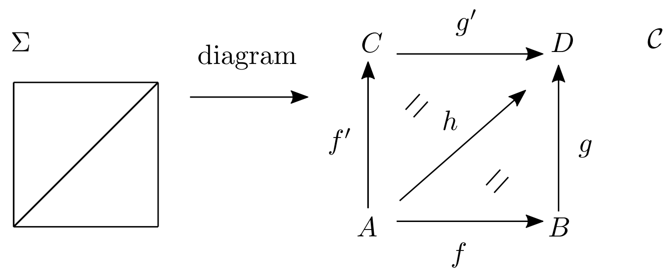


**Figure 9:** *A commutative square as a diagram shaped by a suitable simplicial set.*

These strings, which are the $n$-simplices, can be regarded as abstract triangulations, as displayed in Figure 7. In this case, the analogue of a gesture is a natural transformation from a simplicial set $\Sigma$ to the nerve, which sends vertices (0-simplices) and arrows (1-simplices) of $\Sigma$ to objects and morphism of $\mathcal{C}$, the relations expressed by simplices of higher dimensions being translated into different levels of commutativity relations. This analogue is just a (possibly) *commutative diagram in* $\mathcal{C}$; see Figure 9. In particular, when $\mathcal{C}$ is the category of modules on a commutative ring with affine morphisms between them, which include the natural symmetries that occur in music like transposition and inversion, we obtain the diagrams of transformational theory [26].

We thus conclude that the introduction of simplicial sets in the definition of gesture solves in a simple and unified way, on the one hand, the problem of the higher-dimensional representation of the human body in the topological world and, in the categorical world, the problem of considering diagrams with possible commutativity relations for transformational theory.

We have discussed gestures in the categories of topological spaces, topological categories, locales, categories, and Grothendieck topoi. This justifies a more comprehensive definition of gestures. The notion that unifies all the previous manifestations is that of *abstract gestures*. We define an *object of gestures*, shaped by a simplicial set, in a given category. In mathematical terms, given a simplicial set $\Sigma$ and a *simplicial object* $S$ (presheaf on $\Delta$ with values in a category $\mathcal{E}$), the *object of* $\Sigma$*-gestures with respect to* $S$ is the *cotensor product* $\Sigma \pitchfork S$, which is an object of $\mathcal{E}$, whenever it exists. The most important features of this object are the following:

- As a cotensor product, it has an associated *adjunction* and a *universal property*. This means

that it is a categorical *concept* that has an intrinsic naturality.
- It is the dual concept of that of *tensor product*, and more precisely, for the presence of the simplicial category, it is the dual of the *realization* concept. In the case of topological spaces, the latter is the *geometric realization* of a simplicial set $\Sigma$, which transforms the combinatorial model $\Sigma$ of a space into the space. *Thus, the concept of gestures is the dual of realization.*
- We do not define *a gesture* as a primitive notion, since the objects of the category $\mathcal{E}$ need not have points, like locales.
- However, this definition includes the previous ones of individual topological gestures and diagrams. Topological gestures are elements (points) of the space of gestures $\Sigma \pitchfork \mathbf{Top}(\Delta^{(-)}, X)$, where the singular complex is regarded as a presheaf whose values are topological spaces (*function spaces*). Diagrams are objects (points) of the category of gestures $\Sigma \pitchfork N(\mathcal{C})$, where the nerve is regarded as a presheaf whose values are categories (*functor categories*).

Therefore, gesturality is not inherent to spaces, but can be found in any category.

## VI. Sheaves of gestures

The profound notion of *sheaf* embraces the notions of *number* and *magnitude*. At the same time, it unifies the visions of Galois and Riemann, as follows. On the one hand, the notion of scheme, a suitable gluing of affine schemes (sheaves on the prime spectrum of a commutative and unitary ring whose fibers are local rings), can be regarded as a generalization of that of algebraic variety (number) in the line of Galois-Hilbert-Zariski. On the other hand, the notion of Grothendieck topos, which is made up of sheaves that resemble the sheaf of germs of holomorphic functions in complex analysis, can be regarded as a generalization of that of topological space (magnitude) in the line of Riemann-Leray-Cartan. A great discovery of Grothendieck is that the notion of sheaf is not inherent to topological spaces but it can be formulated in terms of coverings of morphisms in any category, leading to Grothendieck topologies.

In Fernando Zalamea's words, sheaves are the simplest mathematical structures that allow the transit from the local to the global. Given a site, a sheaf is (informally) a correspondence assigning to each object $C$ of the category a set, whose elements are called sections over $C$, and to each morphism a restriction function, in such a way that coherent families (under restriction) of sections on the domains of morphisms of a covering of the Grothendieck topology glue together to form a unique section. The main application of sheaves (in Grothendieck's sense) to our discussion is that gestures can be glued together to produce new ones, thus forming sheaves [3, Theorem 3.12.1]. In mathematical terms, this means that the *presheaf of gestures* $(-) \pitchfork S : \left(\widehat{\Delta}\right)^{op} \longrightarrow \mathcal{E}$ obtained from the object of gestures construction, where $\widehat{\Delta}$ is the category of simplicial sets, is a sheaf for the Grothendieck topology of epimorphic sieves.

We can interpret the sheaf of topological gestures as follows. If we have a cover of a skeleton by skeleta that project exactly on it and a coherent family (indexed by the skeleta of the cover) of gestures, then the family yields a unique gesture, built from the initial family. This is an endless process that extends throughout all possible skeleta. We represent a fragment of the sheaf of gestures by means of Figure 10.

**Example 4.** The construction of a global gesture from local ones can be observed in the fourth variation in Mozart's K. 331; see Figure 11. The melody is obtained by taking the harmonic tones in the theme (Example 1 and Figure 4) and a melodic and rhythmic uniformization. Remarkably,

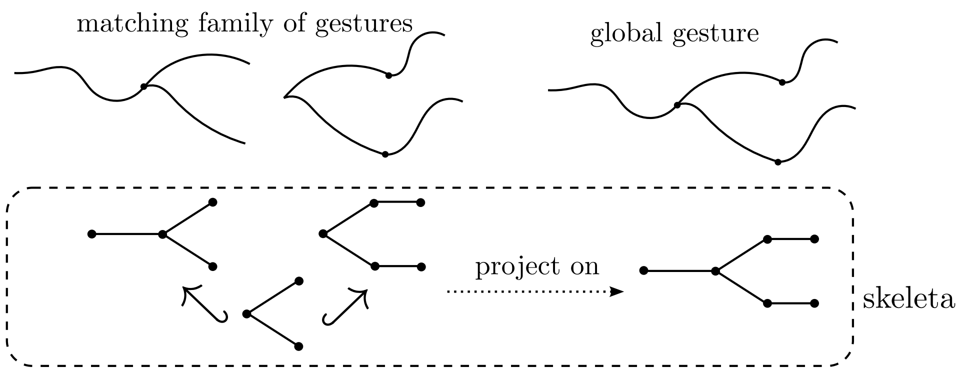


**Figure 10:** *The sheaf of gestures: local gestures that are compatible extend to global ones.*

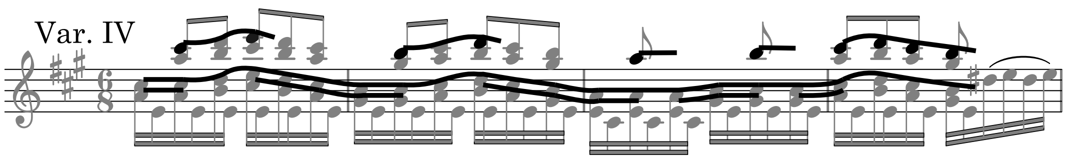


**Figure 11:** *A global gesture in the fourth variation in Mozart's K. 331 (middle) from local melodic gestures (top and bottom). The harmonic tones are the black dots. We omit the digraphs, which are of the form* $\bullet \to \bullet \cdots \bullet \to \bullet$*, for simplicity.*

the melodic fragments (gestures) in the top of each measure are coherently pasted (transposed by an octave) to form a global gesture that accompanies the melody. ♣

Conversely, the definition of sheaves can be given in terms of cotensor products [3, Section 3.12]. Probably, this could express that *a sheaf is made up from gestures*: in fact, for each object of the category, each covering of it, and each coherent family, a sheaf expresses a sort of movement of individual sections (limbs), with the configuration given by the covering involved, an objective (to form a section), and a modality (uniqueness of the section). In this way, sheaves can be related to the human gestures of solidarity and will, which oppose chaos.

The existence of sheaves of gestures also has a philosophical interpretation because it gives a precise mathematical proof of a previous intuition due to Cavaillès: the importance and possibility of multiplication of gestures, their solidarity, and their *analytic continuation* in Riemann's sense, as basis of understanding. Recalling the recurrent reference to Cavaillès in Mazzola's works [27]: "*Comprendre est attraper le geste et pouvoir continuer.*"

Presheaves of gestures or *gestural presheaves* [29, Section 62.7] were also used to give a version of Yoneda lemma [29, Theorem 39] for gestures on topological categories. However, the result gives an embedding of categorical digraphs (analogue to topological and localic digraphs) of topological categories into gestural presheaves, which need not guarantee an embedding of topological categories into gestural presheaves.

Finally, there is another interesting way of addressing the problem of the local/global in gesture theory, namely the *global gesture* construction [29, Section 66.5] for gestures on topological categories, which is analogue to the global composition concept [29, Section 66.1]. In this case we give a cover of a digraph and a different gesture shaped by each member of the cover. But in this case the gestures are not defined in the same topological category, so we paste them by means of isomorphisms between the topological categories involved.

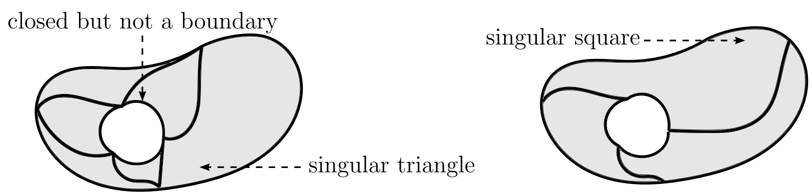


**Figure 12:** *Triangulation (left-hand side) and squaring (right-hand side) of a surface with a hole.*

## VII. Homology

Singular *homology* of a topological space is a tool created by Poincaré [34] so as to associate with the space certain invariants under continuous deformation (*homotopy*), namely Betti numbers, which are, intuitively, numbers of holes of different dimensions in the space. The development of this concept gave rise to the powerful language of *homological algebra*, which allows the correct formalization of Poincaré's ideas (*homology modules*) and the effective computation of Betti numbers. See [12] for a detailed account of the birth of homology theory. Later, the study of the dual notion of homology, that is, cohomology, led Eilenberg and Mac Lane to discover *category theory* via *natural transformations*; see the notes in [17, p. 29]. In fact, category theory is the natural environment of homological algebra, as witnessed, for example, by Grothendieck's Tôhoku paper [15], which achieved a unified treatment of (co)homology.

If the space is a surface in Euclidean space, the idea of homology is *to perform* a *triangulation* of the surface, that is, an approximation by means of simpler objects, by using singular triangles (in the surface) so as to detect holes by identifying closed trajectories that are not boundaries of singular triangles; see the left-hand side of Figure 12. Thus, homology can be regarded as a measure of how far a space is from being an ideal object (triangle). The procedure is generalized to solids, by detecting holes with tetrahedra, and then to higher dimensions, by using simplices.

However, these ideal objects can be changed, so as to produce other homologies. For example, if we consider squares (Figure 12, right-hand side) and cubes instead of triangles and tetrahedra, we will arrive to a homology that is essentially equivalent to the first. After all, these objects are equivalent for the purpose of detecting holes.

Another way of interpreting homology is inspired by the *Yoneda lemma*: we take a good subset of (ideal) domains to try to classify objects. For the category of sets, it is sufficient to consider the singleton domain, for digraphs it is the singletons and the one-arrow object. In homology, we try to classify spaces by using domains that are simplices.

Now, squares and cubes are instances of hypergestures in dimensions 2 and 3, so we could use general hypergestures to define homology [25]; see Figure 13. We would hope to gain new invariants of a space that were not grasped by classical homology or different computations of the latter by gestural means.

But hypergestural homology has a structural drawback. It is a particular case of cubical homology of a certain semi-cubical module, so it should be refined by means of a certain normalization process, based on degeneracies, so as to obtain trivial homology modules of contractible spaces. This point remains obscure since we have not a clear definition of degenerate hypergestures in this context. Some alternative definitions of hypergestural homologies, which are perhaps more natural, were given in [8] but they share the same problem.

The fact that simplicial homology does not require a normalization process, and that we have defined higher-dimensional gestures in the simplicial context, are powerful enough reasons to try a notion of simplicial gestural homology. This notion, as well as that of infinity-category of

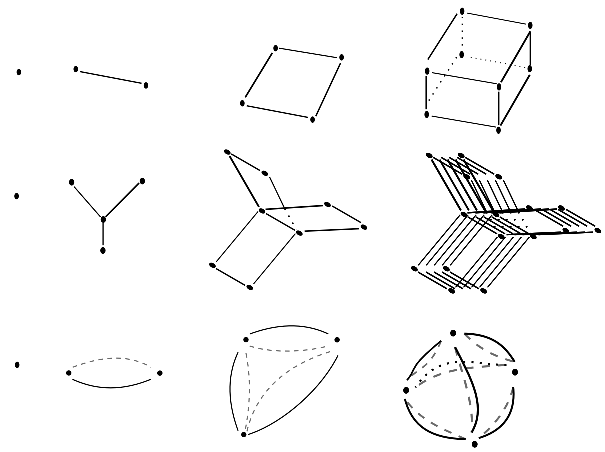

**Figure 13:** *Hypercubes (first row). Hypergestures (second row). Gestures of higher dimensions, in the simplicial approach, associated with a certain cosimplicial space (third row). Compare with Figure 7.*

gestures, is based on the observation that singular simplices can be regarded as gestures, under our definition in Section V. In fact, each continuous map $\Delta^n \longrightarrow X$ corresponds to a unique $\mathbf{y}_n$-gesture in $X$, where $\mathbf{y}_n$ is the combinatorial simplicial set modeling the space $\Delta^n$. Thus, classical simplicial homology is gestural and can be generalized by changing the models $\mathbf{y}_n$ for more general shapes. This leads to the notion of *cosimplicial space* (a functor $T : \Delta \longrightarrow \widehat{\Delta}$), which generalizes the Yoneda embedding $\mathbf{y} : \Delta \longrightarrow \widehat{\Delta}$ whose values are all models $\mathbf{y}_n$ for $n \in \mathbb{N}$, and hence to a notion of *gestural homology*, defined as the homology of the simplicial set $\widehat{\Delta}(T(-), \mathbf{Top}(\Delta^{(-)}, X))$ called *gestural complex*, for each cosimplicial space $T$. Some gestures of higher dimensions, which generalize the singular simplices from Figure 7, are pictured in Figure 13.

At once this notion of homology is invariant under homotopy equivalence of spaces, and hence the homology modules of contractible spaces are trivial since gestural homology modules of the point space are. Details in [7].

## VIII. Homotopy

There are two main facts that lead to the study of simplicial homotopy applied to gesture theory. First, thanks to the generalization of gestures to the simplicial context, we can capture more retractions between (digraph-shaped) spaces and locales of gestures. This is due to the fact that by extending a digraph to a simplicial set we have identity arrows and they help collapse many digraphs that were not contractible with plain morphisms of digraphs. An important application of this fact is the determination of new examples of non-spatial locales of gestures, by finding retractions of them to familiar non-spatial locales.[15] Second, the invariance of gestural homology with respect to homotopy equivalence of spaces, as a particular case of simplicial homology, can be obtained via simplicial homotopies of gestural complexes induced by homotopy equivalences of spaces.[16] See [7] for details.

**Example 5.** Let us apply the simplicial retraction to the first movement in Mozart's K. 331.

[15] This reasoning is based on the fact that spatial locales are closed under retracts, so if the retract of a locale is non-spatial, then the original locale is non-spatial.

[16] More precisely, a homotopy equivalence of spaces induces a homotopy equivalence of simplicial sets (gestural complexes), which, in turn, induces a homotopy of chain complexes and isomorphisms of homology modules.


**Figure 14:** *The descending gesture of the first variation in Mozart's K. 331 as an extension, by homotopy and section, of the original one.*

First, consider the vertex digraph • and the arrow digraph • → •. The extension (Definition in [7, Section 2.3]) to simplicial sets of these digraphs are the representable preshehaves $\Delta(-,[0])$ and $\Delta(-,[1])$ on the simplical category $\Delta$, the former being the final object in the category of simplicial sets. Moreover, we have that the topological spaces of gestures $\Gamma@X$ and $L(\Gamma) ⫛ S_X$ (where $L(\Gamma)$ is the simplicial extension of $\Gamma$) are homeomorphic [3, Theorem 3.5.2] for each digraph $\Gamma$ and each topological space $X$, and in our cases they are the space of continuous paths $X^I$ and $X$, respectively. The unique natural transformation $! : \Delta(-,[1]) \longrightarrow \Delta(-,[0])$ induces a continuous map (the presheaf functor $(-) ⫛ S_X$ is a contravariant one to topological spaces) $f : X = \Delta(-,[0]) ⫛ S_X \longrightarrow X^I = \Delta(-,[1]) ⫛ S_X$. However, the same reasoning is not possible with digraphs since there is no digraph morphism from • → • to •; it must send arrows to arrows, but the bullet digraph has no arrows.

Actually, the map $f$ sends an element $x \in X$ to its constant path with value $x$ and has as left inverse the evaluation at 0 map. This means that $f$ is a *section*. Similarly, we can find sections from $\mathbf{n}@X$ to $\mathbf{m}@X$, where $n < m$ in $\mathbb{N}$, and $\mathbf{n}$ denotes the digraph $\bullet \to \cdots \to \bullet$ with $n$ arrows.

In the first variation in Mozart's K. 331, fourth measure, we can observe how the original descending gesture in the score space (Example 1) is extended; see Figure 14. First, the last two notes (C♯ and B) suffer a contraction in their durations, which is just a continuous path, or homotopy, in the space $\mathbf{4}@\mathbb{R}^2$. Then the descending gesture in the last measure extends to one in $\mathbf{7}@\mathbb{R}^2$ by means of sections and homotopies. ♣

## i. Infinity-categories of gestures

Infinity-categories are a convenient language for categories with morphisms of higher dimensions, avoiding the long lists of data and axioms that are usually needed for defining them. For instance, in the 3-category of 2-categories we have pseudofunctors (relating categories), pseudonatural transformations (relating pseudofunctors), modifications (relating pseudonatural transformations), and a lot of compatibility axioms between them. On the other hand, in a topological space, we have elements, paths linking elements, path homotopies, and so on, together with notions of composition. These two analogous manifestations can be synthesized in the definition of *infinity-category*, which is a simplicial set satisfying certain lifting properties that allow to define the usual categorical operations.

In particular, the two most important simplicial sets that we considered in our discussion of abstract gestures (Section V) are examples of infinity-categories: the nerve of a category, related to possibly commutative diagrams (gestures) for transformational theory, and the singular complex, related to the performer's gestures.

**Example 6.** This example is inspired by [20, 19]. Let $\Sigma$ be a simplicial set modelling a human body (Figure 8). Consider a topological space $X$, where the movement of the body occurs (for example, a subspace of the usual Euclidean space $\mathbb{R}^3$), and the topological space $\Sigma \pitchfork S_X$ of all $\Gamma$-gestures in $X$ (Section V). The singular complex $\mathbf{Top}(\Delta^{(-)}, \Sigma \pitchfork S_X)$ is an example of $\infty$-category.

The categorical structure is the following. Objects are $\Sigma$-gestures in $X$, which are just *poses of the body*. The 1-morphisms are continuous paths of poses in $\Sigma \pitchfork S_X$, which can be interpreted as *movements of the body*. The 2-morphisms are paths in the path space $(\Sigma \pitchfork S_X)^I$, that is, paths between movements of the body, etc. The composition of morphisms is carried out modulo homotopy equivalence. ♣

Based on cosimplicial spaces, we can construct many gestural complexes (Section VII). Since the singular complex is a particular case of a gestural complex, this suggests that, under suitable conditions, gestural complexes can be $\infty$-categories. In such cases, we have a notion of composition of gestures, which was an initial motivation for introducing $\infty$-categories in the gestural context [4, p. 103]. Several examples of $\infty$-groupoids [17] of gestures, other than the singular complex, are given in [7].

The introduction of infinity-categories in gesture theory also responds to Mazzola's fundamental problem of rebuilding algebraic operations from gestural instances. In [27, Section 6.1], the fundamental groupoid of a space was acknowledged as an important tool for reconstructing a certain gestural substance (based on loops and their homotopies) behind group operations. In a more general way, $\infty$-groupoids, $\infty$-categories, and, in particular, categories could be identified as $\infty$-categories of gestures, thus providing fully gestural information on their algebraic operations. For example, by a central theorem of Quillen [14, Theorem 11.4], we know that in an appropriate sense, each $\infty$-groupoid is equivalent to the singular complex of a certain topological space, so we can regard operations in an $\infty$-groupoid in terms of homotopies related to gestures shaped by combinatorial simplices.

# IX. Applications and relations to other fields

## i. Mathematics

Within mathematics we have the following incidences of gesture theory.

### Locales

In locale theory, a way of exhibiting examples of non-spatial locales is to take suitable exponentials and limits. Since the construction of locales of gestures requires these constructions, many locales of gestures are non-spatial. Thus, gestures are useful for providing examples of non-spatial locales.

### Realizations

It is very curious that, unlike the realization concept, its dual (gestures) has no substantial reference in the categorical literature, up to the knowledge of the authors. For example, geometric

[17] Informally, $\infty$-groupoids are $\infty$-categories whose morphisms are invertible.

realizations and *fundamental categories* are the realizations associated with the singular complex and nerve functors, but the respective gestural notions are overlooked.

Realizations are central in *abstract homotopy theory*. Besides the importance of geometric realizations, fundamental categories of simplicial sets are a basic tool for understanding the usual categorical structure behind $\infty$-categories.

#### Infinity-categories

Some gestural complexes (Section VII) are examples of infinity-groupoids that have a topological flavor, but that are not singular complexes, as shown in [7].

#### Sheaves

The definition of sheaf can be given in terms of cotensor products (see Section VI), which suggests a profound relation between sheaves and gestures.

#### Homology

By [31, Lemma 5], each gestural homology (Section VII) is equivalent to the classical homology of a certain space. This and the plasticity of gestures suggest that we could use gestural homology to compute classical homology modules. For example, in [7], there is a simplified proof of a Hurewicz theorem based on gestural homology.

### ii. Mathematical music theory

In mathematical music theory, where gestures were conceived, we have the following connections.

#### Transformational theory

Diagrams are abstract gestures (Section V). Also, the topological structure of the topological category of continuous paths of a digraph [6] can help understand classical diagrams as continuous processes, near to Mazzola's ideal of recovering gestures from morphisms.

#### Counterpoint theory

One of the main notions of counterpoint, and of Mazzola's model, is that of *interval*. Mazzola's intuition of intervals as gestures is supported by the fact that there is an infinity-groupoid of intervals, which links intervals to paths, via Quillen's vision of infinity-groupoids as singular complexes. Moreover, this groupoid is useful for determining when a partition $\{K, D\}$ of the ring of intervals $\mathbb{Z}_{12}$ has an affine symmetry. This is important, since the existence of such a symmetry is necessary to open up a non-traditional counterpoint world based on consonances (in $K$) and dissonances (in $D$). A recent exposition of mathematical counterpoint theory is [5].

## X. Open and closed problems

#### The diamond conjecture

In essence, *the diamond conjecture* asserts the existence of an adjunction between the category of topological gestures and the category of formulas, the latter understood as diagrams of transformational theory. The adjunction is the composition of two adjunctions linked by a category that would offer a unified comprehension of music [27, Section 9].

However, there are several technical reasons that suggest a review of Mazzola's notion of formula [3, Section 3.11.2]. More generally, the notion of abstract gestures is already a *concept* that unifies all notions of spatial gestures, and transformational diagrams. Hence, the statement of the conjecture could be revisited under this notion. However, the conjecture surmises the existence of a unifying *object*, which has not been found up to now.

### Gestural Yoneda lemma and the problem of rebuilding categories and operations from gestures

As commented before, a solution has been proposed in [29, Section 62.7], in the context of topological categories. The generalization of the representation theorem of categorical digraphs [29, Theorem 39] may be explored for abstract gestures. It is also an open problem whether the notion of gestures on Grothendieck topoi is useful to provide such a lemma. The existence of infinity-categories of gestures also suggests the possibility of rebuilding operations in (infinity-)categories from gestural movements. This deserves a serious study.

### The problem of rebuilding topological gestures from algebraic data

Locales help address the problem of representing the movement of the human body in algebraic terms. In turn, this can be helpful for introducing gestures in a computational language. In fact, the topology of every space of gestures on a sober space is embedded in a locale of gestures, as shown in [3, p. 18], and each individual gesture is a point of a certain locale of gestures [3, Corollary 2.4.3], locales being particular cases of lattices.

The characterization of the singular complex and more general gestural complexes as infinity-categories also shows that gestural movements can be managed at the level of categorical operations. Other approaches to this problem could be considered.

### Exponential presentation of objects of gestures

This problem is inspired by the fact that each topological space of gestures, in the case of digraphs, can be regarded as a function space whose domain is the geometric realization of the digraph involved [4, Theorem 7.3]. Similarly, the category of $\Gamma$-shaped diagrams, for $\Gamma$ digraph, can be regarded as a category of functors whose common domain is the category of paths of $\Gamma$, the latter being a realization in the category of categories [3, p. 81].

The exponential presentation problem consists in determining whether the object of $\Sigma$-gestures in an object $C$ of a category is the exponential $C^{|\Sigma|}$ for a certain realization[18] $|\Sigma|$, where $\Sigma$ is a digraph, or a (semi-)simplicial set. It has been solved in the cases of semi-simplicial sets $\Sigma$ for topological spaces and the geometric realization, of digraphs $\Sigma$ for locales and the induced geometric realization, and for cartesian closed categories; see [3, Section 3.2]. The case of topological categories was initially studied in [6] without a full solution, but with a characterization of topological categories of gestures as topological categories of functors, whose domains are categories of continuous paths of digraphs. It is also worth asking whether this problem can be addressed in a general framework that does not depend on a particular category.

[18] The realization usually has better homotopical properties than the skeleton $\Sigma$.

#### Normalization of hypergestural homologies

A notion of normalization for hypergestural homology would pave the way for a more complete study of it, including its computation and relation to classical homology. It is not clear whether hypergestural homology provides more information about an space beyond classical homology. See [8].

#### Gestural homologies

In the simplicial case, it is an open problem the explicit computation of gestural homologies, which are based on particular simplicial sets, namely gestural complexes. Also, it is an open problem whether gestural homologies offer simplified computations of classical homologies. Details in [7].

#### Spatiality of locales of gestures

In essence, all locales of gestures on a non-spatial locale are non-spatial [3, Corollary 2.5.3]. Thus, the problem of spatiality reduces to studying locales of gestures on spatial locales. There are several possibilities, depending on the space and the digraphs. A summary of results can be found in [3, Section 2.9]. The most important concepts used were exponential presentation, which is specially linked to the theory of injective locales, and retractions.

Some improvements on the preceding results were made by means of the extension of digraphs to simplicial sets and the introduction of simplicial retractions [7]. Nevertheless, a complete theory is far from being completed.

#### Infinity-categories

Up to now, we do not know if there are gestural complexes that are examples of infinity-categories but not infinity-groupoids [7].

#### Philosophy

The discussion in Section II together with [29, Part XV] can give rise to a lot of philosophical developments on gestures.

## XI. Environment

The place of gesture theory, and more generally mathematical music theory, is privileged in knowledge but usually misunderstood by pure mathematicians and musicians. Some mathematicians could regard mathematical music theory as a not serious one or as a branch that is not of general interest for the mathematical community. On the other hand, musicians could assess this theory as excessively formal or even useless.

However, it is undeniable that mathematics and music can benefit each other. There are many mathematical tools, such as group theory, that have a long tradition of use among composers. Also, as the cases of gestures and Grothendieck's motifs [21] show, there are musically motivated concepts that can bear a considerable mathematical interest. Mathematics is an infinite source of tools for composing and understanding music. Music is an infinite source of inspiration for doing mathematics. In this way, the study of the relationship between mathematics and music is important for human knowledge since it helps understand creativity and multiply it in both fields.

And gestures precisely allow the dialogue between mathematics and music as human activities. Recalling Mazzola's fundamental dialectic [23, Chapter 1]: the performer's gestures, inspired by scores, become real music and the mathematician's gestural intuitions become theorems. Every human activity is a gesture. Everyday technology is full of gestural devices. Consequently, gesture theory should be the easiest branch of mathematical music theory to share with the public.

## Glossary

In what follows $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ denote arbitrary categories.

**Free object** Let $U : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor. Given an object $D$ of $\mathcal{D}$, we say that an object $F(D)$ of $\mathcal{C}$ is a *free or universal object of* $D$ if there is a morphism $\eta : D \longrightarrow UF(D)$ such that for each morphism $f : D \longrightarrow U(C)$ there is a unique $h : F(D) \longrightarrow C$ such that $U(h)\eta = f$. The latter condition is called **universal property** of $F(D)$. *Examples*: if $U$ is the forgetful functor from abelian groups (respectively topological spaces) to sets, $F(X)$ is the free abelian group on $X$ (respectively $X$ with the discrete topology).

**Topological gesture** A *digraph* $\Gamma$ is a quadruple $(A, V, t, h)$ such that $A$ and $V$ are sets whose elements are called *arrows* and *vertices*, respectively. Given a topological space $X$, we can construct the *topological digraph* $\overrightarrow{X}$ of $X$, which is the quadruple $(X^I, X, e_0, e_1)$, where $I$ is the usual interval $[0, 1]$ in $\mathbb{R}$, $X^I$ is the space of continuous paths $c : I \longrightarrow X$ with the *compact-open topology*, and $e_0$ and $e_1$ are the continuous evaluation maps at 0 and 1. A $\Gamma$-*gesture in* $X$ is a digraph morphism from $\Gamma$ to $\overrightarrow{X}$. It consists of two functions $u : A \longrightarrow X^I$ and $v : V \longrightarrow X$ such that $e_0u = vt$ and $e_1u = vh$.

**Compact-open topology** The subbasic opens of the *compact-open topology* on the space of continuous paths $X^I$ are those of the form $\{c : I \longrightarrow X \text{ continuous } \mid c(K) \subseteq U\}$, where $K$ is compact (closed) in $I$ and $U$ is open in $X$. This makes $X^I$ an exponential in the category of topological spaces.

**Space of gestures** Let $\Gamma$ and $X$ be as in the topological gesture definition. The *topological space of all* $\Gamma$-*gestures in* $X$, denoted by $\Gamma@X$, is the limit in the category of topological spaces of the diagram consisting of for each $a \in A$ a copy of $X^I$, for each $z \in V$ a copy of $X$, a copy of $e_0$ whenever $z = t(a)$, and a copy of $e_1$ whenever $z = h(a)$. The elements of the space $\Gamma@X$ are in bijective correspondence with the respective topological gestures.

**Locale** Complete lattice satisfying the distributivity of $\wedge$ (meet) with respect to the infinitary $\vee$ (join). A *morphism of locales* from $L$ to $M$ is a function from $M$ to $L$ that preserves finite meets and arbitrary joins. *Examples*: 1. Given a topological space $X$, the poset of its opens $\mathcal{O}(X)$ is a locale. This construction defines a functor $\mathcal{O}$ from topological spaces to locales sending a continuous map $f : X \longrightarrow Y$ to the inverse image map $f^{-1}(-) : \mathcal{O}(Y) \longrightarrow \mathcal{O}(X)$. 2. (Fundamental example) The lattice of subsheaves of any sheaf on a (small) site is a locale. In the case of the final sheaf (constant sheaf with value $\{*\}$), the subsheaves are the opens of the site, and hence the opens of any site form a locale.

**Locale of gestures** Let $\Gamma$ be a digraph with $\Gamma = (A, V, t, h)$ and $L$ a locale. Consider the continuous endpoint inclusions $i_0, i_1 : \{*\} \longrightarrow I$ of the interval $I$ and the induced morphisms of locales $\mathcal{O}(i_0), \mathcal{O}(i_1) : \mathbf{2} \longrightarrow \mathcal{O}(I)$, where $\mathbf{2}$ denotes the final locale. We can construct the *localic digraph* $\overrightarrow{L}$

of $L$, which is the quadruple $(L^{\mathcal{O}(I)}, L, e_0, e_1)$, where $L^{\mathcal{O}(I)}$ is an exponential[19] locale, $e_0 = L^{\mathcal{O}(i_0)}$, and $e_1 = L^{\mathcal{O}(i_1)}$. *The locale of all* $\Gamma$*-gestures in* $L$, denoted by $\Gamma@L$, is the limit in the category of locales of the diagram consisting of for each $a \in A$ a copy of $L^{\mathcal{O}(I)}$, for each $z \in V$ a copy of $L$, a copy of $e_0$ whenever $z = t(a)$, and a copy of $e_1$ whenever $z = h(a)$.

**Point** If $\mathcal{C}$ has a final object $\mathbf{1}$, we define a *point* of an object $C$ of $\mathcal{C}$ as a morphism $\mathbf{1} \longrightarrow C$. *Examples*: 1. Points in the categories of sets and topological spaces correspond to elements. 2. Points of the locale $\mathcal{O}(X)$, where $X$ is a Hausdorff (or more generally, sober) space, are just elements of $X$.

**Presheaf** *A presheaf on a category* $\mathcal{C}$ is a contravariant functor from $\mathcal{C}$ to a suitable[20] category of sets. All presheaves on $\mathcal{C}$ form a category $\widehat{\mathcal{C}}$ whose morphisms are natural transformations between presheaves. *Examples*: 1. The **representable functors**, which are of the form $\mathcal{C}(-, C)$ for $C$ object of $\mathcal{C}$, are presheaves on $\mathcal{C}$. 2. Any **simplicial set**, which is a presheaf on the simplicial category. 3. The **singular complex** functor $\mathbf{Top}(\Delta^{(-)}, X)$ of a topological space. 4. The **nerve** of a category $\mathcal{C}$ of a category $\mathcal{C}$, which is the presheaf $\mathbf{Cat}(-, \mathcal{C})$ on the simplicial category that sends $[n]$ to the functor category $\mathbf{Cat}([n], \mathcal{C})$ regarding $[n]$ as a poset category, and $\alpha : [n] \longrightarrow [m]$ to $(-) \circ \alpha : \mathbf{Cat}([m], \mathcal{C}) \longrightarrow \mathbf{Cat}([n], \mathcal{C})$ regarding $\alpha$ as a functor.

**Yoneda lemma** Given a presheaf $P$ on $\mathcal{C}$ and an object of $\mathcal{C}$, it establishes a natural bijection between the sets of natural transformations from $\mathcal{C}(-, C)$ to $P$ and $P(C)$. Explicitly, the bijection sends such a natural transformation $\tau$ to $\tau_C(id_C)$.

**Yoneda embedding** The functor $\mathbf{y} : \mathcal{C} \longrightarrow \widehat{\mathcal{C}}$ sending an object $C$ to the representable functor $\mathcal{C}(-, C)$ and a morphism $f : C \longrightarrow D$ to the natural transformation defined by composition with $f$ in each component. This functor is full and faithful by the Yoneda lemma.

**Sieve** Let $C$ be an object of a category $\mathcal{C}$. A *sieve* on $C$ is a set of morphisms with codomain $C$ that is closed under right composition. *Examples*: 1. The **maximal sieve** $t(C)$ consisting of all morphisms with codomain $C$. 2. The **sieve generated by a set** $X$ of morphisms with codomain $C$, defined as the closure of $X$ under right composition. 3. The **restriction sieve** $h^*(S)$ of a sieve $S$ on $C$ along a morphism $h : D \longrightarrow C$, defined as the set of all morphisms $f$ with codomain $D$ such that $hf$ is in $S$.

**Grothendieck topology** A *Grothendieck topology* $J$ on a category $\mathcal{C}$ consists of for each object $C$ a set of *covering sieves* $J(C)$ such that i) the maximal sieve $t(C)$ is in $J(C)$, ii) if $S$ is in $J(C)$, then all possible restriction sieves of $S$ are covering sieves, and iii) if all possible restriction sieves of a given one $S$ are covering sieves, then $S$ is a covering sieve. *Examples*: 1. Let $T$ be a topology (of a topological space) regarded as a category (category of a poset). The sieves generated by open coverings of opens in $T$ are the covering sieves of a Grothendieck topology on $T$. 2. Consider the category $\widehat{\mathcal{C}}$ of presheaves on $\mathcal{C}$. The **epimorphic sieves** of a presheaf $P$ are the sieves $S$ on $P$ such that for each object $C$ of $\mathcal{C}$ the set of images $\{\mathrm{Im}(\tau_C) \mid \tau \in S\}$ covers $P(C)$.

**Site** Category with a Grothendieck topology $(\mathcal{C}, J)$.

[19]The locale $\mathcal{O}(I)$ is exponentiable since it is a continuous lattice and hence this exponential and its derivative constructions exist. Also, the final object $\mathbf{2}$ is exponentiable (in any category) and the exponential $L^{\mathbf{2}}$ is isomorphic to $L$. Details in [2] and [3, Section 2.3].

[20]That is, the sets are the members of a Grothendieck universe [9, Exposé I].

**Sheaf** A presheaf $F$ on a site $(\mathcal{C}, J)$ is a *sheaf* if for each object $C$ of $\mathcal{C}$ and each covering sieve $S$ in $J(C)$, given a family of *local sections* $\{x_f \mid f \text{ in } S \text{ and } x_f \in F(\mathrm{dom}(f))\}$ such that $F(h)(x_f) = x_{fh}$ whenever the composite $fh$ exists, there is a *unique* $x \in F(C)$ such that $F(f)(x) = x_f$ for each $f$ in $S$. In such a case we say that $x$ is a *global section*. *Examples*: given the site of the usual topology of $\mathbb{R}$ (respectively $\mathbb{C}$), the presheaf with $P(U)$ defined as the set of all (continuous or differentiable) functions defined on $U$ with values in $\mathbb{R}$ (respectively $\mathbb{C}$) is a sheaf.

**Grothendieck topos** Category (equivalent to one) of sheaves on a site, which is a full subcategory of the associated category of presheaves. *Examples*: 1. All categories of presheaves are Grothendieck topoi since presheaves are sheaves on the **trivial topology** whose covering sieves are the maximal ones, in particular the categories of digraphs and semi-simplicial sets are. 2. Categories of sheaves on the examples of Grothendieck topologies.

**Exponential presentation** We can state the problem, in the case of simplicial sets (for short), as follows. Let $T : \Delta \longrightarrow \mathcal{C}$ be a functor, where $T$ has all its images exponentiable in $\mathcal{C}$. Given an object $C$ of $\mathcal{C}$, we define $S_C$ as the functor $C^{T(-)} : \Delta^{op} \longrightarrow \mathcal{C}$. This object generalizes the internal versions of the singular complex and the nerve. Given a simplicial set $\Sigma$, we ask whether the object of gestures $\Sigma \pitchfork S_C$ coincides with the exponential $C^{|\Sigma|_T}$, where $|\Sigma|_T$ is the realization of $\Sigma$ with respect to $T$, whenever these objects exist. In the case when $\mathcal{C}$ is Cartesian closed, we solve the problem immediately, since $C^{(-)}$ transforms the colimit $|\Sigma|_T$ into the limit $\Sigma \pitchfork S_C$. Otherwise, the solution is not guaranteed. See [3, Section 3.2] for details.

**Simplicial category** Denote by $[n]$ the ordered set (ordinal) $\{0, 1, \ldots, n\}$ for $n \in \mathbb{N}$. The *simplicial category* $\Delta$ has as objects all $[n]$ for $n \in \mathbb{N}$ and as morphisms all order-preserving maps between them.

**Standard simplex (functor)** For each $n \in \mathbb{N}$, we define the *standard n-simplex* $\Delta^n$ as the set in Equation 3:

$$\{(t_1, \ldots, t_n) \mid 0 \le t_1 \le \cdots \le t_n \le 1\}. \tag{3}$$

The standard $n$-simplex is a subspace of $\mathbb{R}^n$ and this construction defines a *standard simplex functor* $\Delta^{(-)}$ from the simplicial category to the category of topological spaces, which sends an order preserving map $\alpha : [n] \longrightarrow [m]$ to the appropriate continuous map $\Delta^\alpha : \Delta^n \longrightarrow \Delta^m$ sending the $i$th vertex (with $n - i$ zeros) to the $\alpha(i)$th one. *Examples*: $\Delta^0$ is a singleton, $\Delta^1$ is the interval $[0, 1]$ in $\mathbb{R}$; $\Delta^2$ is the triangle with vertices $(0, 0)$, $(0, 1)$, and $(1, 1)$ in $\mathbb{R}^2$; and $\Delta^3$ is the tetrahedron with vertices $(0, 0, 0)$, $(0, 0, 1)$, $(0, 1, 1)$, and $(1, 1, 1)$ in $\mathbb{R}^3$.

**Tensor product** Suppose that $\mathcal{E}$ has small hom-sets and small colimits and that $\mathcal{C}$ is small. Suppose given a functor $T : \mathcal{C} \longrightarrow \mathcal{E}$. Consider the functor $\mathcal{E}(T, -) : \mathcal{E} \longrightarrow \widehat{\mathcal{C}}$ that sends $E$ to the presheaf $\mathcal{E}(T(-), E)$. The *tensor product* $P \otimes_{\mathcal{C}} T$, where $P$ is a presheaf on $\mathcal{C}$, is the free object of $P$ with respect to the functor $\mathcal{E}(T, -)$ if it exists. It can be computed via the colimit[21] in Equation 4:

$$\mathrm{Colim}\left(\int P \xrightarrow{\pi} \mathcal{C} \xrightarrow{T} \mathcal{E}\right). \tag{4}$$

[21] Here, $\int P$ denotes the *category of elements* of $P$. Its objects are of the form $(C, p)$ where $p \in P(C)$ and $C$ is an object of $\mathcal{C}$. A morphism from $(C, p)$ to $(C', p')$ is a morphism $f$ of $\mathcal{C}$, from $C$ to $C'$, such that $P(f)(p') = p$. We denote by $\pi : \int P \longrightarrow \mathcal{C}$ the natural projection functor.

Actually, this construction defines a left adjoint $(-) \otimes_{\mathcal{C}} T$ to $\mathcal{E}(T(-), E)$, which means that there is a bijection, as in Equation 5:

$$\mathcal{E}(P \otimes_{\mathcal{C}} T, E) \cong \widehat{\mathcal{C}}(P, \mathcal{E}(T(-), E)), \tag{5}$$

natural in $P$ and $E$. *Examples*: 1. If $\mathcal{C}$ is the simplicial category $\Delta$, the tensor product $\Sigma \otimes \Delta^{(-)}$, where $\Sigma$ is a simplicial set, is the **realization** $|\Sigma|_T$ of $\Sigma$ with respect to $T$. 2. In the case when $T$ is the standard simplex functor $\Delta^{(-)}$, $\Sigma \otimes \Delta^{(-)}$ is the **geometric realization** $|\Sigma|$ of $\Sigma$.

**Cotensor product** It is the dual of the tensor product notion—we change $\mathcal{E}$ for its opposite in the above definition. Thus, the difference is that now we require that $\mathcal{E}$ has small limits. We give a functor $S : \mathcal{C}^{op} \longrightarrow \mathcal{E}$ and consider $\mathcal{E}(-, S) : \mathcal{E} \longrightarrow \left(\widehat{\mathcal{C}}\right)^{op}$. The *cotensor product* $P \pitchfork S$, where $P$ is a presheaf on $\mathcal{C}$, is the limit in Equation 6:

$$\operatorname{Lim}\left(\left(\int P\right)^{op} \xrightarrow{\pi^{op}} \mathcal{C}^{op} \xrightarrow{S} \mathcal{E}\right). \tag{6}$$

The corresponding natural bijection is given in Equation 7:

$$\mathcal{E}(E, P \pitchfork S) \cong \widehat{\mathcal{C}}(P, \mathcal{E}(E, S)). \tag{7}$$

*Example*: If $\mathcal{C}$ is the simplicial category and $\Sigma$ is a simplicial set, then $\Sigma \pitchfork S$ is the **object of $\Sigma$-gestures with respect to** $S$. This notion includes all gesture definitions mentioned in this paper.

**Simplicial homology** Let $S$ be a simplicial set and $R$ a commutative ring. For each $n \in \mathbb{N}$, *the nth homology of* $S$ is the quotient $R$-module $\operatorname{Ker}(\partial_n)/\operatorname{Im}(\partial_{n+1})$, where $\partial_n = \sum_{i=0}^{n} (-1)^i d_i$ for $n \geq 1$ and $d_i : R^{S[n]} \longrightarrow R^{S[n-1]}$ is the linear extension (to free modules) of the corresponding face of $S$. The $R$-homomorphism $\partial_0$ is the unique from $R^{S[0]}$ to the trivial module $\{0\}$. *Example*: The simplicial homology of the singular complex of a topological space $X$ is the classical homology of $X$.

**Spatial locale** A locale is *spatial* if it is isomorphic to that of opens $\mathcal{O}(X)$ of some topological space $X$. Otherwise we say that it is **non-spatial**.

**Infinity-category** Simplicial set $S$ such that given $n \in \mathbb{N}$ and $k$ with $0 < k < n$, for each subset $\{a_i \mid 0 \leq i \leq n;\ i \neq k\}$ of $S([n-1])$ satisfying the identities of Equation 8:

$$d_i(a_j) = d_{j-1}(a_i) \ \ (i < j;\ i, j \neq k), \tag{8}$$

there is an element $a \in S([n])$ such that $d_i(a) = a_i$ for $i \neq k$. If this property also holds for $k = 0$ and $k = n$, then we say that $S$ is an **$\infty$-groupoid**. *Examples*: 1. The nerve of a category is an $\infty$-category. 2. The singular complex of a topological space is an $\infty$-groupoid.

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# Modeling, Listening, Analysis, and Computer-Aided Composition

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***Abstract:*** *The main objective of this short paper is to present results and a computational modeling process as a tool to aid musical composition. These results are related to the composition of the piece "Intermezzi and Capriccio" from a Fantasia, an ensemble of piano solo pieces worked from a cross analysis of the first Intermezzo of Johannes Brahms' Four Pieces for Piano Op. 119. In this way, the musical analysis I propose aims to compositional results, which in turn guides, the analysis itself. In this sense, the objective is not that of an analysis and modeling in order to reconstruct an original, but of analysis and modeling that allows it to unfold into new objects.*

***Keywords***: *Composition. Brahms. Musical modeling. Computer-aided analysis. Max/MSP. zl objects*

## I. From analysis to patch design

First, it is important to present here what I am considering as musical analysis. I consider that all musical analysis is a fable. It is the invention of connection operators, sometimes evident or not. It is also important to say that I consider the analysis to be something not restricted to the score, and that it involves what I would call a listening scene: a heterogeneous field where music, space and performance are intermodulated. If there is a primacy of the score, it is because the whole space will be modulated by its realization; it would be the structured and repeatable domain of this plan.

For me, as a composer, analyzing is something like finding not only connection operators but also the series of nonsense that emerge in the listening fable and, from there, finding variables that allow the analyzed work to unfold outside itself. In this sense, everything in the scene of listening, performance, reading, participates in what we call the object of analysis.

It was in this sense that I programmed a series of patches, in Max/MSP environment, based on the "analysis of the score reading" of the first Intermezzo of Brahms' Op. 119, considering

*The Brahms analysis presented in this paper was part of the examples showed at the "Porto International Symposium on the Analysis and Theory of Music: Musica Analítica 2019", under the title "Repeating Difference: Music analysis and rewriting". Here, in this paper, I try to present more detailed aspects of the computer modeling used as example at Porto Conference. The author thanks the funding agencies FAPESP and CNPq.

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constants in figural level (intervals, sense of phrases, harmony) and gestural (accents, realization speeds etc.), where each constant opens to a variable.

The proposal is based on studies carried out in the 1970s by André Riotte (composer) and Marcel Mesnage (computer engineer) . Mesnage and Riotte understand that the analysis must be "independent of a particular theory or style, consisting of progressively leading the initial complexity to a simpler combination of entities". I therefore observe that the more defined the material, the more formed and closed in precision relations, the harder the material for its compositional unfolding. In view of a compositional decision, the objective is not to find an "analytical elegance", such as the search for out-of-time symmetries, but processes that have the most open potential for unfolding.

The starting point of this method of analysis is to define the relevance of parameters and processes, and to observe whether they are pertinent to the whole work or just a section and if such pertinence concerns the work or a singular listening to this work. The focus thus is not analysis with an end in itself, but analysis as a compositional objective, with a view to automating the realization and updating of results. Automation here constitutes a prototype of the composition from which some compositional solutions can be maintained or changed.

The modeling carried out that I performed from Brahms' work was implemented in Max/MSP, and is based on the language of lists as present in software from the 1990s such as Patchwork and later OpenMusic. The list language used in modeling that I propose part of Lobjects and ListOps, a series of objects capable of operating with lists in Max/MSP that James MacCartney (ListOps) and Peter Elsea (Lobjects) programmed in the early 1990s to work with lists in Max/MSP. The zl Objects then allowed actions similar to some functions present in Open Music and Patchwork (and PWGL), with the possibility of working in real time of performance.

For the analysis stage prior to the implementation, I made use of a reading of objects and traces of the perceptible surface of Brahms' work, leading to a simple analysis which could kept the main affective elements of the first Intermezzo of Op. 119 by Brahms (first 15 measures are shown in Figure 1), with a compositional rewriting purpose.

1) *The slow descending cascades*: Brahms works with two interval sets in order to draw arpeggios and short melodic structures. One first, formed by the descending sequence of thirds (intervals 3 and 4) and another, also descending formed by seconds (intervals 1 and 2).
2) *The interval universe*: There is a clear interval circularity, an interval constancy and a cyclic set of transpositions that is extremely evident at the first glance at the score: a simple third reiteration algorithm (intervals 4 and 3) within the key you chose, a reiteration that implies the idea of alternation of the two intervals 4 and 3, being able to incorporate in the set of chords the augmented $(4-4-4(-4))$ or diminished $(3-3-3-3)$ sequences, as well as greater or lesser.
3) *The flow continuity always with slightly oscillating progress*: With regard to the temporal structure, Brahms reiterates sixteenth notes, which apparently means a constant, but in a letter to Clara Schumann notes that: "Every measure and even every note must sound like a great rhythm."
4) *The separation between two layers with the presence of a melodic line latent to the acute inflection of the resumes of the arpeggios*: In the specific case, the inflection and accentuation of the higher note (or notes) of the arpeggio enhances latent melodies, which also contributes to the positioning of the note at precise inflection points of the measure or even by prolonging its duration. A compositional strategy, constantly present in this and other works by Brahms.

This implementation step sought not to restrict the interval cycles to the tonal field and to consider only what I am calling the interval color used by Brahms. Out of tonality restriction, it was then possible to find a total of 16 possible permutations of thirds. In order to program a generative algorithm, a generated interval lists were designed to trigger sequences of intervals, with cycles that exceed the 4 intervals of the original and, from them, the sequence of notes is generated, calculated from an inflection note (high or low).

Aiming at a new composition born from the analysis of Brahms, the lists (interval, notes, time) can be transformed resulting in a gradual anamorphosis of the initial gesture (the descending cascades), in order to maintain the initial musical affection in another musical result. The main procedures employed in this regard are: (1) reversal of the arpeggio sense (acute-severe, severe-acute), (2) anamorphosis by expanding the intervals, (3) changes in time with variations around an average time, (4) redistribution of dynamic accents allowing the appearance of latent melodies through highlights, (5) increase or decrease in the interval cycle, (6) inflection note control from an external signal obtained by a musical instrument. One of the most radical elements of anamorphosis is the substitution of the inflection note by a group of notes, in case an arpeggio in a quick appoggiatura from a fragment of the same interval structure employed at the time of automation[1].

## II. Patch steps and algorithm design

Even employing list objects provided for Max/MSP, this language which is quite developed in software such as PatchWork, PatchWorkGL and Open-Music, is nevertheless quite incipient in Max/MSP. There are some list libraries, but more complex operations such as those found in other software are not native to the program and do not cover all needs for musical programming. In this sense, a large part of the operations are carried out through the packaging and unpacking of lists with intermediate calculations.

The intervallic compositional thought found in Brahms is quite simple and is inherent in the play of arpeggios, but for compositional and post-tonal objectives it was, as I already said, considered only as a regular sequence of intervals, which can be changed in half step.

In the series of images below, I expose the Max/Msp modules in detail together with the interval analysis carried out from the original by Brahms:

1) Arpeggio sequence analysis, transformation in cascades, Max-MSP modules with the possibility of working at different intervals. In this way, in addition to the pattern of thirds $(3-4)$ taken from Brahms, the patch opens an interval variable that allows through a panel of cursors (multislider), to configure sequences of thirds, as well as seconds (direct opening up to tritons) or open intervals from one tenth. This mechanism has the function of both partially redoing the Brahms model and expanding it with compositional intent. In view of a limitation of tessitura to that of a normal piano, the patch delimits bass and treble with which when using very open intervals the result generated is of small descending or ascending cycles, thus comprehending the breadth of possibilities of the model learned from Intermezzo of Brahms (Figure 2).
2) Since the purpose of designing the patch was not only to perform an analysis of Brahms' work, but to create a compositional device from Brahms, it caught my attention when observing the arpeggios the possibility of converting cascades of more than five or six notes. In the figure below I present the arpeggios and short melodies (joint degrees) present in

[1] The developed patch allows for a composition in interaction between live instrumentalist (captured instrument or any MIDI instrument) and an automatic composition performed by an object of uninterrupted generation in irregular time cycles of inflection notes based on a simple attractor. For this mode of operation, attractors were also implemented for gradual and close variations of time, number of notes per arpeggio, dynamic variation.

**Figure 1:** *Manuscript draft of the original from Brahms Op. 119.*

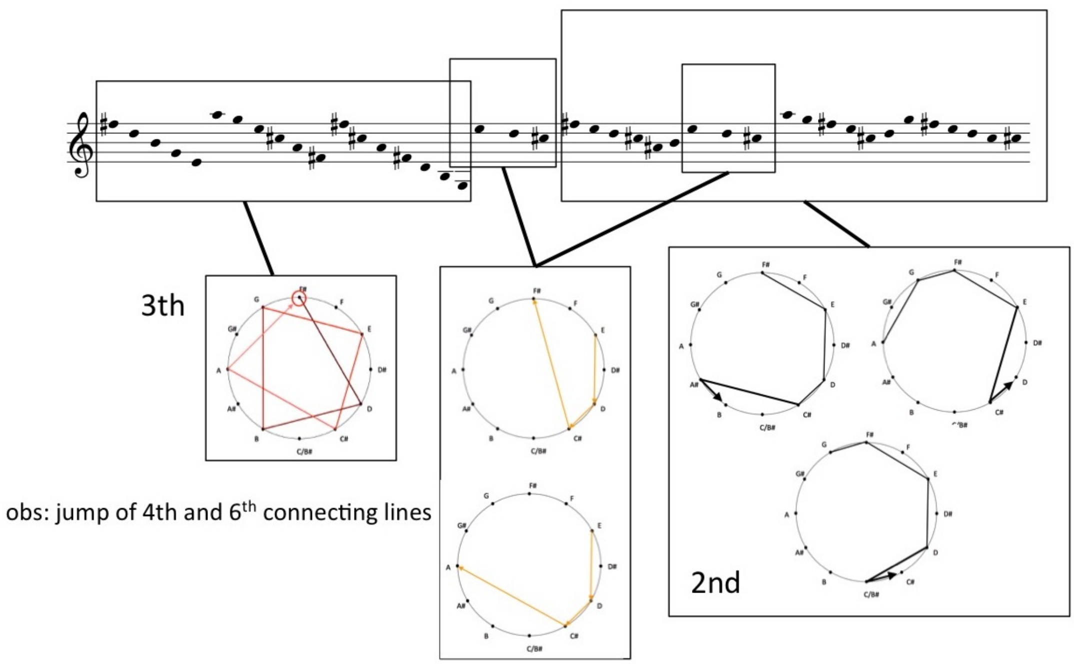


**Figure 2:** *The arpeggio sequences with chromatic clock analysis.*

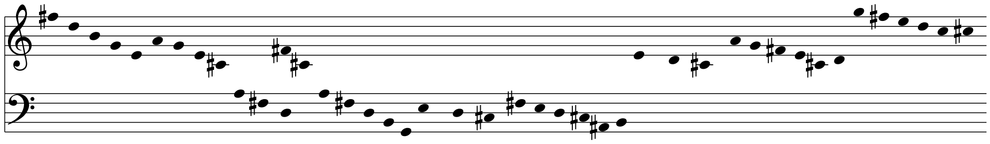

**Figure 3:** *The conversion of the arpeggios and melodic lines into descending cascades.*

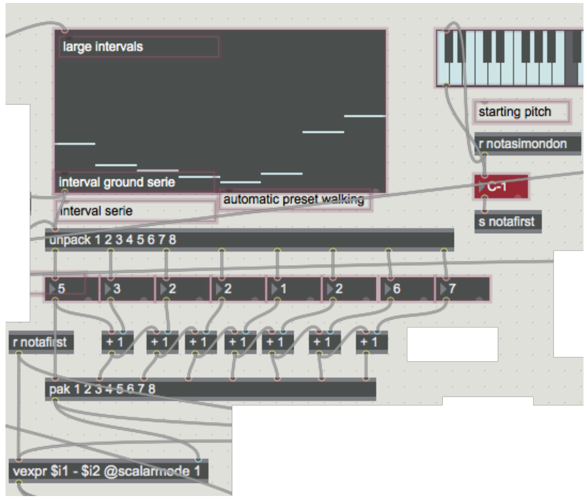


**Figure 4:** *The central interval module, multislider/intervals/plusminus module/sum from a first note (which can come from a MIDI keyboard or a pitch-tracker, allowing the patch to be associated with a signal input audio).*

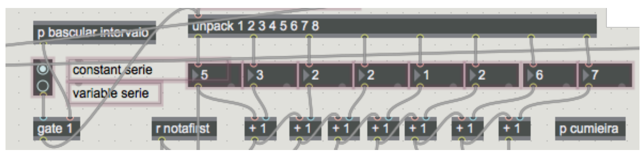


**Figure 5:** *Plusminus modules, which randomly add or subtract halftone to the intervals according to the arpeggios model that alternate intervals 4 and 3 and the diatonic melodic model that alternates intervals 1 and 2). In this module it is possible to work with a constant series of intervals or with a variable series via plusminus.*

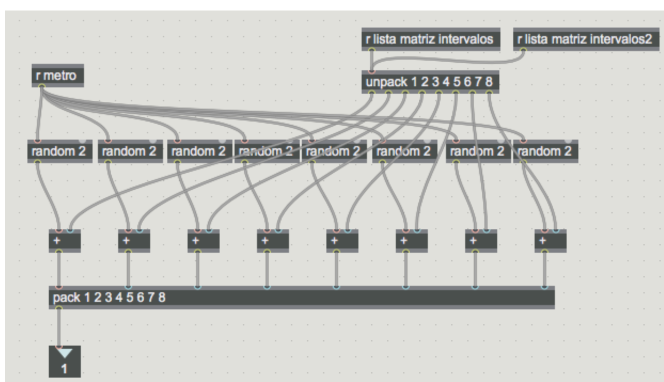


**Figure 6:** *Continuation of plusminus modules.*

Brahms and their conversion into cascades where some notes can be taken as pivots, like the G at the end of the first arpeggio, resumed as the beginning of the next and the F♯ at the end the second arpeggio also resumed to the octave as the beginning of the next (Figure 3).
Another important element in this analysis is the use of latent melody that Brahms performs in the upper voice. In this sequence I also observe an interval constancy and even a regularity and repetition mechanism. This data was used to draw the subpatch <cumeeira> (Figure 9),

which updates each new cascade sequence with a new high note according to interval regularity. From a cascade cycle calculated based on the first note, comes a new note that has a melodic interval relationship with that first.
For the design of this patch it was necessary to design a calculation subpatch x → dx and dx → x that allows calculating a series of notes from a series of intervals and vice versa, native operator in software such as OpenMusic and Patchwork (Figures 7 and 8).

**Figure 7:** *Converting arpeggios to cascades with transpositions based on supposed pivot notes.*

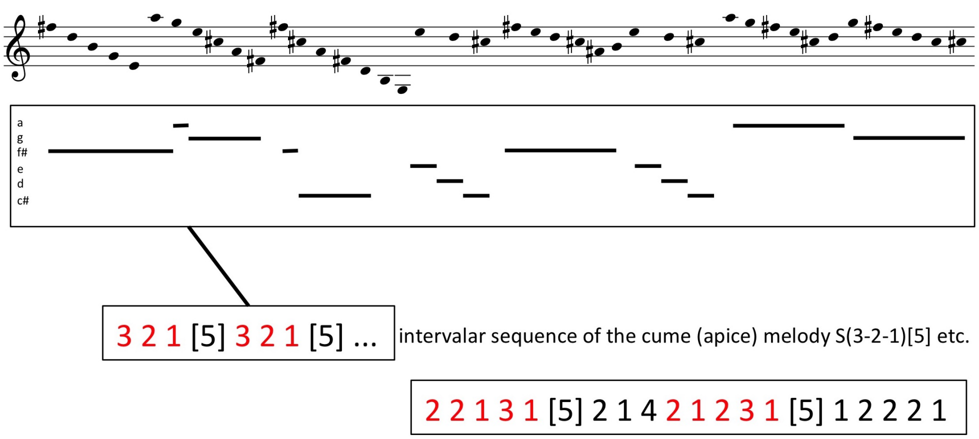


**Figure 8:** *Ridge (which has a correspondence with the <cumeeira> as commented above and in Fig. 5) notes that make up latent melody and its interval sequence.*

3) Still with respect to a compositional project resulting from the analysis and algorithmic implementation in Max-Msp, the patch comprises two important extensions of the original model. The first refers to extremely fast appoggiature performed either with an ascending model or with a descending one coming from the interval system towards the ridge note. In view of this direction, this change of trend of the cascade of notes, it was possible to determine an inflection point in relation to the note derived from the analysis of audio data or sending via MIDI. In general, this direction reversal module only reverses the reading order of the note sequence (Figures 4, 5, and 6).
4) When performing Brahms' modeling, it was of great importance to read Brahms' letter to the composer Clara Schumann, in which he specifically comments on this Intermezzo

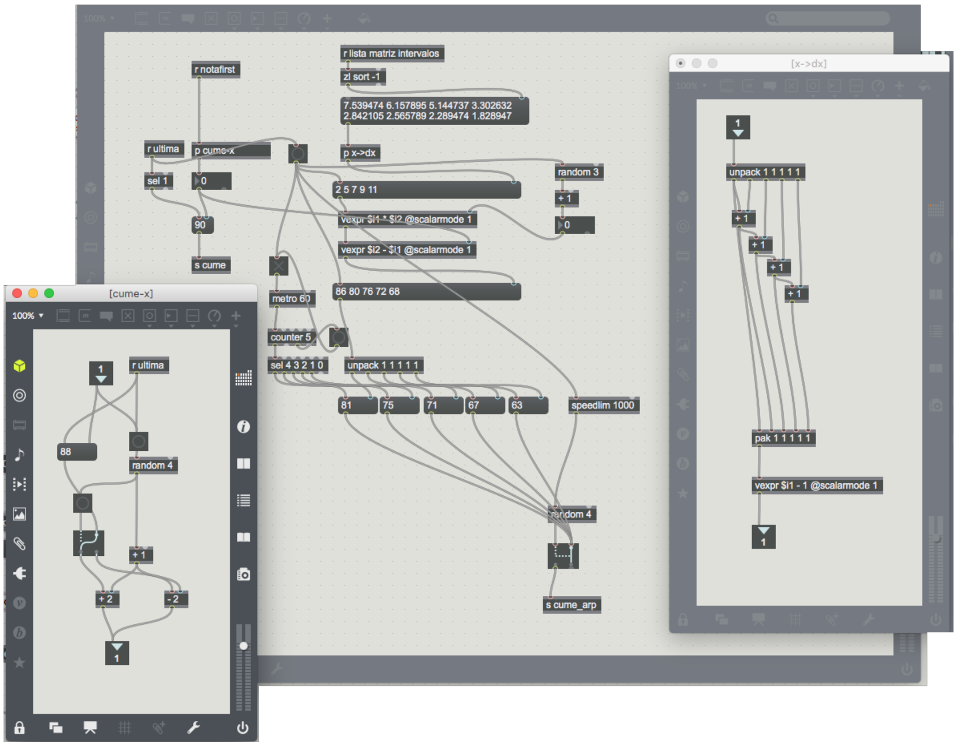


**Figure 9:** *Module (subpatch) [cumeeira] and its subpatches [cume-x] and [x → dx]): receiving and scaling (sort list) the list of intervals in the cascade, which will be applied to the new peak note from a calculation, performed on [cume-x].*

from Op. 119: "Every measure and even all notes must sound like a great rhythm". In this sense, I proposed to write the patch with two interchanging meter structures: stable meter and changeable meter. Through a checkbox it changes to a stable meter or to a changeable one, according to the result of the addition and statistical subtraction of values that make it possible to feel the change in tempo between notes or groups of notes. What is observed is the greater interest in the musical result. For the design of this and other timing devices in the patch I used the idea of pulse, which is expressed by the object <metro>, measure expressed by cycles counted via object <counter>, and which can be changed at any time, and rhythm, adding to the <counter> object a selector that defines which pulses should or should not pass and trigger the final note (Figures 10 and 11).

## III. Final remarks

It is important to note as a conclusion that other elements were considered in the design of the patch, always keeping in mind that each data inserted from the score analysis behaves as a variable according all the time with the opening points that the Brahms system allows. Increase or decrease the number of notes of each downward current (Brahms arpeggio now 5 now 7 notes), modify the tempo until the point where the arpeggios become abrupt or even chords, and even the most unusual version of inversion of the direction (descending to ascending), expanding intervals to the point of working the edges of the piano tessiture (with great jumps) and dismantling the figures

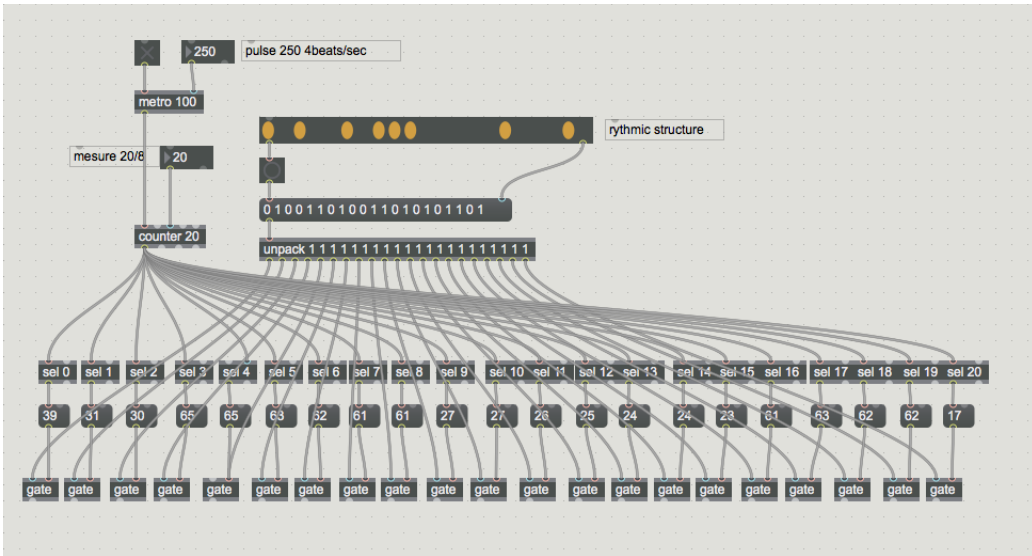


**Figure 10:** *Patch to change tempo (metro) and rhythmic structures.*

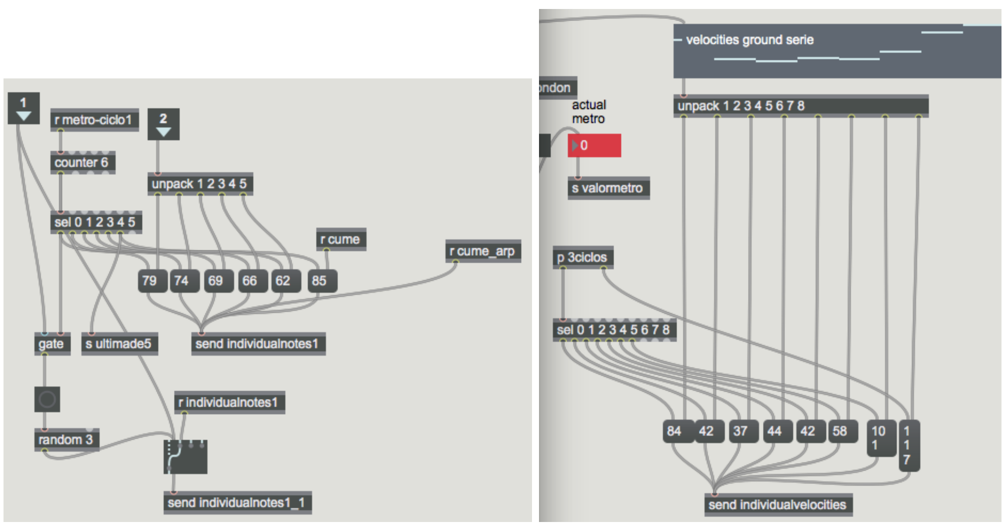


**Figure 11:** *Two modules from the patch created for modeling Brahms Intermezzo. The first example, related to one of the cycles (here with 6 notes) and the second related to the dynamic parameter that will be assigned to each new note in the arpeggio set.*

of descending cascades.

By having each note programmed as an individual element, it was also possible to control accents for each one of those notes, thus causing cycles of a certain size to be sectioned by ways of accentuating peak notes or even notes in meaningless positions. Such modifications allowed changes in the sonority of the piece, giving rise to the appearance of quick arpeggio gestures, chord structures, overlapping lines, amplification of tessitura, design of complex rhythmic structures.

In a compositional point of view, the patch follows a former idea of cycle superimposition. Each cycle have a different compass, and each one operates with different gesture elements: notes and intervals to compose the circular cascades, the introduction of accents assigning cascades apices or points of inflexion, the incrustation of fast apoggiattura gestures, large intervals changing allowing the born of new gestures, etc.

Finally, it is important to say that these open variable implementation of the patch, served as an experimentation field to develop compositional strategies I implemented when writing a composition of two "Intermezzi and Capriccio" for solo piano, which are part of a cycle entitled Fantasia for piano, I wrote in 2013 (Figure 12). In this piece I tried to start as close as possible to Brahms model and expand it with incrustation of new gestures and changing of basic (but pertinent) parameters leading to gesture changing. In this last figure I show tow moments of Intermezzo I, where a simple changing of intervals could transform radically the texture, leading to the birth of a new gesture in the entire compositional flow.

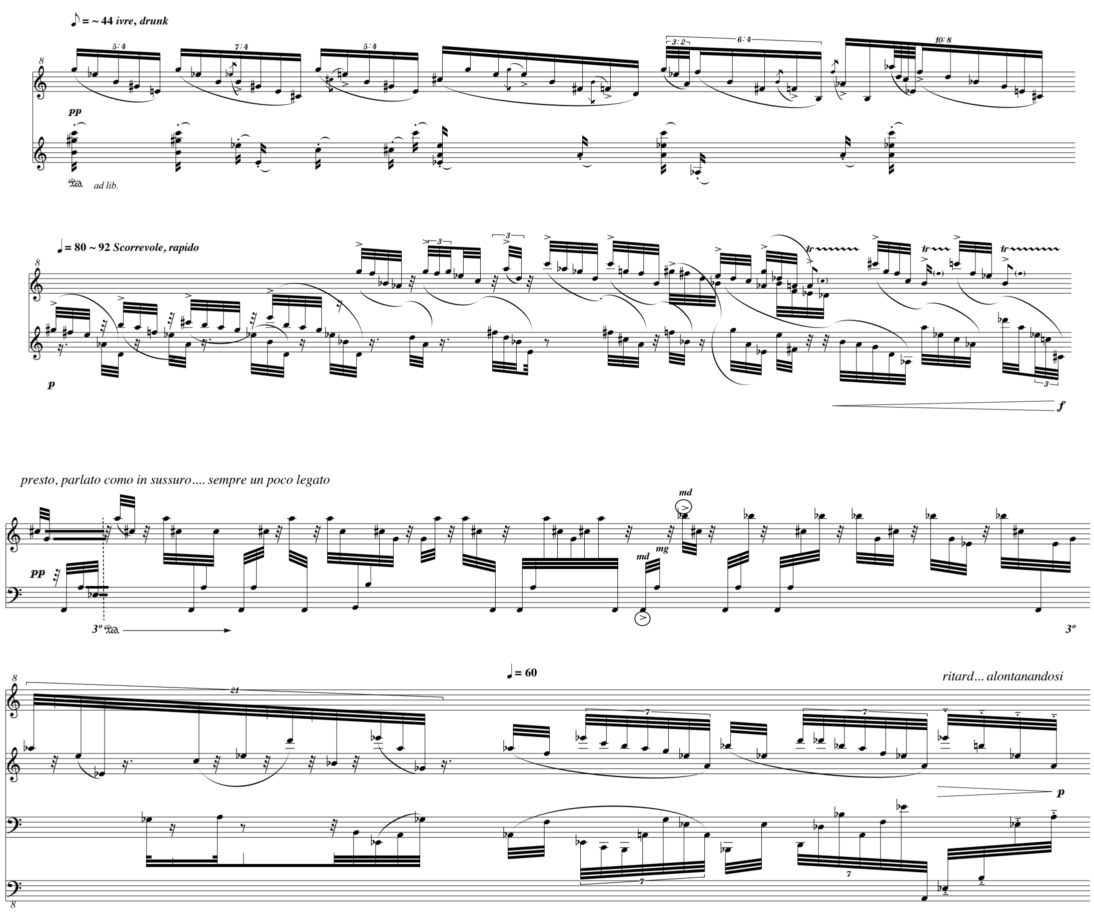


**Figure 12:** *Four excerpts from the score of Fantasia for solo piano (Intermezzo I and II). It shows different results from the same patch for a modeling from Brahms Intermezzo Op. 119.*

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# Measuring the Amount of Freedom for Compositional Choices in a Textural Perspective

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***Abstract**: In this paper I discuss the relation between the number of available compositional choices and the complexity in dealing with them in the scope of musical texture. First, I discuss the paradigm of compositional choice in light of the number of variables for a given situation. Then, I introduce the concept of compositional entropy—a proposal for measuring the amount of freedom that is implied in each compositional choice when selecting a given musical object. This computation depends on the number of available variables provided by the chosen musical object so that the higher the compositional entropy, the more complex is the choosing process as it provides a high number of possibilities to be chosen. This formulation enables the discussion of compositional choices in a view of probability and combinatorial permutations. In the second part of the article, I apply this concept in the textural domain. To do so, I introduce a series of concepts and formulations regarding musical texture to enable such a discussion. Finally, I demonstrate how to measure the compositional entropy of textures, considering both the number of possible textural configurations a composer may manage for a given number of sounding components (exhaustive taxonomy of textures) and how many different ways a given configuration can be realized as music in the score, considering only textural terms (exhaustive taxonomy of realizations).*

***Keywords**: Compositional entropy. Musical texture. Textural layout. Exhaustive taxonomy. Probability and combinatorial permutations.*

## I. Introduction

As Marisa Rezende ([27, p. 77]) states: "composing means, among other things, to make choices." [1] This means that *choosing* is an intrinsic aspect of the compositional process as the endemic characteristic of a piece is determined by the composer's idiosyncratic choices of materials (pitches, pitch classes, rhythmic structures, dynamics, timbre, textures, etc.),

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[1] *Compor significa, entre outras coisas, fazer escolhas.*

aesthetic orientation, compositional techniques, sonic means, and the like. Each choice within the compositional process may contribute to unfolding the musical form so that being aware of the creative implications of it is a crucial expertise for any composer. In this paper I am concerned with mapping the number of compositional choices that are available for composers during the compositional process in a textural perspective. To do so, I will first discuss the relation between the number of choices and the complexity in dealing with them, which can be measured by what I call *compositional entropy*. Then, I will present the theoretical framework of musical texture that underlies this paper so that the set of available choices a given texture may hold, that is, their exhaustive taxonomy to be implemented as music, can be introduced.

7

## II. Compositional Choices

In general, I call *compositional choice* a decision made by the composer that contributes to advance the construction of a piece of music. It may involve from the simple selection of materials to the very definition of the compositional strategy to be used. Composers deal with compositional choices all over the creative process in a more or less controlled (or conscious) way. From a temporal perspective, compositional choices can occur in either in *real-time* or *out of time*. The difference between both applications is in the order of bottom-up/top-down compositional approach. Real-time compositional choices involve definitions in a linear perspective, i.e., the composer defines over time how to advance a piece of music considering the possibilities that are suitable to his/her aesthetic orientation. Therefore, each choice is conditioned in part by the context it is made considering its contribution to unfold the musical form. Out-of-time compositional choices, on the other hand, consists of defining a priori how a piece of music shall be constructed. In other words, it involves prior decisions to define materials, structures, processes, or any other instance that may lead the construction of a piece. In any case, compositional choices can be related to creativity given that the more possibilities of choosing composers are aware of, the more creative and imaginative solutions they may explore in their music. For each musical parameter, there is a possible compositional choice to be made so that a piece of music may be understood as the sum (or combination) of all compositional choices made during the creative process (Figure 1).[2]

In this paper, I will focus on out-of-time compositional choices, discussing their creative implications and potentialities within the compositional process. Thus, I shall henceforth refer to them as just compositional choices to avoid wordy expressions.

Most often, compositional choices involve the construction of a compositional design (or plan) to organize the various elements of the piece beforehand. This stage is prior to the very process of writing the chosen elements in the musical score. In this context, compositional choices consist of the process of defining an abstract musical object in the scope of specific musical parameters (or attribute[3]) from its set of possibilities.

Assume, for example, that $H$ is a set of given musical parameter that hold three discrete objects ($H = \{a, b, c\}$). If $H$ is a sequence of notes, then compositional choices concerns choosing any number of notes from the set $H$ from one (a unitary object) up to three (the whole set or the *universe*

[2]Note that the musical parameters in the figure are not ordered hierarchically, that is, the top-down disposition does not necessarily define order each choice was made. Moreover, in a wider compositional spectre, these compositional choices could be interpreted as the actual piece without the urge of being combined with the others.

[3]In the present paper, musical texture is not understood as a musical parameter, as it emerges from the combination of them. Rather, it is a musical attribute that concerns organization. See section IV

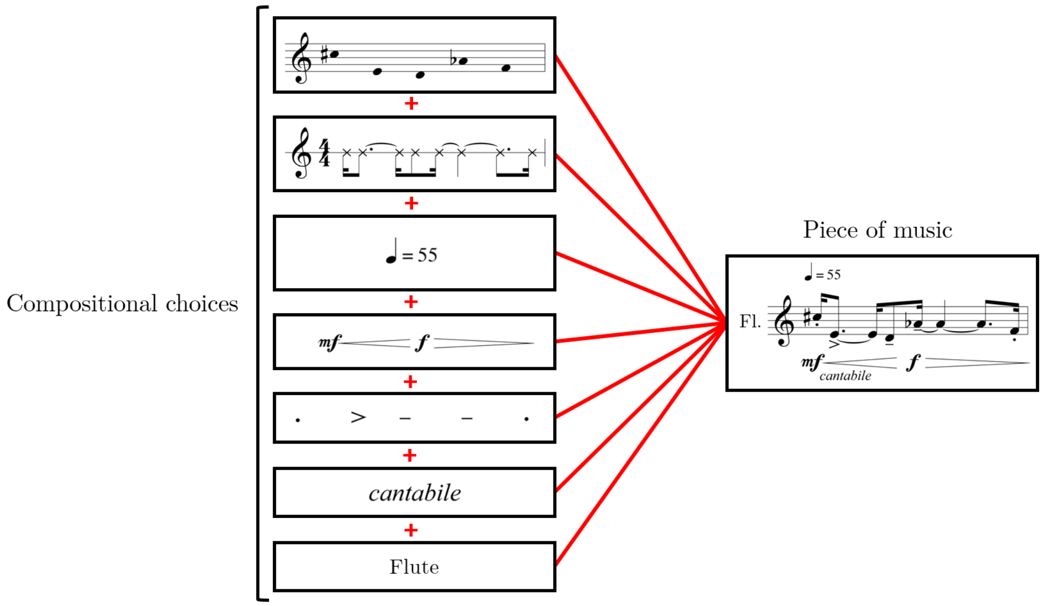


**Figure 1:** *The combination of various compositional choices applied to different musical parameters resulting in a piece of music.*

*set*[4]). If the compositional choice involves choosing a single object, then there are three possible choices available for the composer: $\{a\}$, $\{b\}$, or $\{c\}$. A compositional choice of two objects, in turn, may also provide three possibilities of choice, considering the pairwise combination: $\{a,b\}$, $\{a,c\}$, and $\{b,c\}$.[5] Finally, a compositional choice of three objects in $H$ comprises a unique possibility of choice that is the set $H$ itself.

The calculation of the number of possibilities available in a compositional choice is given by the combination formula $C(n,r)$ (read as "$n$ choose $r$"), where $n$ is the number of variables of the set and $r$ is the number of components to be chosen so that $r$ must be equal to or less than $n$ (Equation 1):

$$C(n,r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}, \text{ for } \ r \leq n. \tag{1}$$

Table 1 shows the application of this formula in the scope of pitch classes, with the number of compositional choices ($r$) ranging from one to twelve.[6] Besides the number of possibilities for $r = 6$, all possibilities have a pair in a different number of compositional choices. This happens because they are the complement to one another, that is, for each possibility $K$ of a given $r$

[4]In Set Theory, the universe set, noted as $\mathbb{U}$, stands for a collection that contains all possible variables of a given element to be considered in a specific situation or in a given purpose. Any set of elements can be understood as $\mathbb{U}$ so that it is taken as a reference to its possible subsets. Note that the definition of a $\mathbb{U}$ is not applicable in any contexts whatsoever. Within the scope of Zermelo-Fraenkel. It is possible to define the more general object so-called *category*, but it is outside of the scope of this paper to deal with such objects. Furthermore, when $\mathbb{U}$ contains all possible variables of a given element, then it is indeed the exhaustive taxonomy of that element.

[5]Note that each one of them is a subset of $H$.

[6]Note that this number of possibilities does not consider class principles defined in pitch-class set theory formulation as number of prime forms of pitch-class set is considerably lower. See [5].

**Table 1:** *The relation between the number of compositional choices (r) and the number of possibilities for the set of twelve pitch classes (n) given by $C(n,r)$.*

| Number of choices ($r$) | Number of possibilities |
|---|---|
| 1 | 12 |
| 2 | 66 |
| 3 | 220 |
| 4 | 495 |
| 5 | 792 |
| 6 | 924 |
| 7 | 792 |
| 8 | 495 |
| 9 | 220 |
| 10 | 66 |
| 11 | 12 |
| 12 | 1 |

there is a possibility $J$ so that their union contains all twelve pith classes ($\mathbb{U}$).[7] As could not be otherwise, the compositional choice with the lower number of variables to choose corresponds to the whole set of pitch classes. One may note therefore that exhaustive taxonomies are of greatest interest for composers within compositional choices since the wider is the $\mathbb{U}$ of variables, the more compositional choices can be made from that.

Compositional choices may be cumulative within the same musical parameter. After choosing a pitch class, for example, the next compositional choice may involve defining its position in the register (pitch), its duration, its timbre, the way it shall be articulated and, so on. Each stage of compositional choice leads toward the realization of the piece as music.[8] A distinction shall be made among the various stages of compositional choices. In the present work, the process of defining a musical object is referred to as a compositional choice of the first instance so that any choice made afterward is of the second instance. Thus, the second instance concerns decisions to be made considering the available options provided by objects defined in the first instance. This implies that a composer may choose a musical object instead of other in the first instance based on the number of possible realizations it provides for the second instance. In other words, the number of possible ways of realizing a given musical object as music may be the most decisive criteria for choosing it beforehand.

Consider, for example, the information provided by Table 1. One may say that the compositional choice involving the selection of twelve pitch classes leaves no room for compositional choices since the unique option to be chosen is the whole itself. Nevertheless, in the second instance several other compositional choices may coordinate it in a myriad of ways. For example, by simply including an ordering factor to organize it sequentially in time, a composer can create up to 479,001,600 of different twelve-tone rows (12!). Furthermore, its realization also involve cumulative compositional choices concerning various musical aspects, such as pitches, their absolute duration, their dynamic, articulation, timbre, and so on. As a consequence, the number of possibilities within

[7] Note that the complement for $r = 12$ is $r = 0$.

[8] In this paper, the idea of *realizing as music* expresses process thereby a musical object (or abstraction) is written in the score, considering their temporal organization, as well as the inclusion of a series of other variables that are not implied by them. Therefore, music that is not supported by musical notation is out of the scope of the present paper. Additionally, I shall consider the realization only in the terms of traditional notation, which excludes graphical indications, open musical forms, interpretative indeterminacy, and the like.

each compositional choice to achieve its realization is astronomic, given the multiple variables and their possible combinations. In this paper, for the sake of simplicity, the discussion of the possible realizations of musical objects will consist of only an intermediary stage between the compositional design, where the objects are to be chosen, and their actual realization in the music score. Thus, when I refer to a possible realization of a given musical object, I am, in fact, speaking about its available possibilities in the second instance in terms of this intermediary stage. A intermediary stage may be as simple as the example above regarding the inclusion of order for the set of pitch classes.

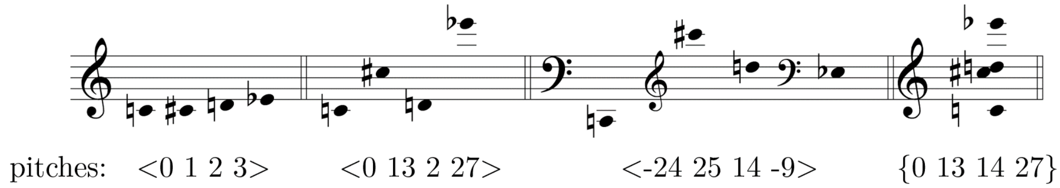


**Figure 2:** *Four realizations of the ordered set of pitch classes <0123> considering pitches.*

Based on the information provided by Table 1, a composer may choose four pitch classes among 495 possible compositional alternatives. A compositional choice of the second instance may involve the simple inclusion of order. The number of compositional choices in this second instance is defined by the possible permutation of notes, which is equal to 24 (4!). Another compositional choice would involve the articulation of this sequence as pitches—a note in a specific register—so that the number of available choices is defined by the number of possible realizations of these pitch classes as pitches. This computation is not as simple as in compositional choices of the first instance or as that one regarding order as several factors shall be considered. For example, it is necessary to consider the range of the instrumental mean to calculate how many different pitches are available for each pitch class of the sequence. Within an octave, the number of available compositional choices for their realization as pitches in a sequential order is equal to the number of choices involved in the order (24). In these terms, the sequence of pitch classes may have different possibilities depending on the chosen instrument to be realized. Therefore, the very choice of the instrumental mean is a crucial factor for computing the number of possibilities in the instance of realization. If the composer defines the sequence will be written in a flute, for example, the number of possibilities would be considerably smaller than if he/she chooses a piano solo given their difference in the range of available pitches. This number may also increase by including the possibility of pitches coinciding in time.

Figure 2 illustrates some of the possible realizations of the set of pitch classes [0123], considering both order and pitch.[9] In all realizations, the order of pitch classes is preserved. In the first measure, the notation of pitch classes and pitches is equivalent; pitch classes are arranged from low to high in both time and register from the middle-C.[10] In measures two and three, pitch classes are ordered only in time as pitches are placed in several registral spans. Finally, in the last measure, all pitches are articulated at once in a single temporal span, which means the order is defined only in the register from low to high.

[9] The notation of pitch classes consists of ascribing an integer for each pitch class so that 0 refers to C-natural, 1 stands for C-sharp or D-flat, 2 corresponds to D-natural, and so on up to 11 that refers to B-natural.

[10] In the notation of pitches, each note is written according to its intervallic relation with middle-C, that is labeled as 0. Then, the D-flat (or C-sharp) one semitone above is 1, the adjacent D is 2, and so on. The B directly below middle-C is defined as –1, the B-flat (or A-sharp) a semitone below is –2, and so on. See [22].

In the context of other musical parameters, computing the number of possible realizations in the second instance would involve different factors suitable to their specific natures. If the musical object is a duration, for example, its variables could imply, for example, the definition of its metric position in the temporal span. Some compositional aspects are naturally non-practical in terms of realizations of the second instance; the very definition in the first instance provides its realization. Consider, for example, the definition in the first instance of a given tempo marking (e.g. quarter note equals to 100bpm). This choice already defines its unique realization that is the tempo marking itself. So there are no variables to be chosen in the second instance.

It should be clear now, from the discussion provided so far, how compositional choices operate in the creative process to define the elements and their possible realizations. On the face of it, a composer may argue that the wider the set of musical objects to be chosen and/or their possible realizations in the second instance, the better for compositional choices as it provides more options to be chosen. Although this might be true, an extensive number of possibilities may also intricate the compositional process.

Concerning the way people deal with the choosing process, Sheena S. Iyengar and Mark R. Lepper [9, pp. 999-1000], based on various researchers, say that in front of an extension of choices some people:

> may actually feel more committed to the choice-making process; that is, that they may feel more responsible for the choices they make because of the multitude of options available. These enhanced feelings of responsibility, in turn, may inhibit choosers from exercising their choices, out of fear of later regret. In other words, choice-makers in extensive-choice contexts might feel more responsible for their choices given the potential opportunity of finding the very best option, but their inability to invest the requisite time and effort in seeking the so-called best option may heighten their experience of regret with the options they have chosen. If so, choosers in extensive-choice contexts should perceive the choice-making process to be more enjoyable given all the possibilities available. They should at the same time, however, find it more difficult and frustrating given the potentially overwhelming and confusing amount of information to be considered.[9, p. 1000]

This situation is perfectly suitable to the compositional context, in which the choice overload may lead to a creative block—a state in which the composer is unable to decide a path to follow in the face of the universe of possibilities. The so-called "dilemma of the blank page", where the composer has no idea on how to start his piece or what kind of music he/she wants to compose, is a trivial example of this creative block. This situation can be explained by the complexity of dealing with a wide set of variables at once. This can be a critical issue for composers to manage both objects and their realizations.

Liduino Pitombeira proposes the idea of an *organized complexity*, "a category in which is located the vast majority of human problems, including music analysis and composition." ([25, pp. 39-40]). Based on this idea, Pitombeira defines a continuum of complexity in such a way that organized complexity is located between *organized simplicity*, which involves simple deterministic problems holding up to four variables, that "can be handled, for example, by calculus and differential equations", and *disorganized complexity*, "which involves the use of probability and statistics to deal with an astronomical number of variables." (Figure 3).[11]

In the compositional process, organized complexity may be understood as the realm in which composers create their music in a more or less restricted way. Disorganized complexity is therefore

[11] This proposal is based on the work "Science and Complexity" by Warren Weaver.

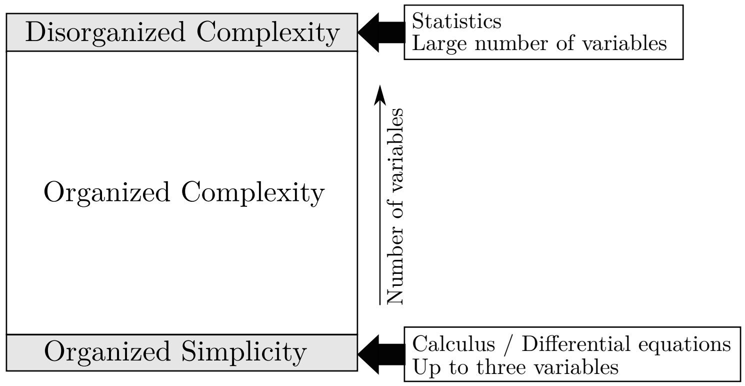


**Figure 3:** *Continuum of complexity from organized simplicity to disorganized complexity ([25, pp. 40]).*

the most chaotic compositional situation, where the number of available choices tends to infinite—the blank page. Organized simplicity, in turn, may be understood as the most strict situation, in which there is only one possible compositional path to be taken. In terms of compositional choices, the continuum of complexity impacts the number of available possibilities in both instances. This means the higher the number of variables, the more possibilities the composer may choose, and, consequently, more complex will be the very choosing process. This complexity, of course, impacts both instances.[12] Let me illustrate the creative implications of the number of variables in this continuum of complexity by discussing the simple task of creating a list of things.

If a task requires to randomly list ten different things whatsoever, without specifying a category, a topic, or rules, one could be taken too long to even start. Moreover, the choice overload could even inhibit the starting point as a creative block. After all, choosing anything from an infinitude of possibility is not an elementary task. Some people would start listing in a pragmatic way, choosing at once something that is in their sight or the very first haphazard idea that pops up in their mind. Sheena S. Iyengar and Mark R. Lepper relate that this kind of response would be a way to "simply strive to end the choice-making ordeal by finding a choice that is merely satisfactory, rather than optimal". ([9, p. 999]) In any case, the decision for the first item in the list would probably mitigate the choosing process. Presumably, the other elements of the list would be related somehow to the first chosen item, because it provides a guiding principle to be followed. This means the infinitude of possibilities would become now finite and manageable, being confined to guidelines established from the first choice. This explains why a task requiring to list ten different fruits or animals would be probably easier to most anyone as it involves the definition of a specific category, reducing the number of variables in the choosing process.

The idea of reducing the number of variables to be chosen may imply a sense of control. On this, Igor Stravinsky advocates that "the more art is controlled, limited, worked over, the more it is free" ([31, p. 63]). Thus, for him, the limitation is a crucial aspect of the compositional process:

> As for myself, I experience a sort of terror when, at the moment of setting to work and finding myself before the infinitude of possibilities that present themselves, I have the feeling that everything is permissible to me. If everything is permissible to me, the best and the worst; if nothing offers me any resistance, then any effort is inconceivable, and I cannot use anything as a basis, and consequently every undertaking becomes futile.
>
> [. . . ]

[12] In section III, I introduce a way of measuring this complexity in terms of quantity of available choices.

> Let me have something finite, definite matter that can lend itself to my operation only insofar as it is commensurate with my possibilities. And such matter presents itself to me together with its limitations. I must in turn impose mine upon it. [31, pp. 63-64].

In fact, the constraint of creative variables is an effective way of stimulating the composer's imagination as it decreases the number of variables to be chosen. This constraining strategy has been a recurrent procedure throughout the history of Western Classical Music. In the study of common-practice species counterpoint, for example, the complexity increases along with the species. This means the various melodic and harmonic constraints are gradually introduced to students so their comprehension is parsimonious, that is, before learning a new set of rules, the student shall master the previous ones.[13] By doing so, the student shall gradually improve the ability to deal with different musical aspects, such as the independence of voices, the preparation and resolution of dissonances, the harmonic progression holding the tonal sense, the melodic directionality, the vertical (harmonic) resultant of the superposition of voices, and so forth. If all rules were presented at once the study of counterpoint would be probably daunting due to the number of its restrictions. It is interesting to consider that good contrapuntal writing urges from a set of limitations so the composer shall master those rules to accordingly create inventive pieces. Concerning the relation between freedom of choice and constraint of possibilities, Stravinsky states that "we find freedom in strict submission to the object" ([31, p. 76]).

> Let us take the best example: the fugue, a pure form in which the music means nothing outside itself. Doesn't the fugue imply the composer's submission to the rules? And is it not within those strictures that he finds the full flowering of his freedom as a creator? Strength, says Leonardo da Vinci, is born of constraint and dies in freedom. ([31, p. 76])

In tonal practices, composing a fugue in a given tonality compels the composer to attend to some invariable features for any fugue, as it involves a specific type of polyphonic writing based on imitative principles. Not following this principle means the final result is anything but a fugue. Yet, despite its limitations, the composition of a fugue is sufficiently flexible to provide rooms for exploring creativity in various ways. Thus, creativity is not necessarily a matter of the number of available compositional choices, but how he/she deals with any possible limitation by inventing creative solutions to them.

It is up to the composer to define the size of the set of variables that he/she is more comfortable in dealing with or that is more suitable to his/her compositional goals and skills. Such a decision may involve either a wide set of choices, leaving rooms for inventive solutions, or a more easily manageable set of possibilities with a limited number of variables. A skilled composer would probably see a wide set of choices in both instances as a wealthy territory to explore his/her imaginative mind. On the other hand, as discussed above, choice overload may baffle composers given the overwhelming amount of available variables to be chosen.

In order to choose a set of variables, the composer shall be aware of the universe of possibilities that are available to him/her. Therefore, mapping how many possibilities are available for the realization of a given compositional choice is a crucial database for musical composition as it gives the composer the freedom to choose in advance the degree of complexity in the continuum he/she would like to manage in the construction of his/her music. This degree of complexity can be measured by what I call *compositional entropy*.

[13] Note that in this strategy, the complexity of compositional choices is also progressive, gradually increasing the number of possibilities.

## III. Compositional Entropy

In the article "A Mathematical Theory of Communication", published in 1948, Claude Elwood Shannon ([30]) proposes the theoretical foundations for what would be known as the Theory of Information. According to Shannon ([30, p. 379]):

> The fundamental problem of communication is that of reproducing at one point either exactly or approximately a message selected at another point. Frequently the messages have meaning; that is they refer to or are correlated according to some system with certain physical or conceptual entities. These semantic aspects of communication are irrelevant to the engineering problem. The significant aspect is that the actual message is one selected from a set of possible messages. The system must be designed to operate for each possible selection, not just the one which will actually be chosen since this is unknown at the time of design.

Assume, for example, that $A$ and $B$ are two different messages one should choose to transmit to someone else. Both messages are equivalent in terms of choice, as any of them can be chosen. Nevertheless, to transmit messages $A$ and $B$, it is necessary to optimize the transmission process for the most economical as possible. For example, if message $A$ is a sequence of zeros repeated thirty thousand times, then it is not necessary to send the entire sequence of message $A$, but only two pieces of information: number zero and thirty thousand. With that, the receiver will be able to understand the message $A$. Now, if message $B$ is, for example, a random noise, this is information is essentially incompressible, which means the entire sequence contained in B must be sent. This means message $B$ is more complex than message $A$ given the amount of information it contains.[14] In a general sense, a message can be understood as the random realization of successive autonomous variables so that to measure the amount of information within a random message, Shannon proposes the idea of *entropy*. The term was borrowed from thermodynamics, where it refers to a molecular system, that is, it measures the degree of disorder of a given system. In terms of information, entropy is a metric that quantifies the degree of predictability of a given message happening in a certain context. This measurement depends on the amount of information so that the greater the amount of information, the greater the entropy, that is, the message is more chaotic or unpredictable (message $B$). Similarly, a low value of entropy indicates more predictable information with less variables to be considered, which evokes a sense of simplicity (message $B$). In his words:

> Suppose we have a set of possible events whose probabilities of occurrence are $p_1, p_2, ..., p_n$. These probabilities are known but that is all we know concerning which event will occur. Can we find a measure of how much "choice" is involved in the selection of the event or of how uncertain we are of the outcome?
>
> If there is such a measure, say $H(p_1, p_2, ..., p_n)$, it is reasonable to require of it the following properties:
>
> 1. $H$ should be continuous in the $p_i$.
>
> 2. If all the $p_i$ are equal, $p_i = \frac{1}{n}$, then $H$ should be a monotonic increasing function of $n$. With equally likely events there is more choice, or uncertainty, when there are more possible events.
>
> 3. If a choice be broken down into two successive choices, the original $H$ should be the weighted sum of the individual values of $H$. ([30, p. 379])

[14]See ([28]).

Based on this idea, I shall define the compositional entropy as the measurement of the amount of freedom a composer has in creating his/her music considering how many compositional options are available to him/her in the scope of a given musical parameter. To put it differently, it expresses how complex a compositional choice can be by considering the number of variables involves in it. This means the higher the compositional entropy, the more complex is the compositional choice as there will be a high number of possibilities to be chosen. On the other hand, a low degree of compositional entropy implies a limited set of compositional possibilities.[15]

The compositional entropy may be used to measure the complexity for compositional choices in both instances so that it expresses the complexity of choosing within the set of possibilities. Therefore, exhaustive taxonomies are crucial for computing compositional entropy as it defines the number of all variables to be considered the compositional choices. For example, let $K$ be a set of musical objects of a given nature. The compositional entropy of $K$, denoted by $H(K)$, is given by the weighted average of the logarithm of the probability of each realization of the objects of $K$, where the weights are the respective probabilities themselves. This is in accordance with Shannon's definition of entropy in Information Theory [30], and this quantity is computed as (Equation 2)[16]:

$$H(K) = -\sum_{i=1}^{n} p_i \log p_i, \quad \text{where } n \text{ is the number of possibilities within } K. \tag{2}$$

Now, consider $G$ is a set of possible realizations for a given musical object defined in $K$. Then, $H(G)$ measures the compositional entropy of available choices in a compositional choice of the second instance. In order to differ $H(K)$ from $H(G)$, henceforth, I shall refer to them as, respectively, *compositional entropy of choices*, that concerns the process of choosing a musical object, and a *compositional entropy of realization*, in which is taken into consideration all available ways to realize the chosen object as music. Therefore, compositional entropy of realization is conditionally related to the nature of the musical object previously defined. Such a formulation is analogous to the idea of conditional entropy, in which the computation of the entropy of a random variable is conditioned to the entropy of another random variable (see [28]).

During the compositional process, all variables have the same probability to be chosen by a composer within both instances of compositional choices. Hence, there are no preferred variables a priori from the set of possibilities.[17] Thus, the formula may be simplified as the following by setting each $p_i$ as equal to $1/n$ (Equation 3)[18]:

$$H(K) = \log(n) \tag{3}$$

[15] This article does not intend to discuss in depth the concept of entropy developed by Shannon [30], but only to use it as a quantifier of uncertainty in compositional choices based on the number of available options. Moreover, the proposal presented here differs from those concerning the application of entropy in musical contexts. Leonard Meyer ([17][16]) was the first to introduce the concept of entropy in music. Other musical applications can be seen, for example, in [13] [10] [11][32][33][12]

[16] It is noteworthy that his is the unique function that satisfies the involved axioms.

[17] Note that a non-uniform probability to choose a variable would involve either a total awareness of the composer's idiosyncratic preferences, which would probably require a compositional maturity, or a statistical observation to catalog the most recurrent variables in the compositional practices.

[18] In the present work, all logarithm base are equal to 2, but for concision, it is omitted. Entropy is usually measured in a logarithm base of 2 in information theory due to the output is in bits—the unit. Nevertheless, any base greater than 1 may be used.

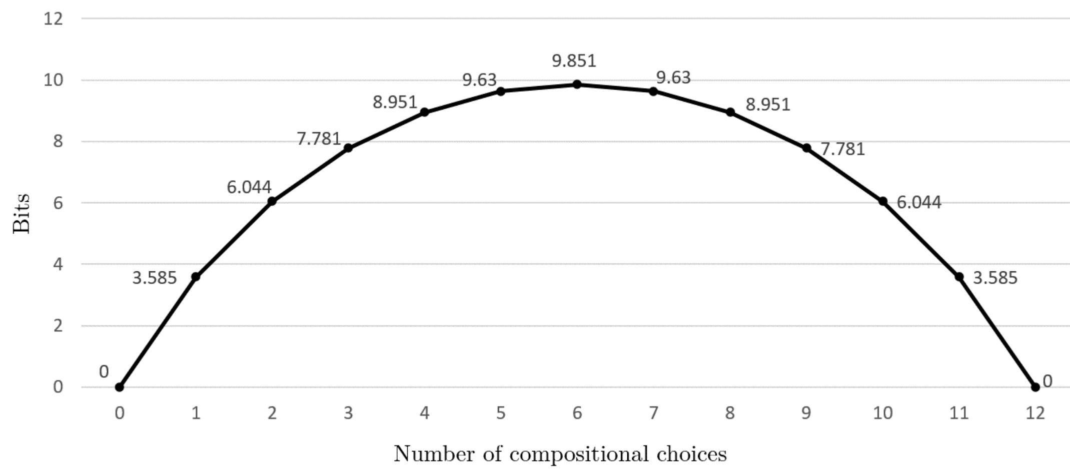


**Figure 4:** *Curve of compositional entropy defined by the number of possibilities demonstrated in Table 1 defined by the formula* $\log(n)$*, where n is the number of possibilities to be chosen from the set of twelve pitch classes.*

If $K$ holds five musical objects, for example, and a composer wants to choose one element of $K$. Then, the compositional entropy of this compositional choice of $K$ is equal to 2.322 (i.e., $H(K) = \log(\binom{5}{1} = 5) = 2.322$). This means the complexity of the compositional choice in dealing with $K$ is relatively low for a situation of choosing a single component. Now, if a composer is compelled to choose two elements of $K$, then the compositional entropy is equal to 3.322 ($H(K) = \log(\binom{5}{2} = 10)$ = 3.322). Of course, for any musical object chosen from K there is also a compositional entropy of realization (i.e., $H(G)$).

Figure 4 provides the curve of compositional entropy for the set of possibilities presented in Table 1. As one would imagine, choosing the $\mathbb{U}$ of available choices constitutes the minimal degree of compositional entropy since it does not involve a choice properly speaking. Also, the highest degree of entropy is related to the set with more available compositional choices to be made (six out of twelve). Even so, in terms of compositional entropy, the difference of its value compared to the others is not as much greater than its difference in terms of possibilities provided in Table combinationspc. Based on the degree of the compositional entropy, the composer may decide the amount of complexity he/she would prefer to manage in their music.

In the following sections I will present the exhaustive taxonomy of musical textures by considering their relations to integer partition, a proposal first introduced by Pauxy Gentil-Nunes[6]. From this definition, I will discuss a way of mapping their possible realization as music, enabling, thereby, the calculus of the degree of compositional entropy of realization in a textural perspective. For this computation, I will assume the premise that the probability of compositional choices is uniform, so the process will be as simple as counting the possibilities by using the formula $\log(n)$, where $n$ is either the number of musical objects to be chosen (first instance) or the number of available realizations of a given object (second instance).[19]

[19] As aforementioned, the consideration of non-uniform probabilities is out of the scope of the present work. Such an investigation would demand examining which are the musical textures or their realizations that are more likely to be chosen by a composer, considering his/her compositional expertise or awareness of most traditional compositional practices.

## IV. Musical Texture

Musical texture has been of great interest for composers since at least the late eighteenth century as an increasingly important musical component to articulate musical syntax and its hierarchical relations. Such poietic development has influenced not only manuals for composition and orchestration, but also the way texture (and music) is perceived. During the twentieth century, texture became even more significant to compositional practices as the urge to overcome the traditional syntax of the tonal system led various composers to address texture and its potential to unfold the musical form. In fact, the way music evolves over time can be conceived as changes of texture and therefore functions to advance the shape or trajectory of a composition.

The definition of the term "texture" is not consensual among musicians and theorists since it has been associated with different musical aspects throughout history. Anne Trenkamp states that "despite problems of definition, the musician recognizes texture as a definable element." [34, p. 14]. Based on the vocabulary used to describe musical texture, the concept of texture seems to converge into two main approaches[3]: a) texture as a musical attribute that concerns the organization of musical materials, which is usually described by traditional labels, such as *monophony*, *homophony*, *polyphony*, and *heterophony*; and b) texture as sonority, whose vocabulary describes the aural perception of registral activities, combinations of timbre, variations on dynamics, and the like, using metaphorical descriptive words, such as thin, thick, dark, light, ethereal, sparse, dense, and so on.

In the present work, texture is understood as an organizational attribute defined by the number of simultaneous vocal or instrumental parts therein (quantitative aspect) and the way they interact to one another to assemble what most people would refer to as the “layers” of texture (qualitative aspect). Formally, a given texture (or textural configuration) may be defined as the organization of $n$ simultaneous musical threads[20] into $m$ textural layers.[21] The criteria for defining how many layers a textural configuration holds are contextual and argumentative according to its specific musical context. If two or more threads share one or more characteristics, such as pitch (or pitch-class), rhythm (exact duration), register span, timbre, dynamics, and the like, they assemble the same layer. Otherwise, each thread stands for a different layer with a thickness of 1.[22]

A piece of music can hold either a single or multiple textural configurations, and the way they are diachronically arranged within the piece is intrinsically related to the perception of musical flow. That is, depending on the contrast between two contiguous textural configurations, it may imply a rupture in the sense of continuity. Of course, this rupture may be associated with a structural segmentation of musical form. Parsimonious motion, on the other hand, may contribute to a smooth musical flow without undermining the sense of unity. In order to discuss these textural properties, each texture may be described by a different integer partition.[23] A partition is a way of representing a number by summing other numbers. Number four, for example, holds five different partitions namely: [4], [1+3], [2+2], [1+1+2], and [1+1+1+1].

Figure 5 demonstrates how each one of the partitions of four may address a specific textural

[20] A musical thread (or simply thread) is the minimal constituent of a texture, which may be a single note (or sound), a series of notes positioned in a register, a pitch within a chord, etc. See [21] for further information.

[21] This definition has a theoretical ground on the seminal work of Wallace Berry ([2]), Pauxy Gentil-Nunes’ proposal called *Partitional Analysis* that relates texture with the Theory of Partitions ([6]), and on some of my previous works ([21]).

[22] In a general sense, the thickness of a layer is determined by the number of threads therein. Based on its thickness, a layer may be classified as either a *line* (a single thread) or a *block* (a group of two or more threads).

[23] The use of numbers to depict textural configurations were first introduced by Berry [2], but its association with partitions was proposed by Gentil-Nunes, which enables a series of further developments, as the access of the exhaustive taxonomy of textural configurations for a given number of threads, as well as their topological relations defined by partitional operators. See [6].

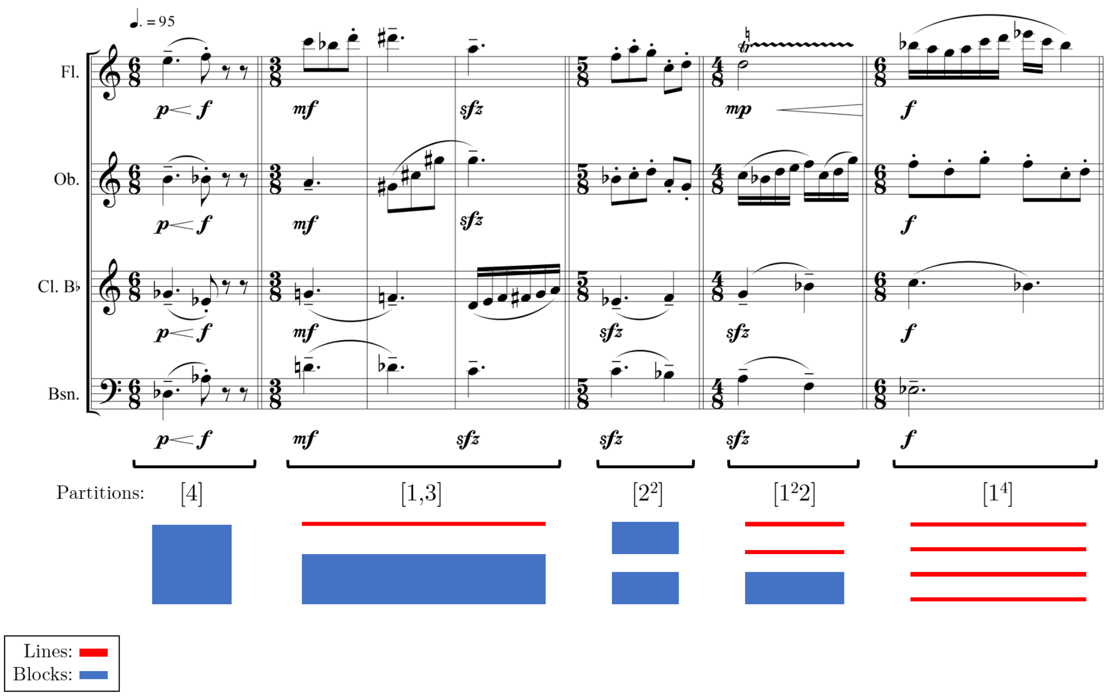


**Figure 5:** *Partitions of four addressing textural configurations. Parts are defined by rhythmic coincidences.*

configuration.[24] Each number stands for a layer and their respective absolute value expresses the number of musical threads therein (thickness). Textural parts can be classified as either a *line* (a unique thread indicated by number 1) or a block (the assemble of multiple threads expressed by any number equal to or greater than 2).[25] Parts are defined by rhythmic coincidences, that is, threads in "rhythmic unison" (where their duration are strictly aligned) assemble the same part. Note that the criterion for segregating textural parts is also related to the diversity of elements for the sake of the analysis. This means that the number of parts reveals the rhythmic diversity of the texture. Similarly, if the criterion were timbral differentiation, the number of parts would reveal the number of different timbres therein. Below each partition, there is an iconic model that portrays the number of parts according to their classification (either lines or blocks).

In the partition [1,3], the line (part 1) shifts its position over the register, moving from one instrument to another within the ensemble. First, the line is presented in the flute. Then, it moves to the oboe, concluding in the clarinet. This change creates internal variations of the textural configuration, without, however, disturbing its morphology—the organization still preserves the superposition of a line (part 1) and a block (part 3). Thus, the variance implies a change in the

[24]Gentil-Nunes presents a series of original concepts and tools concerning the relation between musical elements and integer partitions, which enables a refined analysis of texture, as well as its systematic manipulation for compositional purposes. [6]

[25]For the sake of clarity and conciseness, partitions are noted within brackets in their abbreviated notation, in which the repetition of a number is indicated by a superscript. Also, in order to avoid notational ambiguities, each part is separated by either a comma or by the superscript of the previous part.

spatial order of parts.[26] A partition is, in essence, an unordered set of numbers, which means the order of parts is irrelevant; partitions [1,3] and [3,1] are equivalent. Nevertheless, the inclusion of an ordering factor may portray the registral span of parts, which would perfectly suit to describe the variations of partition [1,3] in Figure 5. [27]

Each combinatorial permutation of the parts within a partition corresponds to an ordered partition. Obviously, the partition must have more than one part; otherwise, the ordered partition is redundant to the partition. The number of ordered partition for a positive integer $n$ is equal to $2^{n-1}$. The five partitions of number four, for example, may be combined into eight ordered partitions: <4>, <1,3>, <3,1>, <$2^2$>, <$1^2 2$>, <1,2,1>, <2, $1^2$>, and <$1^4$>.[28] The order factor indicates the general registral placement of parts so that the left-to-right notation corresponds to the top-to-down position of them in the register. Consider $x$ and $y$ as two different textural parts of a texture. If part $x$ is higher than $x$ in the register, then the ordered partition is written as $< x, y >$; otherwise, the ordered partition is $< y, x >$. Of course, this relation is not absolute since it depends on a comparative observation among all parts. This means the highest part written in the leftmost position within an ordered partition does not necessarily is located in a high register, but it is the highest among all parts.

One may notice that ordered partitions are not suitable to describe *interpolated textures*, i.e., textures in which "threads of a part are interlaced (or interwoven) with threads of another." ([21, p. 81]). To put it differently, ordered partition can only describe textural realizations in which the non-overlapping parts are perfectly stacked to one another. In order to portray the spatial organization of those situations, I have introduced a specific for texture notation that I call *thread-word* (see [21]). A thread-word maps the way threads are posed in the register, conveying their organization into parts. It is based on a one-to-one correspondence between parts, their threads, and letters. For each thread within the texture, is ascribed a letter in such a way that all threads that assemble the same part receive the same letter. The number of different letters corresponds to the number of different parts of the texture, and the sum of all equivalent letters reveals its thickness. The notational principle of thread-words is identical to ordered partitions, so the left-to-right order of letters maps their top-down disposition in the register. For example, partition [$1^3$] can be represented as any of the following thread-words: <*abc*>, <*bcd*>, <*xyz*>, and so on. In the same way, a thread-word noted as <$aba^2$> indicates a texture holding a single line (written as "b") and a block of thickness of three (indicated by the sum of letters "a").[29] Note that this thread-word is equivalent to partition [1,3], expressing an organization an ordered partition is not able to.[30] Figure 6 shows the comparison between ordered partitions and thread-words describing all spatial organizations of threads of partition [1,3]. Textural parts differ from one another in their rhythmic articulations.

Each ordered partition or thread-word provides what I call *textural layout*—a possible compositional choice of the second instance available for the composer to realize a given textural configuration as music, considering the combinatorial permutation of its component parts in the register. Note that the textural realization discussed here deals only with textural factors, that is, it

[26] By spatial order, I refer to the general vertical distribution of threads and/or parts without dealing with their actual registral span—a conception from the same realm of Theory of Musical Contour (see [23][14]). Therefore, this proposal differs from Berry's *texture-space* in which the spatial factor of textures consists of measuring the number of semitones between the outer parts to observe expansions and contractions of register through a textural sequence (see [2, pp. 195-199]).

[27] In mathematics, an ordered partition is called *composition*, but this word can be confusing in musical contexts. For this reason, in the present work, I shall refer to a partition where the order matter as ordered partition. For further discussions on ordered partitions and their features, see [1].

[28] In this article, ordered partitions are enclosed with "< >" to differ them from unordered partitions.

[29] As in partitional notation, thread-words can use a superscript to express the multiplicity of letters.

[30] See [21] for further discussion on thread-words, their particularities, and applications.


| | | | | |
|---|---|---|---|---|
| Unordered Partitions: | [1,3] | [1,3] | [1,3] | [1,3] |
| Ordered Partitions: | <1,3> | - | - | <3,1> |
| Thread-words: | <ab$^3$> | <aba$^2$> | <a$^2$ba> | <a$^3$b> |

**Figure 6:** *Comparison between ordered partitions and thread-words to describe the spatial organizations of threads of partition [1,3]. Parts are defined by rhythmic coincidences.*

is exclusively defined by aspects regarding textural morphology, regardless of the particularities of musical materials that undergird it. In terms of realization, considering materials involved to realize a given texture as music would intricate the compositional process by increasing considerably the number of involved variables, which would, consequently, impact the degree of compositional entropy. Concerning compositional choices, a textural layout refers to the second instance while partitions are defined in the first instance. Therefore, the relation between a partition and its possible textural layouts is of the same nature of the relation between pitch classes and pitches so that the set of all layouts of a given textural configuration enables to do constitutes the exhaustive taxonomy of its spatial realizations.

Prior to the discussion on compositional entropy in a textural perspective, it is necessary to further examine the very notation of thread-word to compute the number of available possibilities, what I call *exhaustive taxonomy of textural layouts*. Moreover, it is important to introduce some concepts regarding textural layouts that are crucial for such a discussion.

## i. Exhaustive Taxonomy of Textural Layouts

In the notation of thread-words, the letters are arbitrary, that is, any letter can be associated with any textural part. Even so, it is possible to attribute a specific letter to a given textural part based on an endemic characteristic of the materials that underlie it.[31] By doing so, it is possible to track the diachronic transformations of textural parts along with contiguous textures. For example, in Figure 7, each letter refers to a different part of the thickness of two that combined forms partition [$2^2$]. The criterion for segmentation is both timbre and pitch-class content as indicated in the score. As letters "a" and "b" are invariably associated with the same parts, thread-letters spell out how threads permute within partition [$2^2$], revealing its textural layout. This shows partition [$2^2$] provides a high degree of compositional entropy as the composer may explore each one of these possible layouts in various creative ways during the compositional process.

[31] According to Berry, this prominent characteristic of textural parts emerges "when materials are of such distinctive textural cast, and when the particular qualities of texture are so vital a factor in identity and interest of thematic-motivic material." [2, p. 254]

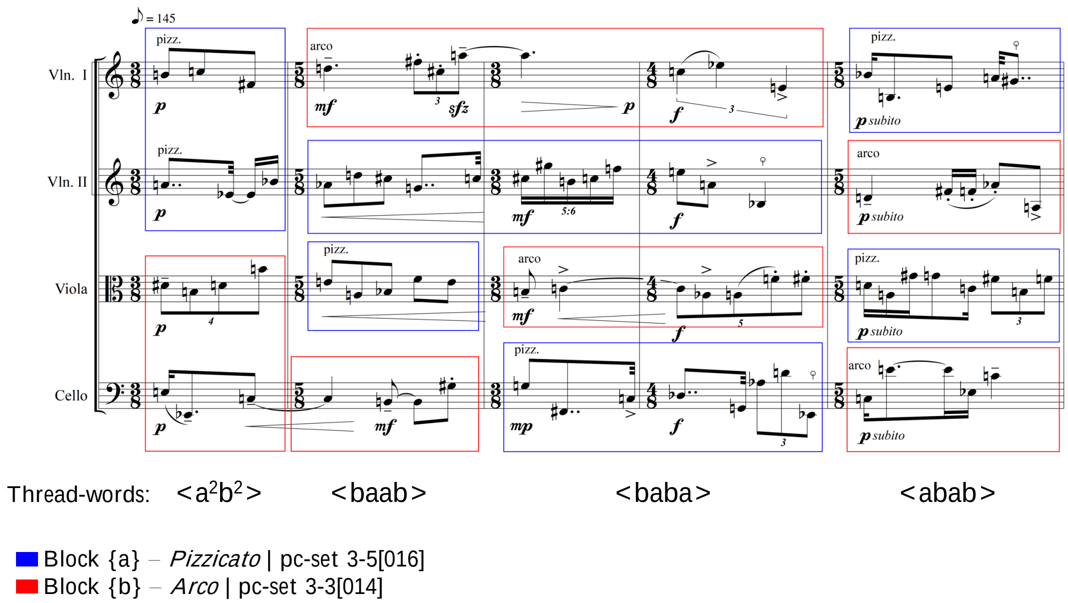


**Figure 7:** *Thread-words portraying different spatial organizations of threads within partition* [$2^2$]*. Parts are defined by both timbre (either* pizzicato *or* arco*) and pitch class content. [21, p. 99]*

Although differing from one another in their notation, the last two thread-words in Figure 7 can be understood as equivalent to each other as they consist of an alternation of threads from both parts. In fact, they correspond to the same *thread-word class* (*tw-class*). A tw-class is a notational convention akin to normal form on pitch-class set theory. It assembles all thread-classes that share the same spatial organization. By convention, the first letter of a tw-class is always "a" and for each new part is ascribed a new letter following the alphabetical order. For example, the thread-word class <$ab^2ac$> comprises thread-words all of the following: <$ba^2bd$>, <$xy^2xz$>, <$ca^2cb$>, etc. Similarly, a thread-word <$x^3yxz^2$> is rewritten as tw-class <$a^3bac^2$>. The main goal of tw-classes is to reduce notational redundancies thereby providing the accurate number of potential realizations of a partition—crucial information for calculating compositional entropy. This property may be clear with the following example. Consider thread-word <*abcd*>. How many permutations is it possible to form from it? A simple factorial of the number of elements reveals thread-word <*abcd*> can hold 24 possible permutations (i.e. $4! = 24$). Thread-words <*dcba*>, <*bacd*>, and <*cbda*> are some of these permutations. However, they are essentially identical in terms of organization: four independent threads (lines) stacked to one another. In this case, these notational variations are only reasonable in a context where each letter express a different musical idea, as demonstrated in Figure 7, which does not concern the spatial organization of texture. Thus, by considering tw-classes, partition [$1^4$] has a unique partition-layout possibility of realization expressed by the tw-class <*abcd*>.

In Figure 7, the tw-class <$a^2b^2$> in the first measure is unique that is equivalent to unordered and ordered partitions. For that reason, tw-class <$a^2b^2$> is called the *thread-word prime class* (or simply *tw-prime*). A tw-prime is a textural adaptation for the concept of prime form in pitch-class set theory. It may be defined as a tw-class where non-intermingled parts are expressed in increasing order so that all layouts of a given textural configuration (partition) are, in fact, permutations (or *anagrams*) of the tw-prime that stands for it. Each one of these anagrams stands

**Table 2:** *Anagrams of tw-prime* <$a^2b^2$> *and their respective notation in tw-class showing redundancies.*

| Anagrams | Tw-classes |
|---|---|
| aabb | <$a^2b^2$> |
| baab | <$ab^2a$> |
| abab | <$abab$> |
| baba | <$abab$> |
| abba | <$ab^2a$> |
| bbaa | <$a^2b^2$> |

for a different textural layout.

In order to eliminate redundancies, all anagrams from a tw-prime must be rewritten in the form of a tw-class. Consider, for example, the thread-word <$b^2a$>. It is an anagram of tw-prime <$ab^2$>. Yet it is not a tw-class given its first letter is "b" instead of "a". In order to access all textural layouts of tw-prime <$ab^2$>, the anagram <$b^2a$>must be rewritten as the tw-class <$a^2b$>. Note that their spatial organization is invariable, but this rewriting process is necessary to compute the number of textural layouts. Table 2 shows this relation between anagrams and their notations as tw-classes to map all layouts of partition <$2^2$>. Despite threads can be arranged into six different anagrams, they provide only three tw-classes, namely: <$a^2b^2$>, <$ab^2a$>, and <$abab$>.

To calculate how many textural layouts are available for a composer from a given tw-prime, it is necessary to compute the possible permutation of its component parts, eliminating possible redundancies. The number of component parts is defined by its *density-number*(DN).[32] So the number of permutations can be accessed by its factorial (i.e., $TL = DN!$, where $TL$ is the quantity of available textural layouts). Nevertheless, not all permutation shall be computed as commuting repeated letters is useless in terms of textural layout. For example, the DN of tw-prime <$a^3$> is equal to 3 so that there are six possible permutations of it ($DN! = 6$). Since all of these permutations are identical the number of textural layouts is one, represented by tw-prime <$a^3$> itself. To eliminate the redundancies of these cases, it is necessary to divide the number of permutations by the factorial of each part (expressed by the sum of each letter). This is expressed in the following formula, where $\{p_1, ..., p_i\}$ are the parts of a partition P and $i \in \mathbb{Z}_+$ (Equation 4):

$$TL = \frac{DN!}{p_1!...p_i!}. \tag{4}$$

For example, the number of layouts of tw-prime <$a^2b^3$> is equal to $\frac{5!}{2!3!} = 10$. Similarly, tw-prime <$ab^2c^3$> provides 60 textural layouts ($TL = \frac{6!}{1!2!3!} = 60$). Note that if the tw-prime comprises a unitary part, the number of layouts will be always one since the thickness of the part is equal to the density-number ($TL = \frac{DN!}{P!} = 1$, where $P = DN$).

None of these examples includes tw-primes with duplicated parts, that is, textural configuration with two or more parts holding the same thickness. These textural configurations are those written with a superscript in the form of ordered partitions. For example, in thread-word <$ab$>, both parts hold a thickness of 1. Given the notational convention of tw-class, this is significant information that impacts the output computation. After all, although <$ab$> and <$ba$> are possible anagrams,

[32] This term was coined by Berry [2] to refer to the sum of all threads within a textural configuration. In the realm of partitions, it corresponds to the positive integer $n$ that is partitioned into various ways.

they are members of the same tw-class <*ab*>. Thus, it is necessary to exclude permutations between and among parts with the same thickness from the computation of textural layouts. To so, the formula presented in Equation 4 shall be divided by the factorial of the number of duplicate parts, as demonstrated in the following formula[33] (Equation 5):

$$TL = \frac{\left(\frac{DN!}{p_1!...p_i!}\right)}{d!}, \tag{5}$$

where $d$ is the quantity of duplicated parts within the textural configuration. For example, in the tw-prime <$abc^2d^2$> there are four duplicated parts: $a$ and $b$, and $c^2$ and $d^2$ so that it provides 45 textural layouts as ($TL = \frac{6!}{1!1!2!2!}/4! = 45$). In the same way, tw-prime <*abcd*> holds a unique partition layout since the density-number is equal to the number of duplicated parts ($TL = \frac{4!}{1!1!1!1!}/4! = \frac{4!}{4!} = 1$). Considering the factorial of zero is equal to one ($0! = 1$), this formula is applicable to any tw-prime.

Table 3 provides the exhaustive taxonomy of textural layouts in the form of tw-prime, as well as their respective anagrams in the form o tw-class, for the partition lexical set for $n = 4$. Each partition can be classified into three distinct types: a) massive partitions (type M), formed by a single block ([2], [3], and [4]); b) polyphonic partitions (type P), whose the number of lines therein is equal to the density-number ([1], [$1^2$], [$1^3$] and [$1^4$]); and c) mixed partitions (type X), when blocks and lines are combined ([1,2], [1,3], [$2^2$], and [$1^2$2]). While type X contains the textural configurations with more textural layouts, in both types M and P the number of textural layouts is equal to their respective tw-prime. In this lexical set, tw-prime <$abc^2$> provides more textural layouts (6).

Figure 8 shows the topological relation among adjacent tw-class within the lexical set for $n = 4$, organized in a structure called *thread-word classes Young Lattice* (TYL). TYL can be understood as an ordered version of Gentil-Nunes' Partitional Young Lattice ([6, pp. 50-51]. It provides the exhaustive taxonomy of textural layouts in the form of tw-classes for $n$ threads. Each square is a different tw-classes so that abreast squares are textural configurations of the same density-number.[34] Lines indicate a different transformational operation that connects them. In summary, *resizing* (m) consists of the increment or decrement of the thickness of a given part by including a singular thread in it (indicated by full blue lines). *Revariance* (v) describes the operation of adding or excluding a unitary part (line) from the textural configuration (indicated by dotted green lines). Finally, *permutation* (T) is an operation where threads swap their spatial position within the textural configuration thereby preserving the density-number (indicated by double red lines).[35] All these operators involves a parsimonious change from a tw-class to another, which means a composer may choose a sequence of textures by moving along edges to ensure a smooth textural flow if this is his/her goal.[36]

[33]Despite its logical construction, this equation is still a conjecture. The proof of its infallibility will be left for future works.

[34]Note that below each tw-class there is a pair of numbers separated by a comma. They indicate the calculus of the binary relations among all threads to express whether they hold a relation of either agglomeration (in cooperation to assemble a textural part) or dispersion (non-congruence or divergence). This pair of indices provides crucial information for the very definition of textural parts. See [6, pp. 33-38].

[35]Permutation is an operation I formulate in my Ph.D. Dissertation, while both resizing and revariance were proposed by Gentil-Nunes to deal with the topological relations among partitions. Hence, their applications in the present article consist of a simple translation of their principles to the realm of thread-word classes. For further information regarding these operations, see [6] [8] and [21].

[36]Obviously, this sense of continuity deals only with textural factors since an actual smooth flow also depends on the materials the composer choose to realize the tw-classes.

**Table 3:** *Exhaustive taxonomy of textural layouts for the partition lexical set for $n = 4$. Revised version from [21, p. 100].*

| Type | Partitions | Density-number | Tw-prime | Tw-classes of Textural Layouts | Total of Textural Layouts |
|---|---|---|---|---|---|
| P | [1] | 1 | $<a>$ | $<a>$ | 1 |
| M | [2] | 2 | $<a^2>$ | $<a^2>$ | 1 |
| P | [$1^2$] | 2 | $<ab>$ | $<ab>$ | 1 |
| M | [3] | 3 | $<a^3>$ | $<a^3>$ | 1 |
| X | [1,2] | 3 | $<ab^2>$ | $< ab^2>$, $<aba>$, and $<a^2b >$ | 3 |
| P | [$1^3$] | 3 | $<abc>$ | $<abc>$ | 1 |
| M | [4] | 4 | $<a^4>$ | $<a^4>$ | 1 |
| X | [1,3] | 4 | $<ab^3>$ | $<ab^3>$, $<aba^2>$, $<a^2ba>$, and $<a^3b>$ | 4 |
| X | [$2^2$] | 4 | $<a^2b^2>$ | $<a^2b^2>$, $<ab^2a>$, and $<abab>$ | 3 |
| X | [$1^2 2$] | 4 | $<abc^2>$ | $<abc^2>$, $<abcb>$, $<ab^2c>$, $<abca>$, $<abac>$, and $<a^2bc>$ | 6 |
| P | [$1^4$] | 4 | $<abcd>$ | $<abcd>$ | 1 |

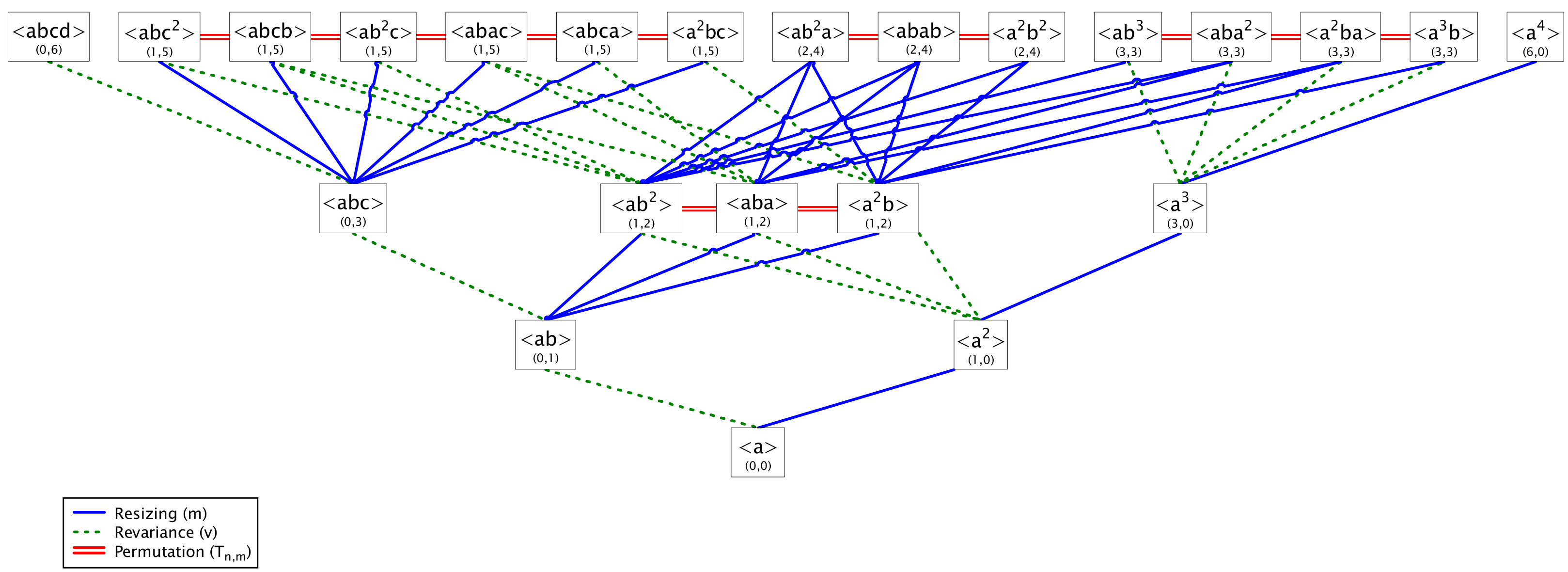


**Figure 8:** *Thread-word Young Lattice (TYL): thread-word classes associated to the lexical set of partitions for $n = 4$. Edges stands for transformational operations. Dotted green lines indicates revariance, full blue lines shows resizing, and double red lines shows permutations. [21, p. 129]*

The number of textural layouts may also consider the number of available threads. For example, suppose a composer wants to use the partition [1,2] in an instrumental set comprising up to four threads. This means that one thread will necessarily be in silence. So the composer may ask how many ways the composer can realize it within the compositional process? To answer this question, it is necessary to compute the combination of the number of available threads (T) and the density-number of the textural configuration (DN) multiplied by the number of its textural layouts (L), as can be observed in the formula (Equation 6):

$$TL = \binom{T}{DN} L = \frac{T!}{DN!(I - DN)!} L, \text{ for } \; T \geqslant DN. \tag{6}$$

Returning to the question below, the number of partition layout for [1,2] is three (i.e., $L = 3$) so that it can be realized in twelve different ways in a context of four of threads by equation$\binom{4}{3}3 = 12$ (Figure 9). In order to indicate which thread is at rest in the realization, I have included the number zero in the notation of tw-class. In this case, instead of mapping the general register span of threads, the spatial order within tw-class is associated with the timbral distribution of threads according to the way they are written in the score so that, the first position within the tw-class corresponds to the first violin, the second position indicates the second violin, and so on. Needless to say that the number of possibilities may increase significantly in contexts where the difference between the number of available parts and the density-number of the texture is greater. Textural parts are defined by timbre (either *pizzicato* or *arco*), rhythmic coincidence, and dynamics.

As discussed above, when the tw-class is a single block (type M), its realization comprises a unique textural layout each that corresponds to their respective tw-prime. Nevertheless, from the analysis of the repertoire, it is possible to observe that blocks usually assume various forms of articulation other than common sense. In my Ph.D. thesis, I proposed five modes of textural realization that probably "cover the most recurrent textural realizations of concert music inferred by various analyses" ([21, p. 166]). One of the modes, called *evolving realization*, deals specifically with the possible realizations of blocks.[37] The main premise of evolving realization is a block can be realized by "the successive superposition of its constituting parts" in such a way that "the block is not understood as such until it is fully constructed; or one can say it is retrospectively defined by the stationary motion of staggered entrance of sustaining notes." ([21, p. 174]).[38] This means the block evolves over time from a polyphonic (or mixed) to a massive presentation. An opposite effect of this construction may involve the "dilution" of blocks by gradually removing its constituting parts, as a "filtering" process. The difference between gradual *construction* and *dilution* of blocks is their onsets and offsets (endpoints). In the first, all threads assembling the block hold the same offset, but differing on their onsets. The latter, in turn, has exactly the opposite relation, with threads aligned on their onsets, but shifting their offsets. On the other hand, the traditional sense of blocks is that where the onsets and offsets of all threads match in time.

Figure 10 exemplifies all three ways of realizing a block with three notes.[39] Below each realization there is an iconic model that portrays their onset/offset relation as described above. Note that both gradual construction and dilution emerge from the cumulative superimposition of

[37] This mode is, in part, based on the proposal of Bernardo Ramos in the analysis of the ways thereby blocks can be articulated in the guitar. See [26].

[38] To better understand the principle of evolving realization, it is important to consider a wide *window of observation*, that is, the temporal frame whereby all components therein are understood as assembling the same texture. This In a wider window of observation

[39] Needless to say that the idea of block is not necessarily restricted to notes as instruments with undefined pitches can also assemble blocks according to given criteria.

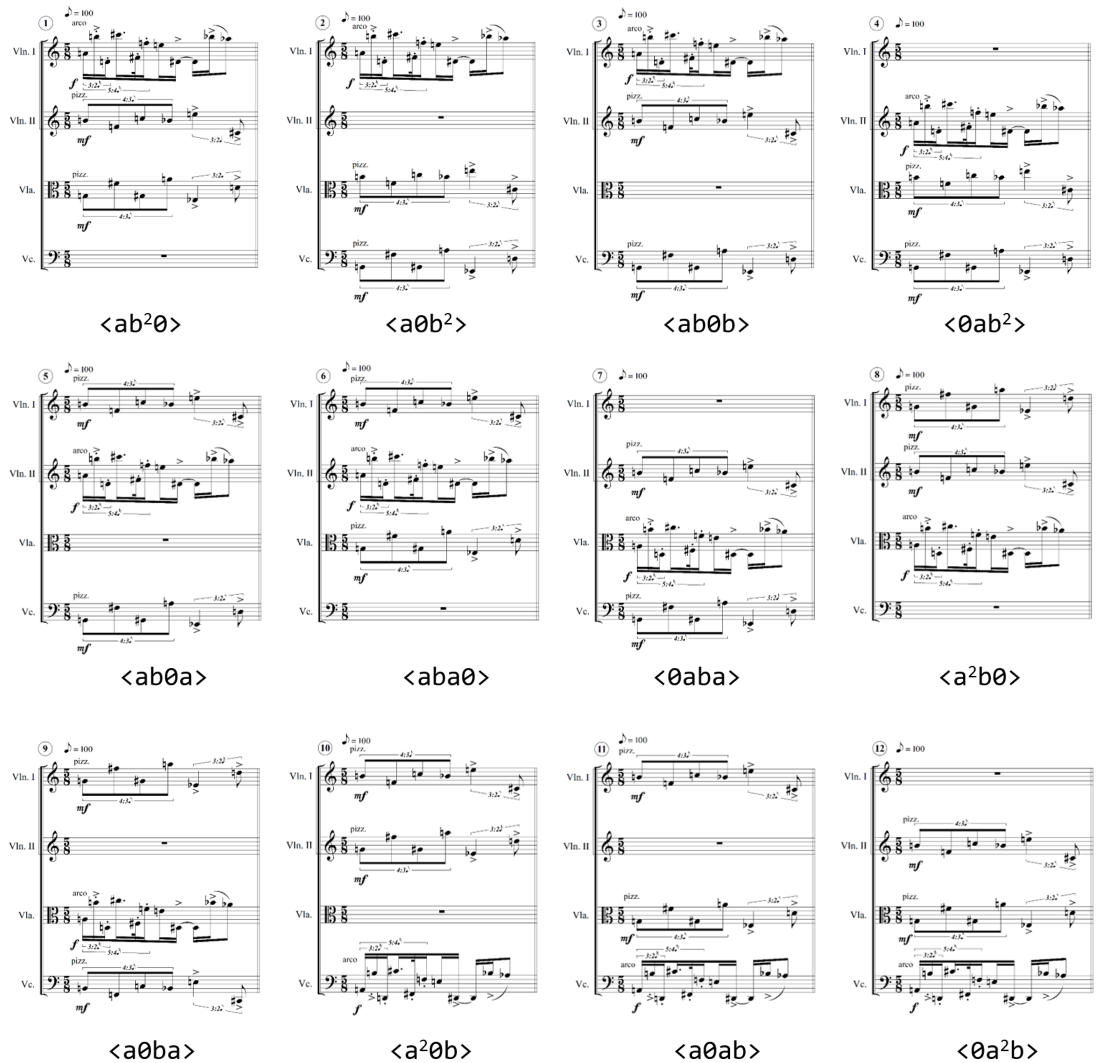


**Figure 9:** *Exhaustive taxonomy of textural layouts for partition [1,2] in a context of four threads. Parts defined by both timbre, rhythmic coincidence, and dynamics.*

each thread, which would be understood, in a strict perspective, as the combination of three lines (partition $[1^4]$).[40]

A block with a thickness $n$ can be presented in evolving realization as any of the partitions of $n$. This means a block with three threads can be presented as any of the following: a) a standard realization of a block (partition [3]), b) three lines ($[1^3]$), and c) a block of two and a line ([1,2]). Just those possibilities are of great significance to composers; however, the spatial factor may be considered in order to increase significantly the set of compositional choices. Hence, the spatial organization of threads within a block, as well as the order of their onsets/offsets may

[40]The music of Edgar Varèse is full of examples of both gradual construction and dilution of blocks. See, for example, the various blocks of *Déserts* (1950/1954)—more specifically mm. 21-26 and mm. 171-174 where this strategy is clearer.


**Figure 10:** *Three ways of realizing a block with three notes considering the evolving realization: standard realization, gradual construction, and gradual dilution. Parts are defined by their stationary motion.*

include textural layouts for textures of type M, expanding, thereby, the set of layouts in tw-classes discussed above.

To compute all possible layouts for a block with a thickness of $n$, it is necessary to sum all layouts of all partitions of $n$. This may be accessed by using the same formula provided in Equation 5, but multiplying the output by $i!$, where $i$ is the number of parts within the partition, and then by two. The first multiplication is meant to consider permutations of threads in all available time-points while the second includes the computation of both gradual constructions and dilutions, (Equation 7)[41]:

$$L(K) = 2\frac{\left(\frac{DN!}{p_1!\ldots p_i!}\right)}{d!}j!\;. \tag{7}$$

With this formula, the computation of all layouts is just a matter of summing all outputs for all partitions of $n$, subtracting one from it to exclude the duplicated standard realization of the block (where $K = n$). This is given by the formula, where $n$ is the thickness of the block to be realized (Equation 8):

$$TL_{(n)} = \sum_{j=1}^{K} L(K_j) - 1. \tag{8}$$

For example, a block of three can be presented in 25 different textural layouts considering evolving realization, as demonstrated below (Equation 9):

[41] In the formula $K$ is a partition of $n$, $p_i$ is its parts, $i$ is the number of parts, and $d$ is the quantity of duplicated parts within the partition.

$$
\begin{aligned}
P_{(3)} &= \{[3],[1,2],[1^3]\} \\
L([3]) &= 2\frac{\left(\frac{3!}{3!}\right)}{0!}1! = 2 \\
L([1,2]) &= 2\frac{\left(\frac{3!}{1!2!}\right)}{0!}2! = 12 \\
L([1^3]) &= 2\frac{\left(\frac{3!}{1!1!1!}\right)}{3!}3! = 12 \\
TL_n &= 2+12+12-1 = 25
\end{aligned}
\tag{9}
$$

Figure 11 shows a set of iconic models to portray each one of these 25 textural layouts, relating them with their respective partition.[42] Letter (a) shows the standard realization of [3], with both onsets and offsets of all threads strictly aligned. This corresponds to the textural layout for tw-prime <$a^3$> presented in Table 3. The non-coincidence of either onsets or offsets produces, as mentioned above, gradual constructions (Figure 11b and c) and dilutions (Figure 11d and e), respectively. Below each iconic model has its description as a *contour in duration space* (see [15, pp. 150-167]).

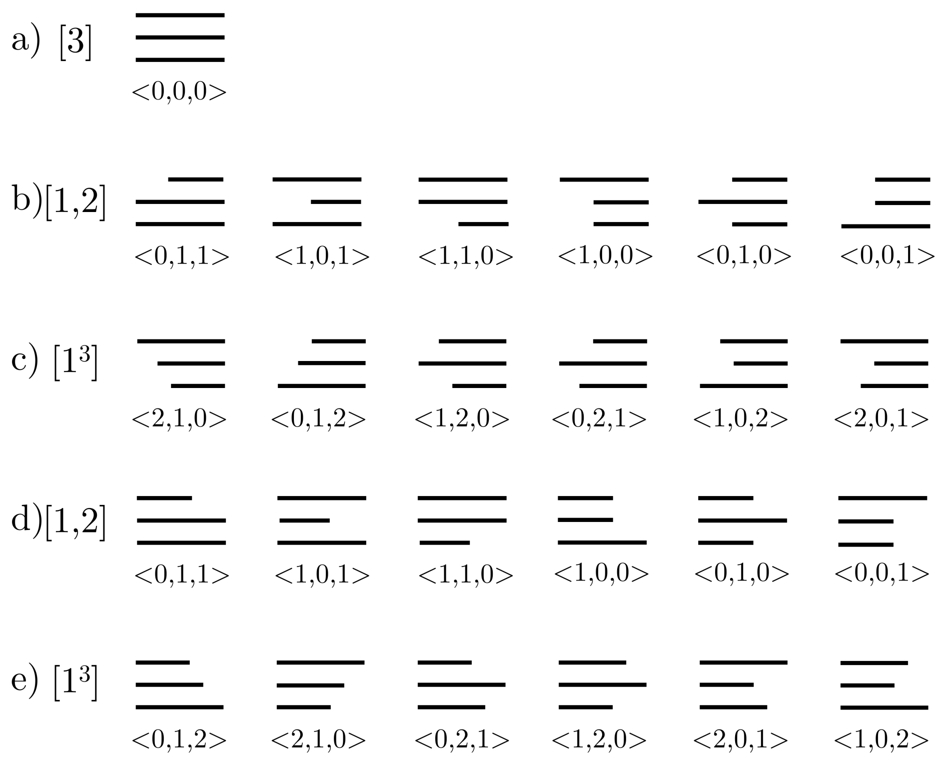


**Figure 11:** *Exhaustive taxonomy of textural layouts of three threads to assemble a block of a thickness of three considering the evolving realization: onset and offset alignment (a); offset alignment (b and c); and onset alignment (d and e). Revised version of [21, p. 176].*

[42]These iconic models are based on Robert Morris organization of sequential tones in time and space (see [22, pp. 295-299][24, p. 346])

Each thread in this textural layout corresponds to a sequence of attack points in the register so the contour expresses their spatial order of presentation according to their relative duration. In the contour, each thread is identified by a number from zero (the shortest duration) to $n-1$ (the longest duration), where $n$ stands for the number of different durations therein.[43] Also, within the contour, the first element refers to the relative duration of the highest note, the second element indicates the duration of the note contiguously above that, and so on, so that the left-to-right order in the contour depicts the top-to-down disposition of threads in the register. Note that all contours in gradual construction have a version in gradual dilution as they are the very retrogradation of each other. In the iconic models, they may be understood as the rotational operation from one another.

During the compositional process, a composer may be compelled to combine gradual constructions and dilutions to produce more complex articulations of the block of three in evolving realization. In this case, there are 144 possible combinations available for the composer. This wealth amount of possibilities may be a fruitful compositional inventory for composers to explore their creativity in dealing with blocks. As could not be otherwise, the greater the thickness of the block, the higher the number of its textural layouts to be considered in evolving realization.

From the computation of an exhaustive taxonomy of textural layout presented here, it is possible to measure the compositional entropy of both the set of musical textures and textural layouts. This is the subject of the next section.

## V. Compositional Entropy Applied to Musical Texture

A compositional entropy applied to texture consists of the measurement of the amount of choices a composer may have to choose both textural configurations in the form of partitions for a given number of threads (the first instance of compositional choices—compositional entropy of choice) and the variety of textural layouts a chosen partition holds (second instance of compositional choices—compositional entropy of realization). In any case, the higher the number of options available for the composer, the higher is the compositional entropy. In order to proceed in this discussion, let me exemplify the relation between textural entropy in both instances: partitions and textural layouts.

By examining all partitions from 1 to $n$, one may notice that the higher the number of threads, the greater will be the set of available partitions for it, and, consequently, the more complex will be the choosing process expressed by the compositional entropy. If a composer chooses to create a piece for a wind quintet, for example, despite multiphonics and other extended techniques, it involves a set of five available threads. Then, there are eighteen different partitions to be chosen in the construction of his/her piece (Table 4).

The compositional entropy involved in the choosing process of a single partition from this set is equal to $\log(18) = 4.17$. Nevertheless, it is not common for an entire piece to comprise only a single partition, except, perhaps, in cases of monophonic pieces as it comprises a single thread. Therefore, the number of compositional entropy may increase depending on the number of partitions to be chosen. A compositional choice involves five different partitions of the set presented in Table 4 denotes a compositional entropy equal to 13.065. Figure 12 provides the curve of compositional entropy of the lexical set of partition for $n = 5$ given a number of compositional choices ranging from one to eighteen so that the composer may decide on how much freedom he/she would like to manage during the compositional process based on the degree of compositional entropy.

[43]In musical contour, the relation among its internal elements is not absolute, but relative, so that two distinct musical structures (e.g., melodies or rhythms) can be depicted by the same contour. Thus, the duration discussed here is based only on the relative proportion among threads, without considering their actual duration.

**Table 4:** *Lexical set of partitions (exhaustive taxonomy) for $n = 5$.*

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| Partitions of 1 | [1] | | | | | | |
| Partitions of 2 | [2] | $[1^2]$ | | | | | |
| Partitions of 3 | [3] | [1,2] | $[1^3]$ | | | | |
| Partitions of 4 | [4] | [1,3] | $[2^2]$ | $[1^2 2]$ | $[1^4]$ | | |
| Partitions of 5 | [5] | [1,4] | [2,3] | $[1^2 3]$ | $[1, 2^2]$ | $[1^3 2]$ | $[1^5]$ |

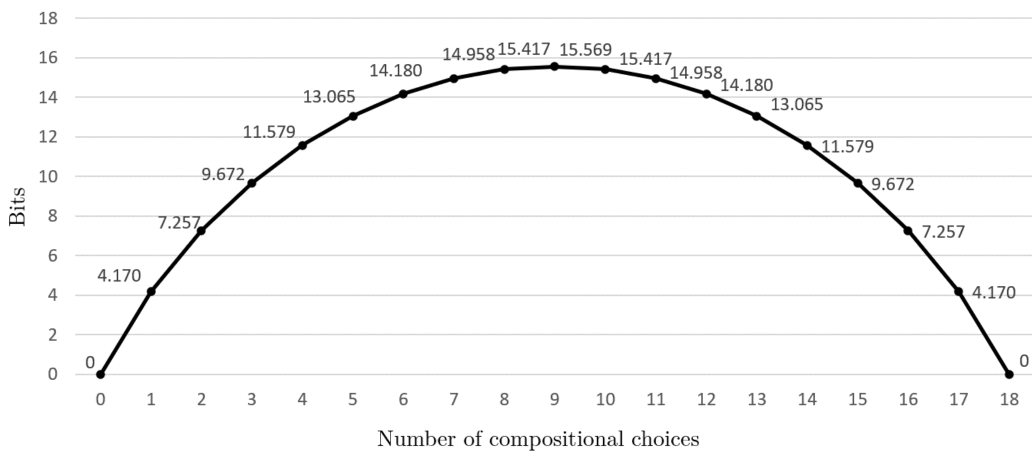


**Figure 12:** *Curve of compositional entropy applied to the 18 partitions provided in of Table 4 considering the number of compositional choices involved.*

Most composers are not aware of all textural possibilities provided in Table 4. Indeed, despite its importance:

> the compositional potential of texture is a topic not often discussed or analyzed in the theoretical literature. Even recent composers (most often) do not explicitly present in their writings on their own music ideas about textural organization; nor do they display any concern about a systematic approach to texture. If their music seems to be constructed out of various textural configurations, these are conceived intuitively, as the outcome of the interaction of the other musical parameters ([21, p. 58]).

Thus, by accessing the exhaustive taxonomy of partitions of a given number of threads is significant to musical compositional as it provides all possible creative paths a composer may take This means that exhaustive taxonomies of partitions enable the composer to be aware of the most recurrent textural configurations in his/her music, thereby bringing texture to a more conscious zone of the creative process. Furthermore, it may allow the composer to explore textural configurations that would probably not otherwise be used in his/her music.

A short musical example of the realization of the sequence of partitions <$[2^2][2,3][4][1^2][2]$> made from partitions of Table 4 may contribute to this still embryonic discussion on both compositional entropy applied to the texture (Figure 13). Textural parts are defined by the rhythmic


| | | | | | |
|---|---|---|---|---|---|
| Partitions: | [$2^2$] | [2,3] | [4] | [$1^2$] | [2] |
| Number options within DN: | 5 | 7 | 4 | 2 | 2 |
| Number of available textural layouts: | 3 | 10 | 149 | 1 | 5 |

**Figure 13:** *A possible realization of the sequence of partitions <[$2^2$][2,3][4][$1^2$][2]> and the number of possible choices involved in both their very definition, considering its probability of being chosen within partitions of its density-number, and the number of their possible textural layouts. Parts are defined by rhythmic coincidence.*

coincidence of threads. Despite the instrumental mean (piano solo) provides the possibility to use more than five simultaneous threads, for the sake of simplicity, the maximum value of $n$ is five. Below each realization is its corresponding partition, the number of possible choices involved in their definition (first instance), considering its probability of being chosen within partitions of its density-number, and the number of their respective possible textural layouts (second instance). Such information is decisive for computing textural entropy of partitions and their respective realizations. Not all available density-numbers were used in this example; only partitions for $n$ equal to either two, four, or five.

A graph can be plotted to compare the difference of compositional entropy of choice and realization (Figure 14). The graph shows each compositional entropy is independent of one another and their value can diverge considerably. This means a partition with a low degree of compositional entropy may imply a high entropy in terms of its realization and vice-versa. Take, for example, the last two partitions of the sequence [$1^2$] and [2]. Both constitute all partitions of $n = 2$ so their compositional entropy of choice is equivalent as they have the same probability to be chosen. Yet, while [$1^2$] has the minimal degree of compositional entropy of realization (zero), which means there is only one way of realizing it, partition [2] has an entropy of realization of 2.322.

A striking feature may be observed in partition [4]. It holds a compositional entropy of choice equals to 2, but provides an entropy of realization equals to 7.219 given its 149 possible realizations, which includes the standard realization of the block of four (the very realization presented in the score) and all its possible gradual constructions and dilutions in evolving realization. In face of that, one may logically conclude that for any value of $n$, a block with a thickness of $n$ is, by far, the most complex situation in terms of realization measured by the compositional entropy. This is quite curious information given the simplicity of the idea of a block may evoke in textural aspects. Indeed, blocks hold the lowest degree of textural complexity within their density-number.[44] In fact,

[44] The complexity of a given textural configuration can be measured by the evaluation of the degree of independence of threads so that polyphonic textural organizations tend to be understood as more complex than massive ones. Of course, this definition is based only in regards of textural aspects. See [18], [20], and [21].

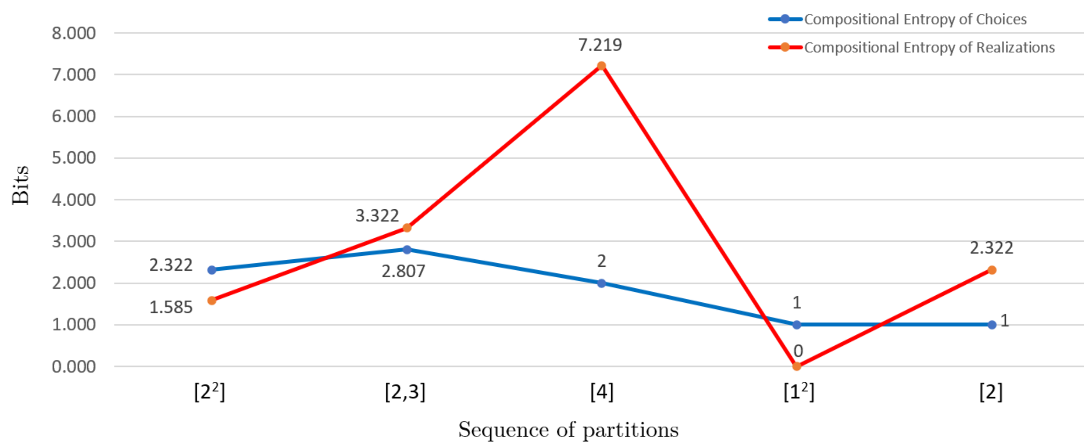


**Figure 14:** *Compositional entropy of choice and compositional entropy of realization for the sequence of partitions <$[2^2][2,3][4][1^2][2]$>.*

the compositional practices of texture show that blocks are commonly associated with opening or cadential gestures, situations in which they are more likely to be understood as simpler than textural those configurations with a higher sense of polyphony (multiple parts).[45] Nevertheless, the compositional entropy brings to lights that blocks are way more complex to be implemented as music than one could imagine, by simply considering their combinatorial permutation of threads in evolving realization. This complexity increases astronomically by combining gradual construction and dilutions.

## VI. Concluding remarks

This article introduces the idea of compositional entropy—a proposal for measuring the amount of freedom a given object in the scope of a musical parameter or attribute provides for compositional purposes so that the more possibilities (variables) to be chosen are given by the object, the higher will be its compositional entropy. The main interest of such a formulation is to discuss compositional choices in a view of probability and combinatorial permutation, considering the number of available compositional choices a composer may operate during the creative process. Further applications in various musical parameters and compositional situations shall be considered in future works to verify all potentialities of this still embryonic proposal.

In order to apply this proposal in the realm of musical texture to verify its potentialities in terms of compositional choices, a series of concepts and mathematical tools regarding textural morphology, as well as their possible spatial organizations, were introduced. This can contribute to the development of the textural field from both analytical and compositional perspectives. Moreover, the exhaustive taxonomy of textural layouts presented in this article opens avenues for compositional and analytical—which shall be further developed in future works.

All examples presented in this article, but those in evolving realization, were given according to a standard mode of presentation, which may be understood as "a simple articulation of all parts of a textural configuration in a strict way within the same time span", constituting, therefore,

[45] See [6].

"a strict one-to-one relation with the referential configuration and its musical realization." ([21, p. 167]). The inclusion of the other modes in the perspective proposed here may expand the number of possible realizations of a given partition. For example, Gentil-Nunes [8] elaborates the idea of a *partitional complex*, in which small deviations of textural parts may produce subsets and other derived parts of the referential partition. Unlike evolving mode, in partitional complex, polyphonic partitions provide a wider set of possible realizations. Hence, taking the formulation of partitional complexes into consideration may expand the number of possible realizations for all partitions, thereby impacting the computation of compositional entropy. Another possibility to be considered is the application of the principles of evolving realization in any block in contexts of multiple parts. For example, each block of partitions [$2^2$] and [2,3] in Figure **??** could be realized as the cumulative superimposition of threads therein. All these possibilities discussed here are a wealthy territory to be explored in future works.

The elaboration of computational tools for automatizing the process of computing the number of possibilities and their compositional entropy may be devised to further improve this discussion. Also, the automatic elaboration of the list of exhaustive taxonomies for textural layouts of a given partition may facilitate their application within the compositional process, also contributing to further theoretical developments from it. Finally, the automatic generation of graphs from a MIDI file may enable analytical applications in a more systematic way. All these computational implementations are left for future works.

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