# Feathered Beams

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***Abstract****: In music, durations are quantized to subdivisions of time in the form of fractions of the inverse powers of 2 (e.g., 1/$2^0$, 1/$2^1$, 1/$2^2$, etc.). All durations that are not involved in tuplets can be represented by sums of these fractions. The gradual transition from one note duration to another through a specified number of intermediate note values requires an accelerando/ritardando beam (i.e., feathered beam). This notation, however, does not indicate exactly how the gradual transition through the intermediate note values is to occur. The various details may be so contradictory that feathered beams may be impossible to realize. Thus, the notation is inherently indeterminate, although it is not often regarded as such. This paper examines these concepts and combines rhythmic nomenclature with a graphing system to deconstruct feathered beams using examples from George Crumb's Night Music I.*

***Keywords****: Feathered Beams. Accelerando/ritardando Beam. Duration. Rhythm. Graph. George Crumb.*

## 1. Feathered Beams

Accelerando/ritardando beams notate the gradual transition from one note duration to another. They are nicknamed feathered beams because in music notation the beams flare out like feathers (see Figures 4, 5, and 16 below for examples from the standard repertoire). This notation specifies note values at the beginning and ending of the beamed group, a gradual transition between those note values through a certain number of intermediate values, and in some cases, a precisely specified total duration. This notation, however, does not indicate exactly how to execute the gradual transition through the intermediate note values. This lack of information means that the notation is inherently indeterminate. However, the specificity of the notation disguises this indeterminacy. Furthermore, in some cases, the various specific requirements of a feathered beam group may be so contradictory that they are impossible to realize in practice. Although this paper examines feathered beams precisely, some composers do not consider feathered beams to be precise, but rather merely mean, for example, from something fast to something slow.

This paper combines rhythmic nomenclature (section II) with a graphing system (section III) to deconstruct feathered beams, and to compare the information inherent in the notation, the composer's intention, and the practical aspects of execution. Examples from George Crumb's *Night Music I* are plotted to explain the feathered beam technique and the theory behind it. First, this paper examines the more general case for when the total duration of the beamed group is not

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specified, and some questions are considered with respect to performance practice and cognition (section IV). Next, a more constrained case is examined, one in which the total duration is specified (section V). In this scenario, however, some problems arise, most notably the impossibility of accurately realizing the notation. Finally, feathered beams are generalized to calculate the full duration of any linear or quadratic feathered beam (section VI).

This paper's focus is intentionally limited in order to demonstrate particular problems with feathered beams, mostly in the abstract. Much work remains to be done to fully understand how feathered beams are executed and perceived within a greater context of the music and phrase, but some ramifications are considered with respect to research that has been done on traditional rhythmic scenarios. Studying the examples in isolation reveals the limitations and complexities of this notation.

## 2. Definitions

### 2.1. Duration

In Western musical notation, the unit of duration is the whole note ($w$) and durations are given as subdivisions of it. The system is organized around note values based on the inverse powers of 2 where $(\frac{1}{2^p}), p \in \mathbb{Z}^+$:

𝅝 $= (\frac{1}{2^0})w = w$

𝅗𝅥 $= (\frac{1}{2^1})w = \frac{w}{2}$

♩ $= (\frac{1}{2^2})w = \frac{w}{4}$

♪ $= (\frac{1}{2^3})w = \frac{w}{8}$

𝅘𝅥𝅯 $= (\frac{1}{2^4})w = \frac{w}{16}$

𝅘𝅥𝅰 $= (\frac{1}{2^5})w = \frac{w}{32}$

etc.

Durations are subject to the normal operative rules of addition, subtraction, multiplication, and division; the commutative and transitive properties hold true with duration operations.

### 2.2. Augmentation Dots

A single augmentation dot adds 50% to the duration of the note that it follows:

𝅗𝅥. = 𝅗𝅥‿♩ $= \frac{w}{2} + \frac{w}{4} = \frac{3w}{4}$

♩. = ♩‿♪ $= \frac{w}{4} + \frac{w}{8} = \frac{3w}{8}$

♪. = ♪‿𝅘𝅥𝅯 = $\frac{w}{8} + \frac{w}{16} = \frac{3w}{16}$

etc.

Consecutive augmentation dots are given by the summation:

$$\sum_{i=p}^{d+p} \frac{w}{2^i} = \frac{(2^{d+1}-1)w}{2^{d+p}}, \tag{1}$$

where $p$ is the exponent of the power of 2 of the corresponding note without the augmentation dots with respect to $w$ (e.g., whole note when $p = 0$, half note when $p = 1$, quarter note when $p = 2$, eighth note when $p = 3$, etc.) and $d$ equals the number of augmentation dots ($p, d \in \mathbb{Z}^+$).

♩. = ♩‿♪ = $\frac{(2^{1+1}-1)w}{2^{1+2}} = \frac{3w}{8}$

♩.. = ♩‿♪‿𝅘𝅥𝅯 = $\frac{(2^{2+1}-1)w}{2^{2+2}} = \frac{7w}{16}$

♩... = ♩‿♪‿𝅘𝅥𝅯‿𝅘𝅥𝅰 = $\frac{(2^{3+1}-1)w}{2^{3+2}} = \frac{15w}{32}$

etc.

When continually applying augmentation dots, the duration limits to 2 times the duration of the note they are on.

## 2.3. Tuplets

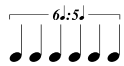
The note values presented so far in this paper can indicate any duration that is evenly divisible by $\frac{1}{2^p}$. In tuplets, however, note values are compressed according to ratios, such as in a traditional triplet of eighth notes that has 3 eighth notes compressed into the duration of 2 eighth notes. The 2 eighth notes in the example that follows are not compressed; their ratio is 2♪ : 2♪ = 1. Below that, the 3 eighth notes are compressed into the duration of 2 eighth notes, yielding a ratio of 3♪ : 2♪ = $\frac{3}{2}$. Each of these individual eighth notes have the duration $\left(\frac{2}{3}\right)\left(\frac{w}{8}\right) = \frac{w}{12}$, which is one-third of a quarter note (i.e., $(w/4)/3$). (Traditional triplets are simply indicated by 3 in a bracket or next to a beam, but this paper is more specific in that it uses the corresponding note values along with their ratio (e.g., 3♪ : 2♪) which is especially applicable to more complicated tuplets.)

$\frac{w}{8} + \frac{w}{8} = \frac{w}{4}$ 2♪ : 2♪ = 1


$\left(\frac{2}{3}\right)\left(\frac{w}{8} + \frac{w}{8} + \frac{w}{8}\right) = \frac{w}{4}$ 3♪ : 2♪ = $\frac{3}{2}$

Some more examples follow:


$\left(\frac{4}{5}\right)\left(\frac{5w}{16}\right) = \frac{w}{4}$ 5𝅘𝅥𝅯 : 4𝅘𝅥𝅯 or 5𝅘𝅥𝅯 : ♩ = $\frac{5}{4}$




$$\left(\frac{5}{6}\right)\left(\frac{6w}{4}\right) = \frac{5w}{4}$$

6♩ : 5♩ $= \frac{6}{5}$

In the above examples and throughout this paper, tuplets have ratios greater than 1 and less than 2 indicating that the note durations have been compressed; however, some composers find it acceptable to use tuplets with ratios of less than 1 augmenting the note durations, such as in Debussy's *Suite bergamasque*, 3. "Clair de Lune", m. 3 (Figure 1).

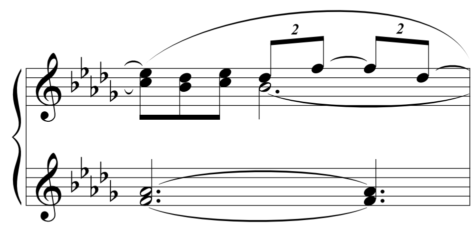
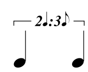



**Figure 1:** *Debussy,* Suite bergamasque*, 3. "Clair de Lune", m. 3.*

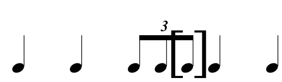
Here, the compound meter has dotted quarter-note beats, and the simple subdivisions of the beat are notated with eighth-note duplets. The ratio is 2♪:3♪= $\frac{2}{3}$, which, according to common notation practice described in many style manuals ([5]), is usually not preferred because $\frac{2}{3} < 1$. Preferably, this example should use 2 quarter notes instead of 2 eighth notes in this duplet, because it makes the ratio 2♩:3♪= 4♪:3♪= $\frac{4}{3}$, which is $1 < \frac{4}{3} < 2$.



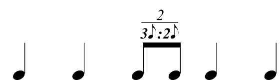

Situations similar to this duplet can be visually misleading in some circumstances. Usually there are simpler ways of notating tuplets and partial tuplets. For instance, the above duplet can be notated more simply as 2 dotted-eighth notes.

## 2.4. Partial tuplets

The individual notes within tuplets can represent many rational durations. For example, the first two notes in a traditional triplet of eighth notes equal the duration $\left(\frac{2}{3}\right)\left(\frac{w}{4}\right)$. Consider a musical context where these two notes occur without the triplet's remaining eighth note. According to traditional musical notation, the remaining eighth note should be included in the score enclosed in brackets as shown in the following rhythm.[8]

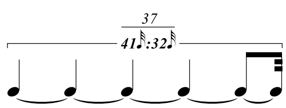



The note in the brackets is not played and its value does not occur in the time of the music. In other words, if the quarter note gets the beat, then the last two quarter notes in the rhythm are shifted a third of a beat forward in time. Only the first two-thirds of the triplet are to be played indicating a partial amount of the tuplet. In this paper, the following notation will be used instead, where the number in the numerator specifies the partial amount of the tuplet.




The tuplet is given as $\frac{z}{x:y}$, which indicates $z$ parts of $x$ notes in the span of $y$ notes where $z$, $x$, and $y$ are positive integers such that $x > y \geq z \geq 1$.[1] Partial tuplets can usually be avoided by using other notation that can express a similar or equivalent musical result. An equivalent rhythm to the one above, but with compound beats and without the partial tuplet, is given below.

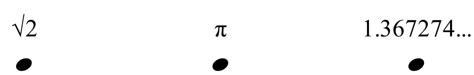
Any rational duration even if it is not evenly divisible by $\frac{1}{2^p}$ can be notated with tuplets or partial tuplets. For example, although musically impractical, the duration $\frac{37w}{41}$ can be notated as follows:




$$= \left(\frac{37}{41}\right)\left(\frac{32w}{32}\right) = \frac{37w}{41} \qquad 41♪ : 37♪ = \frac{41}{37}$$

## 2.5. Irrational durations

All the durations presented in this paper to this point are rational, but sometimes it is necessary to indicate irrational durations. Irrational durations cannot be represented as previously shown in this paper. So, for the purpose of analysis, they will be given as solid stemless noteheads with the durations in whole-note units written above them. Some examples are as follows:




# 3. Duration Graphs

A musical score is like a graph where time is given on the horizontal axis and pitch is given on the vertical axis. Consider the first phrase of Brahms, Variations on a Theme by Haydn, Op. 56 (Figure 2).

[1]This notation may not be conducive to sight reading, but accurately describes the rhythmic phenomena and is useful for the purposes of analysis.

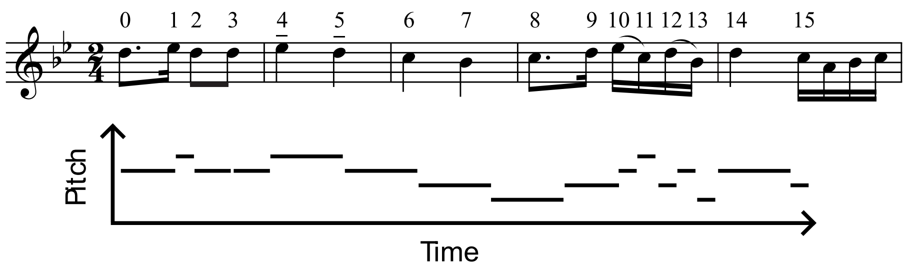



**Figure 2:** *Brahms,* Variations on a Theme by Haydn*, Op. 56, mm. 1–5.*

The notes are indexed beginning with zero. The horizontal bars below the staff are proportional to the durations of the notes. In Figure 3, all those horizontal bars are arranged vertically. The notes are given in temporal order on the horizontal axis according to their indices, and their durations are given on the vertical axis. This way, changes in duration can be plotted.(N.b., the curve fitting in the illustration merely shows that changes in duration can follow a curve.)

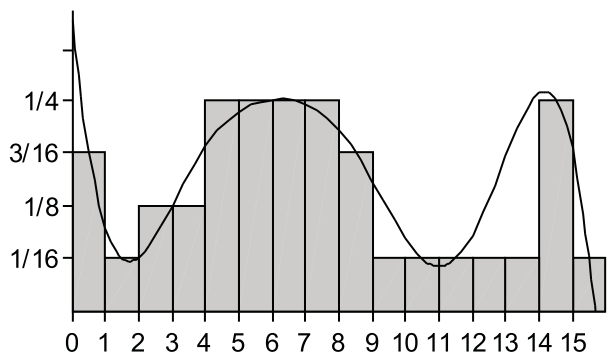



**Figure 3:** *Duration graph of Figure 2.*

In this paper, durations in feathered beams are given as a series of $(i, r)$ coordinates, where $i$ is the index and $r$ is the duration ($r \in \mathbb{Q}$). For the Brahms example above, this is:

$$\left\langle \left(0, \frac{3w}{16}\right), \left(1, \frac{w}{16}\right), \cdots, \left(15, \frac{w}{16}\right) \right\rangle.$$

Changes in duration can be plotted as lines and curves on these graphs, and can be given with standard algebra. Lines are given in the form

$$f(x) = ax + b$$

and quadratic curves are given in the form

$$f(x) = ax^2 + bx + c,$$

where $a, b, c, x \in \mathbb{R}$.

When a series of durations does not correspond to an easily determined equation, a curve may be plotted through the coordinates using splines (piecewise polynomials that match their values and derivatives between the pieces). The first derivative of the lines and curves give the rate of change, and the second derivative gives the acceleration.The integral of the lines and curves approximates the sum of the durations of all the notes. These concepts will be explained more fully in the following section.

## 4. George Crumb, *Night Music* I, "Notturno I", rehearsal 5

In this section of the paper, the following excerpt from the 1976 revised version of George Crumb's *Night Music I*, "Notturno I" is exploited to demonstrate the details of the note-duration graphing system. I chose this excerpt after searching extensively for examples of feathered beams. I found a plethora of examples where the feathered beams do not even closely fit within their corresponding spans of music and/or be approximately performed as notated. In light of these observations, examples in George Crumb's *Night Music I* best suit my needs to illustrate the points in this paper. In "Notturno I" at rehearsal 5, ten notes under a feathered beam group gradually transition from a thirty-second to eighth note [2] (Figure 4).


**Figure 4:** *George Crumb,* Night Music I*, "Notturno I", rehearsal 5.*

The original 1967 version [1] of the score incorporated various improvisatory elements, most of which are notated more precisely in the 1976 revision. Throughout the revised version, there are ritardando or accelerando indications written above the feathered beams that are not in the original version. Other examples, such as the one below from "Notturno III",suggest that the feathered beams should not be further exaggerated by tempo changes, but that the rit. and accel. tempo indications merely reinforce the rhythmic gestures (Figure 5). Thus, the rit. indication in the above example is taken as a redundant instruction.

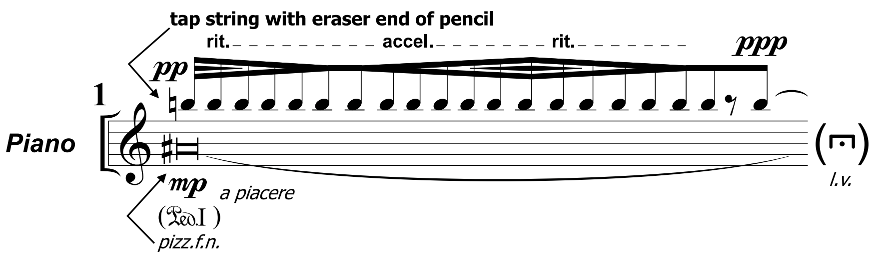



**Figure 5:** *George Crumb,* Night Music I, *"Notturno III".*

The 10-note feathered beam from the "Notturno I" example is in an unmetered context and there are no simultaneous figures to indicate that it spans a specific duration. The ten notes in the feathered beam are identified in temporal order as $n_0, n_1, \cdots, n_9$. The difference between a thirty-second and an eighth note is:

$$\frac{w}{8} - \frac{w}{32} = \frac{3w}{32}$$

and the difference between the note numbers $n_0$ and $n_9$ is

$$9 - 0 = 9.$$

So, assuming that change in duration is linear, each note in succession is

$$\frac{\frac{3w}{32}}{9} = \frac{w}{96}$$

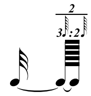
longer than the previous one, such that

$$n_i = n_{i-1} + \frac{w}{96}.$$

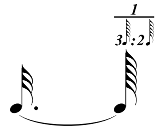
Therefore:

$n_0 = \frac{w}{32}$

$$n_1 = \frac{w}{32} + \frac{w}{96} = \frac{w}{24}$$

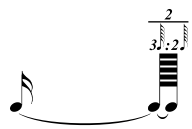


$$n_2 = \frac{w}{32} + 2\left(\frac{w}{96}\right) = \frac{5w}{96}$$

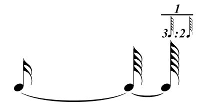


$$n_3 = \frac{w}{32} + 3\left(\frac{w}{96}\right) = \frac{w}{16}$$

$$n_4 = \frac{w}{32} + 4\left(\frac{w}{96}\right) = \frac{7w}{96}$$

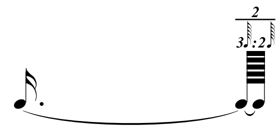


$$n_5 = \frac{w}{32} + 5\left(\frac{w}{96}\right) = \frac{w}{12}$$



$$n_6 = \frac{w}{32} + 6\left(\frac{w}{96}\right) = \frac{3w}{32}$$

$$n_7 = \frac{w}{32} + 7\left(\frac{w}{96}\right) = \frac{5w}{48}$$



$$n_8 = \frac{w}{32} + 8\left(\frac{w}{96}\right) = \frac{11w}{96}$$

$$n_9 = \frac{w}{32} + 9\left(\frac{w}{96}\right) = \frac{w}{8}$$

The addition of a constant value to each note in succession gives the linear equation

$$f(x) = \frac{w}{96}x + \frac{w}{32},$$

which corresponds to the following graph (Figure 6):

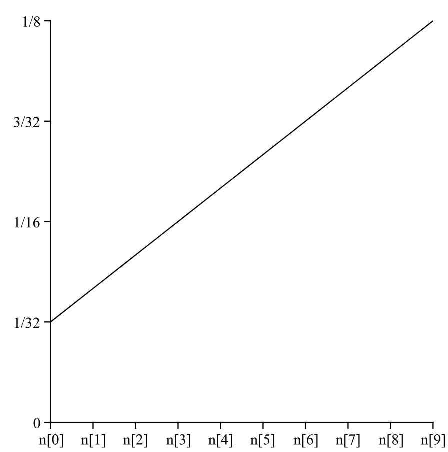



**Figure 6:** *George Crumb, Linear duration graph of "Notturno I", rehearsal 5.*

The linear graph has a constant rate of change or slope, so the first derivative yields a horizontal line. Since the rate of change is constant, there is no acceleration, so the second derivative is zero (Figure 7).

$$f(x) = \frac{w}{96}x + \frac{w}{32}, \quad f'(x) = \frac{w}{96}x, \quad f''(x) = 0.$$

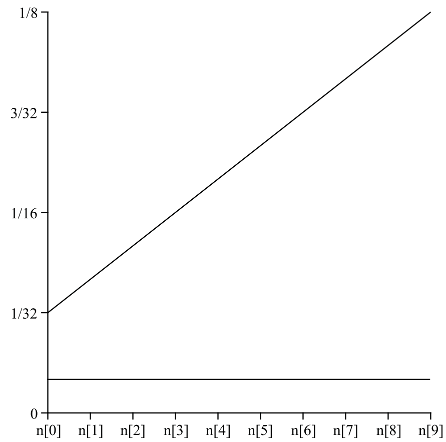
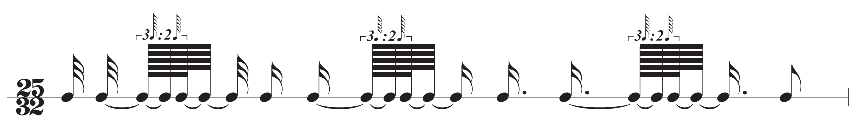



**Figure 7:** *$f(x)$, $f'(x)$, and $f''(x)$of the linear duration graph of "Notturno I", rehearsal 5.*

The sum of the durations of all 10 notes is

$$f(n_0) + \cdots + f(n_9) = \frac{25w}{32}.$$



The integral of $f(x)$ is

$$\int f(x)dx = \frac{w}{192}x^2 + \frac{w}{32}x,$$

which gives the sum of the durations of the notes as they occur, and corresponds to the graph (Figure 8)

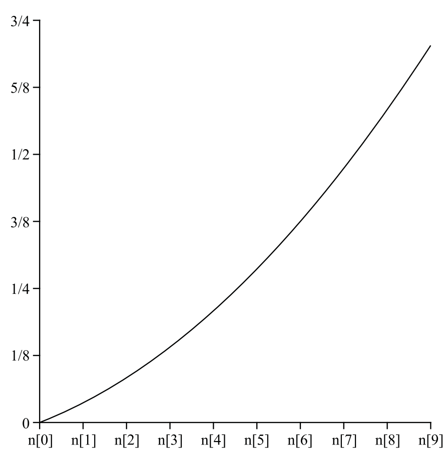



**Figure 8:** *Integral of the linear duration graph of "Notturno I", rehearsal 5.*

Solving the integral from 0 to 10 gives

$$\int_0^{10} f(x)dx = \frac{5w}{6},$$

which is approximately the duration of all 10 notes in the feathered beam. The value is an approximation because $\int_0^{10} f(x)dx$ is equal to the entire area underneath $f(x)$ while the actual duration of all 10 notes is equal to the Riemann sums of $f(x)$.

$$\int_0^{10} f(x)dx = \frac{5w}{6},$$

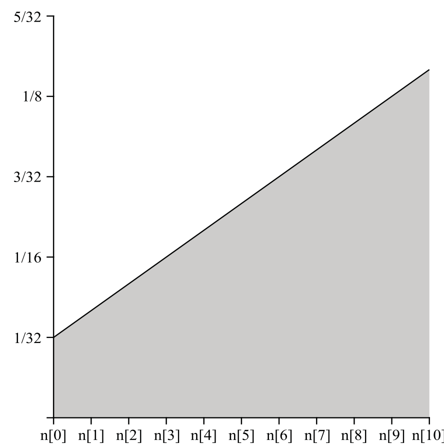



**Figure 9:** *Entire area underneath $f(x)$.*

$$f(n_0) + \cdots + f(n_9) = \frac{25w}{32}.$$

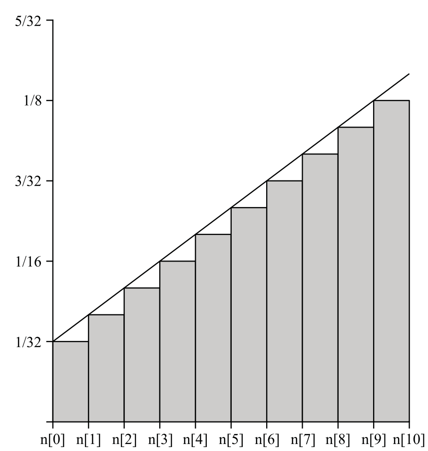



**Figure 10:** *Riemann sums of $f(x)$.*

The linear interpretation, as shown above, is only one of several ways to realize feathered beams. Also useful is the quadratic equation which can lead to multiple interpretations. Ulf

Kronman and Johan Sundberg show that performers' ritardandi at the ends of compositions tend to follow quadratic curves, and Neil Todd uses parabolas to model gestures at the ends of phrases [7], [21],[22]. Furthermore, Bruno Repp suggests that the parabolic curve might "indeed represent a 'natural' way of changing tempo, including both accelerando and ritardando" [11]. For the "Notturno I" feathered beam above, the two endpoints $(0, \frac{w}{32})$ and $(9, \frac{w}{8})$ in conjunction with a third symmetrical coordinate $(-9, \frac{w}{8})$ can be used to solve the quadratic equation

$$g(x) = \frac{w}{864}x^2 + \frac{w}{32}.$$

The third coordinate may be chosen differently, and depending on its value, a wide variety of curves can be made for $g(x)$.[2] The graphs of $g(x)$ and its first and second derivatives are

$$g(x) = \frac{w}{864}x^2 + \frac{w}{32}, \quad g(x) = \frac{w}{432}x, \quad g''(x) = \frac{w}{432},$$

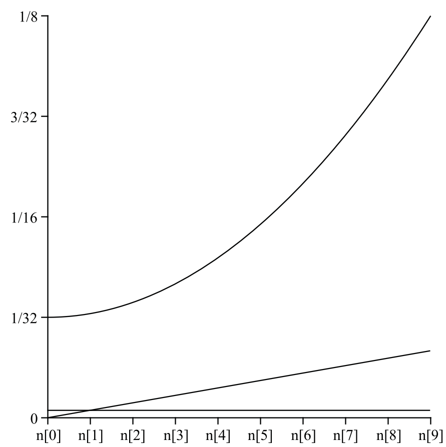



**Figure 11:** *$g(x)$, $g'(x)$, and $g''(x)$ of the quadratic duration graph of "Notturno I", rehearsal 5.*

which give the note values, rate of change, and the acceleration respectively. The integral of $g(x)$ from 0 to 10 is approximately equal to the sum of all 10 notes.

$$\int_0^{10} g(x)dx = \frac{905w}{1296}.$$

[2] Honing ([6]) points out that "the mere fact that the overall shape (e.g. of a square root function) can be predicted by the rules that come with human motion is not enough evidence for an underlying physical model of expressive timing in music performance, however attractive such a model might be." Thus, it is understood that this curve is taken arbitrarily.

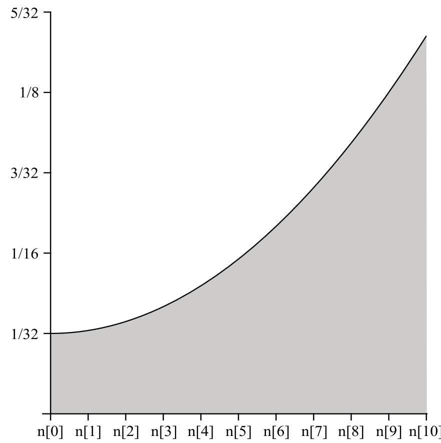



**Figure 12:** *Entire area underneath $g(x)$.*

$$g(n_0) + \cdots + g(n_9) = \frac{185w}{288}$$

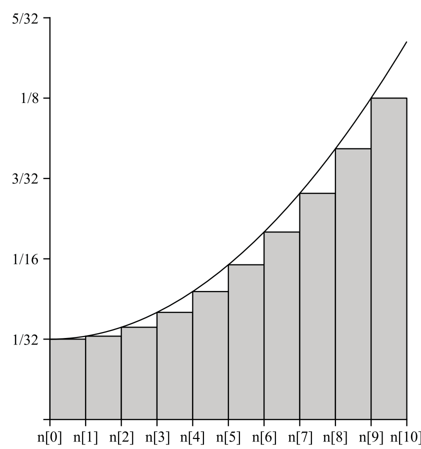



**Figure 13:** *Riemann sums of $g(x)$.*

Just like before, the discrepancy in the durations is due to the difference between the continuous area underneath the curve and the area quantized by the note values.

There are many ways that the notes in the feathered beam can be quadratically interpreted other than the way $g(x)$ shows. For example, consider the quadratic equation solved from the two endpoints $\left(0, \frac{w}{32}\right)$ and $\left(9, \frac{w}{8}\right)$ in conjunction with a third coordinate $\left(-11, \frac{w}{8}\right)$.

$$h(x) = \frac{w}{1056}x^2 + \frac{w}{528}x + \frac{w}{32}$$

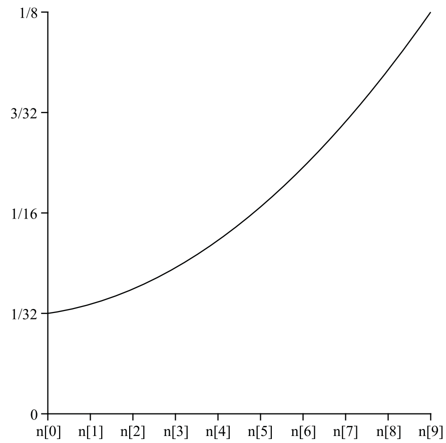



**Figure 14:** *Another possible quadratic duration graph of "Notturno I", rehearsal 5.*

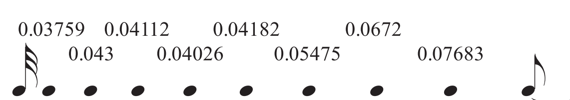
One notable difference between the two quadratic curves is that $g(x)$ begins with a slope of $g'(0) = 0$, while $h(x)$ begins with a greater slope $h'(0) = \frac{w}{528}$.

All of the examples so far have dealt with purely abstract concepts to conceptualize feathered beams. These concepts will now be applied to a real performance. Time points from a sound wave of the *Night Music I*, "Notturno I" excerpt as performed by Speculum Musicae are given below [3]. (N.b., the time points are measured from the highest relative amplitude for each note to 3 significant digits.) The time index begins at 0 seconds, and the times of the grace note and 10 notes in the feathered beam are

| | | |
|---|---|---|
| $g.n. = 0.510s$ | $n_3 = 1.020s$ | $n_7 = 1.785s$ |
| $n_0 = 0.544s$ | $n_4 = 1.195s$ | $n_8 = 2.071s$ |
| $n_1 = 0.677s$ | $n_5 = 1.374s$ | $n_9 = 2.398s$ |
| $n_2 = 0.837s$ | $n_6 = 1.552s$. | |

Since the feathered beam begins with a thirty-second note, the duration of a thirty-second note in seconds can be calculated from the recording by subtracting $n_0$ from $n_1$ ($0.677s - 0.544s = 0.133s$). Furthermore, the tempo is *senza misura, quasi improvisando* (♩ = ca.40, but very free). At ♩ = 40, a thirty-second note is $0.1875s$, which differs significantly from the $n_1 - n_0$ calculation. Admittedly problematic, discrepancies such as this one will be discussed later in this paper. For now we will consider a thirty-second note to be $0.133s$, in order to purely demonstrate the application of splines to feathered beams. With a thirty-second note corresponding to $0.133s$, the notes under the feathered beam have the following durations (in whole-note units to 5 significant digits):

$$\left\langle \begin{array}{c} \left(n_0, \frac{w}{32}\right), (n_1, 0.03759w), (n_2, 0.043w), (n_3, 0.04112w), (n_4, 0.04206w), \\ (n_5, 0.04182w), (n_6, 0.05475w), (n_7, 0.0672w), (n_8, 0.07683w)) \end{array} \right\rangle$$



The duration of the final note $n_9$ is not shown in the series of durations because there is no immediate onset after it for which to calculate its duration. Furthermore, $n_9$ has resonance slurs indicating that it is to be sustained. With the 9 coordinates, the spline $j(x)$ is solved as [3]

$$j(x) = \begin{cases} 0.03125w + 0.00599wx + 0.00035wx^3 & x < 1 \\ 0.03054w + 0.00705wx + 0.00106w(x-1)^2 - 0.00269w(x-1)^3 & 1 < x < 2 \\ 0.04084w + 0.00108wx - 0.00702w(x-2)^2 + 0.00406w(x-2)^3 & 2 < x < 3 \\ 0.04346w - 0.00078wx + 0.00516w(x-3)^2 - 0.00344w(x-3)^3 & 3 < x < 4 \\ 0.04516w - 0.00078wx - 0.00515w(x-4)^2 + 0.00569w(x-4)^3 & 4 < x < 5 \\ 0.01189w + 0.00599wx + 0.01192w(x-5)^2 - 0.00497w(x-5)^3 & 5 < x < 6 \\ -0.03467w + 0.0149wx - 0.003w(x-6)^2 + 0.00054w(x-6)^3 & 6 < x < 7 \\ -0.00658w + 0.01054wx - 0.00137w(x-7)^2 + 0.00046w(x-7)^3 & 7 < x \end{cases}$$

and corresponds to the graph (Figure 15):

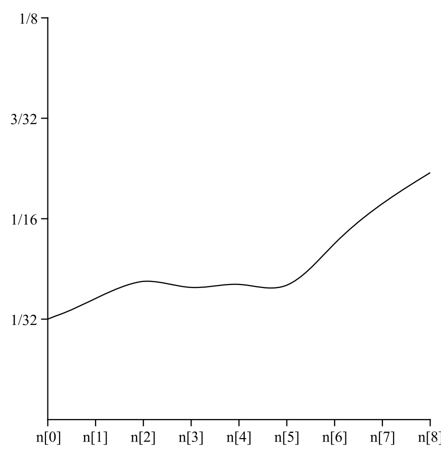



**Figure 15:** *Duration graph of Night Music I, "Notturno I",rehearsal 5 as performed by Speculum Musicae.*

The notes $n_3$ and $n_5$ are shorter than the notes that immediately precede them. Furthermore, with the duration of a thirty-second note being calculated by the time difference between $n_0$ and $n_1$(i.e., $n_1 - n_0 = 0.133s = w/32$),the corresponding eighth note at $n_9$ would be significantly longer than $n_8$ according to the graph, contradicting what the feathered beam suggests.

The linear $f(x)$, quadratic $g(x)$, and spline $j(x)$ are three of many possible representations of the example from *Night Music I*, "Notturno I". The purpose of these three equations given in this paper is to demonstrate some technical aspects behind the feathered-beam technique. In the linear and quadratic representations, the note $n_9$ was assigned the duration of an eighth note even though its resonance slurs indicate that it is to be sustained. In the recording, $n_9$ was not measured because there is no event immediately following for which to calculate its duration. These seemingly arbitrary decisions were made to best compare different representations of the feathered beam excerpt. As one scours the repertoire for examples well suited to the analysis

[3] I would like to thank Michael Wester for his assistance in calculating the splines. It is not necessary to have full understanding of splines in order to grasp their relevance to this paper, but the reader may find further details about them in [4].

techniques demonstrated above, it is difficult to find feathered beams that do not require some arbitrary adjustment. The feathered beam from "Notturno I" is taken out of context to demonstrate technical aspects of feathered beams (as discussed in the previous few paragraphs) because the greater musical context introduces many more variables. Several questions remain to be pursued, especially about the complex ramifications for perception and performance.

The deviation from an exact representation of a feathered beam to a real performance, such as the one by Speculum Musicae, are known as "expressive microstructure." Deviations may not even be perceptible if they are not very severe. Under controlled circumstances with regularly repeating rhythms, deviations need to be at least 10 msec before they are perceptible, but deviations this small may not even be perceptible in a feathered beam because of the irregularity of the rhythm. It is difficult (or impossible) to generalize these deviations because, according to Bruno Repp, "the expressive microstructure of a musical performance reflects general, composer-specific, performer-specific, and piece-specific factors." In addition, "actual performances are likely to contain 'noise' in the form of random and planned deviations. . . , as one should expect from a human performer." "Musicians are usually only dimly aware of these variations, which they control intuitively rather than deliberately. Similarly, listeners perceive the structure and expression conveyed by these variations without being aware of the microstructure as such."[11, p. 222]

A feathered beam would likely be heard as a single expressive unit. The performance timing pattern of a feathered beam gesture probably needs to be compatible with the larger temporal organization [11, p. 225], which further complicates the interpretive restrictions placed upon it. According to Repp, the interpretation of expressive timing in general is due to cognitive analysis and mental representation of musical structure, and to feeling and expressive characterization, so to fully understand a feathered beam would require it to be considered within its context and not just as an isolated rhythm. In the greater context, tempo significantly affects the execution and perception of timing of rhythms [15, p. 389], especially when they comprise more than two interval durations [14, p. 590], such as with feathered beams. Much of the work that has gone into understanding how we execute and perceive rhythms has been done on interval durations based on simple ratios,[17, p. 63] but not much is known about how this happens with rhythms that rely on gradual changes of duration. Generalizations are hard to make even with simple traditional rhythms because, according to Bruno Repp, Justin London, and Peter Keller, "simple interval structure of rhythms seems to have an influence on the nature and degree of ratio distortion, and there are considerable individual differences as well." [17, p. 74] In regards to research on rhythm that has already been performed, the same authors point out that "It remains to be seen whether this result will hold up for sequences that have more complex (nonmetrical) interval ratios." The way in which we hear feathered beams is intrinsically complicated, and little research has been done to understand it. There cannot be a universal way to hear them because, according to Bruno Repp, John Iversen, and Aniruddh Patel, there is no universal way to hear rhythm in general:

> Although it is not uncommon to find that different listeners (or even the same listeners at different times) arrive at different metrical interpretations of the same rhythm, for any given listener at any given time his or her interpretation is considered to constitute a single, momentarily optimal solution to the informational jigsaw puzzle, with other possible solutions being discarded along the way or never even being considered. [19]

The anomalies in the splines example above may be perceptually insignificant, and the human element must certainly factor in, but there must be a point at which discrepancies in general can be considered errors. Another recording of this same passage or an investigation of other feathered beams in the same recording would likely produce drastically different results, and further study of more passages might reveal further conclusions. In general, much more work remains to be done to fully understand these implications. Feathered beams are notation given by a composer, which are to be interpreted by a performer, and then heard by a listener and interpreted based on that hearing. In light of this observation, several questions come to mind. When a composer writes a feathered beam, how exact are they being with respect to the sound that they are hearing or wants to occur? When a performer reads a feathered beam, how exact are they being according to what the composer wants, and what the performer is trying to do? How consistent is the performer across performances? And when listeners hear feathered beams performed, how sensitive are they to variations in them?[4] For some composers, the preciseness of a feathered beam is irrelevant because they may merely mean from something fast to something slow.

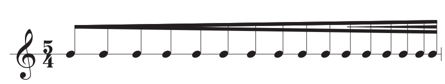
The most notable discrepancy between the above linear, quadratic, and spline representations is their total durations. Although there are complications with exact comparison (e.g., how to handle $n_9$), the durations of $f(x)$, $g(x)$, and $j(x)$ from $n_0$ to $n_8$ as given in this paper are

$$\begin{aligned} f(0)+f(1)+\cdots+f(8) &= \frac{21w}{32} = 0.65625w \\ g(0)+g(1)+\cdots+g(8) &= \frac{191w}{352} = 0.54261\overline{36}w \\ j(0)+j(1)+\cdots+j(8) &= \frac{27,781w}{50,000} = 0.43562w \end{aligned}$$

The differences between the full durations show that problems arise when fitting a feathered beam group within a specific duration of metered music. This dilemma will be tackled in the next section of this paper.

## 5. The Full Duration of Feathered Beams

There exist feathered beams that exactly fit within a specific number of beats in commonly used meters. For example, consider a feathered beam group with 16 notes that linearly transitions from an eighth to a thirty-second note.



$$\begin{aligned} f(x) &= \frac{w}{160} + \frac{w}{8} \\ f(0)+f(1)+\cdots+f(15) &= \frac{5w}{4} \end{aligned}$$

The sum of all 16 notes from $n_0$ to $n_{15}$ is equal to 5 quarter notes. Feathered beams that work out this nicely are uncommon. When clarification for the full duration of a feather beam is needed, particularly in a metered context, Kurt Stone recommends placing a horizontal bracket over the feathered beam with a notehead indicating the full duration ([20, p. 124]). This section of the paper primarily focuses on the discrepancy between the sum of the note durations and the specified

[4] I would like to thank David Bashwiner for helping me arrive at these questions.

metered duration they span. Linear representations of the feathered beams are given in all cases, although the results can be duplicated for quadratic, spline, or some other representation of them. The conclusions drawn rely on the general concepts of feathered beams rather than exact calculations, so similar results can be duplicated with representations other than the linear case.

The following excerpt from the 1963 version of *Night Music I*, "Notturno V" has two feathered beams; each has 5 notes that gradually transition from a thirty-second to eighth note in the duration of 1 metered quarter note (Figure 16).

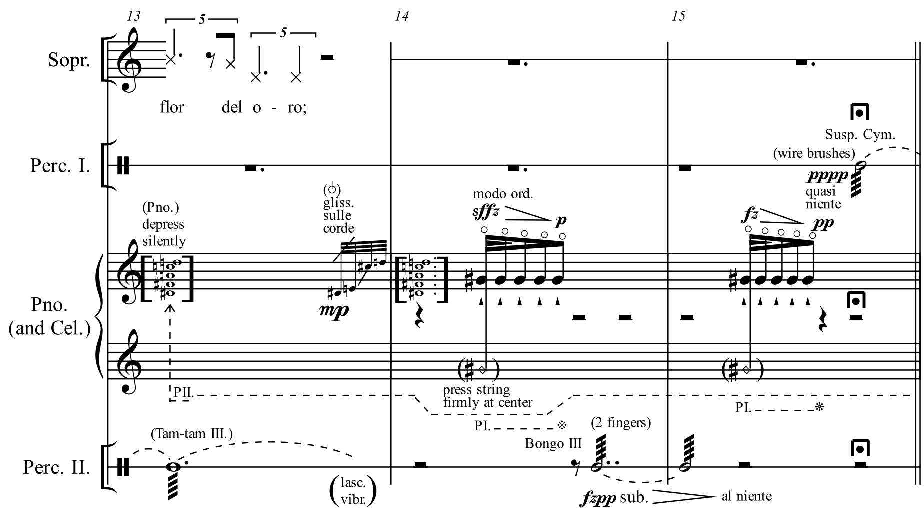



**Figure 16:** Night Music I, *"Notturno V", mm.13–15*

The 5 notes are indexed $n_0$ to $n_4$. Subtracting the first $(0, \frac{w}{32})$ and last $(4, \frac{w}{82})$ notes from the total duration equals

$$\frac{w}{4} - (\frac{w}{8} + \frac{w}{32}) = \frac{3w}{32}$$

The feathered beam suggests that the remaining 3 notes between $n_0$ and $n_4$ are longer than a thirty-second note and shorter than an eighth note $(\frac{w}{32} < n_1, n_2, n_3 < \frac{w}{8})$; however, the allotted space they have is not long enough $f(n_1) + f(n_2) + f(n_3) > \frac{3w}{32})$. Given the slope

$$\frac{\frac{3w}{32}}{4} = \frac{3w}{128},$$

the linear equation that corresponds to the feathered beam is

$$f(x) = \frac{3w}{128}x + \frac{w}{32}$$

and fits in the duration of

$$f(0) + f(1) + \cdots + f(4) = \frac{25w}{64}$$

The total duration $\dfrac{25w}{64}$ is $\dfrac{9w}{64}$ longer than the $\dfrac{w}{4}$ space allotted. Therefore, the notes must be shortened, and there are many ways this can be done. Three ways are given below that act uniformly on the linearity.

- $g(x)$: $n_0$ remains $\frac{w}{32}$ in duration, and the following 4 notes are shortened proportionally to decrease the slope.
- $h(x)$: $n_4$ remains $\frac{w}{8}$ in duration, and the preceding 4 notes are shortened proportionally to increase the slope.
- $j(x)$: All 5 notes are shortened by the same amount, and the slope remains unchanged.

The linear equations for $g(x)$, $h(x)$, and $j(x)$ are

$$g(x) = \frac{3w}{320}x + \frac{w}{32}$$
$$h(x) = \frac{3w}{80}x - \frac{w}{40}$$
$$j(x) = \frac{3w}{128}x + \frac{w}{320}.$$

If desired, the reader may verify these equations by substituting any known values for $n_0$, $n_4$, or $a$ into the sum of the linear equation $(ax + b)$ from $x = 0 \cdots 4$.

$$\frac{w}{4} = (0a + b) + (1a + b) + \cdots + (4a + b)$$
$$n_0 = 0a + b = b$$
$$n_4 = 4a + b$$
$$a = \frac{n_4 - n_0}{4}$$

The total durations of all three linear representations equals $\frac{w}{4}$ (this does not take into account that a performer may make the feathered beam longer than a quarter note).

$$g(0) + g(1) + \cdots + g(4) = \frac{w}{4}$$
$$h(0) + h(1) + \cdots + h(4) = \frac{w}{4}$$
$$j(0) + j(1) + \cdots + j(4) = \frac{w}{4}$$

$f(x)$, $g(x)$, $h(x)$, and $j(x)$ are plotted on the graph in Figure 17:

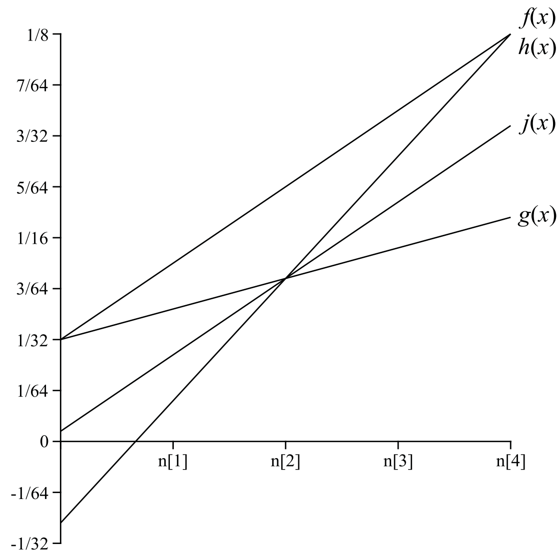
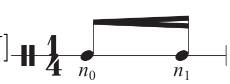



**Figure 17:** *Fitting the feathered beam into $w/4$ by leaving the first note unchanged $(g(x))$, the last note unchanged $(h(x))$, and the slope unchanged $(j(x))$.*

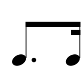
When $n_0$ remains unchanged as in $g(x)$, $n_4$ is nearly a sixteenth note, which contradicts the eighth note in the score. When $n_4$ remains unchanged as in $h(x)$, $\frac{w}{4}$ is not long enough for the remaining notes to fit linearly, thus $n_0$ has a negative duration. Each note in a traditional tuplet is compressed by the same amount, so shortening each note in the feathered beam by the same amount seems like the intuitive choice, as in $j(x)$ but, with this choice, $n_0$ is nearly a two-hundred-fifty-sixth note, which is extremely fast at the tempo. Of the three lines whose durations sum to $\frac{w}{4}$, only $g(x)$ is performable, but it does not represent the final note duration in the feathered beam. A performance of this feathered beam would require something different than what is notated, indicating the subjectivity necessary to execute the rhythm. In the method that follows, this subjectivity is examined more definitively in several feathered-beam examples.

All of the feathered beams that follow are encapsulated within $\frac{1}{4}$ measures. For the first scenario, consider a feathered beam with only two notes.

[H] 

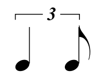

$$n_0 + n_1 = \frac{3w}{16} < \frac{w}{4}$$

There are no intervening notes to make the change in duration from the first to last notes gradual, thus it does not suffice for a feathered beam. If it is taken as a feathered beam, however, its duration needs to be increased to fill out the measure. Lengthening only the first note gives the rhythm:

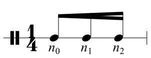


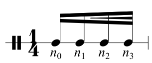
Lengthening only the last notes gives the rhythm:

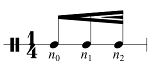
And lengthening both notes by the same amount gives the rhythm:

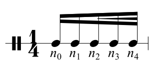


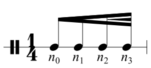

Thus, the feathered beam notation is superfluous in this case. The following table shows 8 more feathered-beam scenarios labeled a–h that in their literal linear interpretations most approximate the duration $\frac{w}{4}$.

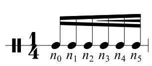
a)  $n_0 + ... + n_2 = \frac{9w}{32} > \frac{w}{4}$

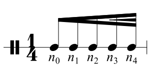
e)  $n_0 + ... + n_3 = \frac{3w}{16} < \frac{w}{4}$

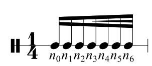
b)  $n_0 + ... + n_2 = \frac{15w}{64} < \frac{w}{4}$

f)  $n_0 + ... + n_4 = \frac{15w}{64} < \frac{w}{4}$

c)  $n_0 + ... + n_3 = \frac{5w}{16} > \frac{w}{4}$

g)  $n_0 + ... + n_5 = \frac{9w}{32} > \frac{w}{4}$

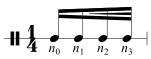
d)  $n_0 + ... + n_4 = \frac{25w}{64} > \frac{w}{4}$

h)  $n_0 + ... + n_6 = \frac{21w}{64} > \frac{w}{4}$

With 1 intervening note between the eighth and sixteenth notes, such as in *scenario a*, the total duration of the measure is exceeded by $\frac{w}{32}$ and the notes must be shortened to fit within the measure. Similarly in the other scenarios, the durations of the feathered beams are shorter or longer than the $\frac{w}{4}$ duration allotted. *Scenarios b*, *c*, and *d* transition from the durations of an eighth to a thirty-second over 3, 4, and 5 notes respectively. *Scenario b* is shorter than $\frac{w}{4}$ by $\frac{w}{64}$, and *scenarios c* and *d* are longer than $\frac{w}{4}$ by $\frac{w}{16}$ and $\frac{9w}{16}$ respectively. *Scenarios e*, *f*, *g*, and *h* transition from the durations of a sixteenth to a thirty-second over $3, 4, 5$, and 6 notes respectively. *Scenarios e* and *f* are shorter than $\frac{w}{4}$ by $\frac{3w}{16}$ and $\frac{15w}{64}$ respectively, and *scenarios g* and *h* are longer than $\frac{w}{4}$ by $\frac{9w}{32}$ and $\frac{21w}{64}$ respectively.

For the eighth/thirty-second note feathered beams, *scenarios b* and *c* straddle the $\frac{w}{4}$ duration, and for the sixteenth/thirty-second note feathered beams, *scenarios f* and *g* straddle the $\frac{w}{4}$ duration. The other scenarios differ from $\frac{w}{4}$ by a greater amount. *Scenarios b* and *c* are as close to $\frac{w}{4}$ as the eighth/thirty-second note feathered beams can be, while *scenarios f* and *g* are as close to $\frac{w}{4}$ the sixteenth/thirty-second note feathered beams can be. Figures 18, 19, 20, and 21 shows, respectively, the graphs of *scenarios e*, *f*, *g*, and *h* for $f(x)$. The other lines modify the note durations in $f(x)$ so that the feathered beams precisely fit within the $\frac{1}{4}$ measures: $g(x)$ leaves the first note unchanged, $h(x)$ leaves the last note unchanged, and $j(x)$ leaves the slope unchanged. (In each graph, $f(x)$ is bolder than the other lines.)

*Scenario e*



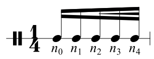
$$f(x) = -\frac{w}{96}x + \frac{w}{16}$$

$$g(x) = 0x + \frac{w}{16}$$

$$h(x) = -\frac{w}{48}x + \frac{3w}{32}$$

$$j(x) = -\frac{w}{96}x + \frac{5w}{64}$$

*Scenario f*



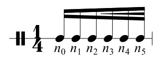
$$f(x) = -\frac{w}{128}x + \frac{w}{16}$$

$$g(x) = -\frac{w}{160}x + \frac{w}{16}$$

$$h(x) = -\frac{3w}{320}x + \frac{11w}{160}$$

$$j(x) = -\frac{w}{128}x + \frac{21w}{320}$$

*Scenario g*



$$f(x) = -\frac{w}{160}x + \frac{w}{16}$$

$$g(x) = -\frac{w}{120}x + \frac{w}{16}$$

$$h(x) = -\frac{w}{240}x + \frac{5w}{96}$$

$$j(x) = -\frac{w}{160}x + \frac{11w}{192}$$

*Scenario h*

$$f(x) = -\frac{w}{192}x + \frac{w}{16}$$

$$g(x) = -\frac{w}{112}x + \frac{w}{16}$$

$$h(x) = -\frac{w}{672}x + \frac{9w}{224}$$

$$j(x) = -\frac{w}{192}x + \frac{23w}{448}$$

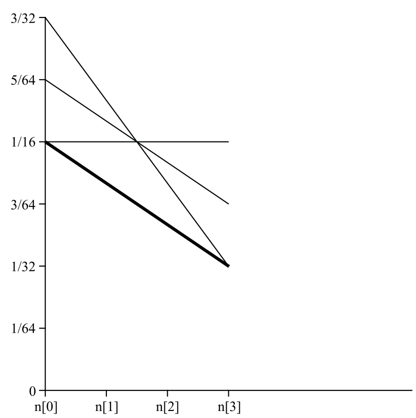



**Figure 18:** *Duration graph of* scenario e.

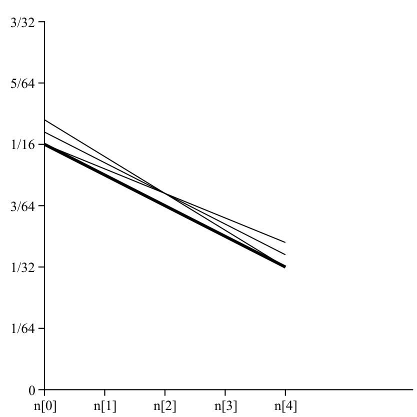



**Figure 19:** *Duration graph of* scenario f.

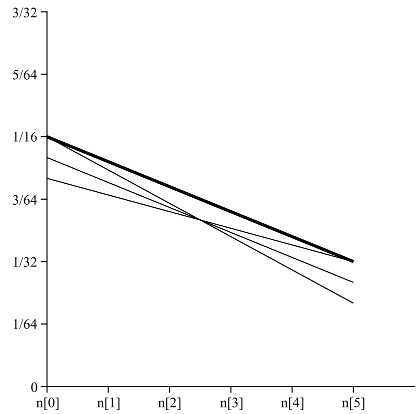



**Figure 20:** *Duration graph of* scenario g

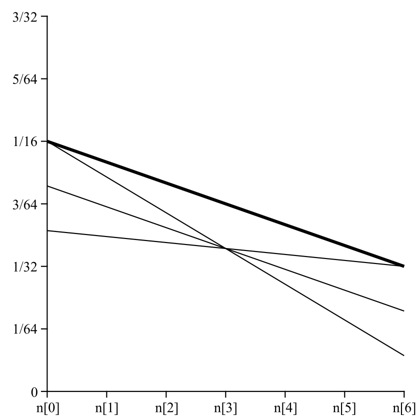



**Figure 21:** *Duration graph of* scenario h.

In *scenarios e* and *f*, the feathered beams are shorter than $\frac{w}{4}$, so $g(x)$, $h(x)$, and $j(x)$ are above $f(x)$. In *scenarios g* and *h*, the feathered beams are longer than $\frac{w}{4}$, so $g(x)$, $h(x)$, and $j(x)$ are below $f(x)$. Of these 4 situations, *scenario f* is the closest in duration to $\frac{w}{4}$, so its notes need to be altered less than the other scenarios to fit within the measure. As the difference between the total duration of the feathered beam and $\frac{w}{4}$ grows, the notes need more alteration to fit within the measure. This can visually be seen in the graphs: $g(x)$, $h(x)$, and $j(x)$ move further from $f(x)$ as the difference increases. It is obvious that a greater discrepancy requires a greater modification. However, the graphs clearly provide specific information about—as well as a general visual representation of—the amount of compression or expansion involved in the performance modification.

## 6. Feathered Beams Generalized

Although the notation is imprecise, the full duration of a feathered beam can give a general idea about its precision. Of the numerous ways they can be interpreted, the general linear case is examined in this section of the paper because it is the simplest and most straight forward case. Although similar results can be duplicated with other representations of feathered beams, the linear case is taken here as mathematically representative.

Given the number of notes in the feathered beam group and the durations of the first and last notes, the equation for the full duration of all linear feathered beams is derived as follows.With the linear equation $f(x) = ax + b$, the full duration of the feathered beam is the sum of all the individual note durations

$$f(n_0) + f(n_1) + \cdots + f(n_i),$$

which expands to

$$(0a + b) + (1a + b) + \cdots + (ia + b),$$

and corresponds to the summation

$$\sum_{x=0}^{i} ax + b = \frac{i(i+1)a}{2} + (i+1)b. \tag{2}$$

The number of notes in the feathered beam is $(i + 1)$. So that they are eighth notes or shorter, the first and last notes are in the form $\frac{1}{2^p}$ for $p > 2$, and they are

$$\begin{aligned} f(n_0) &= 0a + b = b \\ f(n_i) &= ia + b \\ f(n_0) &\neq f(n_i). \end{aligned}$$

Let the last note equal c so that the slope $a$ is

$$a = \frac{c - b}{i}.$$

Substituting $a$ into the summation gives

$$[H]\frac{i(i+1)(c-b)}{2i} + (i+1)b.$$

Given the durations of the first and last notes ($b$ and $c$ respectively) and the number of notes in the feathered beam $(i + 1)$, the equation can be expressed as a function of $b, c$, and $i$:

$$[h]f(b, c, i) = \frac{i(i+1)(c-b)}{2i} + (i+1)b,$$

which gives the full duration of linear feathered beams for all cases. In the above function, substituting values for the first note duration, last note duration, and number of notes in the feathered beam ($b$, $c$, and $i$ respectively) simply gives the full linear duration. In application, for example, the full duration of the "Notturno V" excerpt earlier in this paper is

$$[H]f\left(\frac{w}{32}, \frac{w}{8}, 4\right) = \frac{4(4+1)\left(\frac{w}{8} - \frac{w}{32}\right)}{2 \cdot 4} + (4+1)\frac{w}{32} = \frac{25w}{64}$$

The reader may choose to derive the full length of the quadratic feathered beam. It is given as follows, assuming a third symmetrical coordinate.

$$[H]f(b,c,i) = -\frac{b(i+1)^3}{3i^2} + \frac{b(i+1)^2}{2i^2} - \frac{b(i+1)}{6i^2} + b(i+1)$$

The citations earlier in this paper suggest that the quadratic representation may be most useful because it most naturally reflects the human ritardando or accelerando. The general case is useful because it calculates the full duration, which can be compared to the allotted space in the score. Furthermore, the Java executable jar that can be accessed in the link `https://musmat.org/wp-content/uploads/2021/12/FeatheredBeamCalculator.zip` also quickly provides the full duration, in addition to providing other valuable information. When the difference between the full duration and the allotted space in the score is relatively small, the rhythm can be executed similarly to what the feathered beam suggests. The greater the difference, the less the notation reflects what can actually be played. Composers may find this information useful when implementing feathered beams.

## 7. Conclusion

As musicians, we are trained to precisely execute traditional triplets. Tuplets with more complicated ratios are usually more difficult to execute. Although the one in the measure below may be harder to negotiate than a traditional triplet, its rhythm exactly corresponds to the notation.


In tuplets, the usual circumstance is to compress all the notes according to exact ratios rather than augmenting them. To make the execution of feathered beams possible, there seems to be no preference for compressing the notes over augmenting them, and that individual circumstances dictate the choice. Bruno Repp found that in expressive microstructure variability in traditional rhythms, "lengthening is a common strategy, but accelerations beyond a maximal local tempo serve no expressive purpose and suggest poor timing control,"[12, p. 288] which implies that whether a feathered beam is compressed or augmented may have profound implications on how it is executed and perceived. The *scenarios a* through *h* earlier in this paper show a few ways the notes within feathered beams can be compressed or augmented to fit them within a prescribed duration of music. Furthermore, the examples show that some methods will not work in all circumstances, such as $g(x)$ in *scenario e* where there is no change in duration at all and $h(x)$ in the "Notturno V" exaLmple where there is a negative duration. Feathered beams in nearly all cases require some modification. The circumstances vary, and there is usually no single way in which they can be executed.The possibilities are bountiful, which is perhaps why Kurt Stone writes, "Besides, the gradual increase or decrease in the number of beams makes exact indications of beat units impossible."[20, p. 124] Even though the notation gives precise durations for the first and last notes in feathered beams, their execution is imprecise.

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