Gestural Presheaves: From Yoneda to Sheaves

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DOI: 10.46926/musmat.2021v5n2.103-120

Abstract: *We study gestural presheaves in the language of abstract gestures and, in particular, a general gestural version of the Yoneda embedding, based on a Mazzola's idea, and gestural sheaves. We apply the latter to Mozart and Beethoven.*

Keywords*: Gestures. Melodic contour. Category theory. Yoneda lemma. Sheaves.*

I. Introduction

Mathematical gestures are present in mathematical music theory in several forms. First, they intend to model the configuration and movement of the performer's body when playing an instrument (topological gestures). Second, melodic contours are also topological gestures on the score. Third, all diagrams of transformational theory, as any diagram, are abstract gestures in categories. Introductions to gesture theory are [5, 3, 15]. More general advances include Mazzola's [16], Mannone's [13, 12], and Arias' [2].

This article explores the gestural presheaf construction (Section II.i) in the general language of abstract gestures (Sections II and III), especially the relation thereof to two main developments of presheaves in category theory: the Yoneda embedding and sheaves.

Regarding the first, which represents any category as a subcategory of its associated category of presheaves, it is also relevant for gesture theory since we would like to have a *gestural Yoneda embedding* [14, p. 33] that now represents the given category in a suitable category related to

*This work was supported by Programa de Becas Posdoctorales en la UNAM 2019, which is coordinated by Dirección de Asuntos del Personal Académico (DGAPA) at Universidad Nacional Autónoma de México.

Received: July 20th, 2021
Approved: November 30th, 2021

gestures and enables the recovery of a certain gestural intuition behind a morphism. For example, this embedding should help recover the gesture of a linear transformation associated with a rotation matrix, which only takes into account an argument and its image but not the intuitive movement between them. Similarly, following Lewin [9, p. 159], it should help to find the gestures behind musical transformations. We locate conceptually and generalize (Section V) Mazzola's gestural version of Yoneda lemma, which was originally formulated in the category of topological categories, to more arbitrary categories. In particular, we refine these results to obtain *an embedding of a category into a category of gestural presheaves, under certain conditions*. A previous version of the gestural Yoneda embedding was given in [14, 3.4], however it was not full.

With respect to sheaves, in this article we motivate from a musical perspective their introduction in gesture theory. The motivation is essentially based on the melodic contour gestures and the fact that in some musical works these gestures glue together to form new ones. These procedures could explain some passages obtained by variation in Mozart and Beethoven (Section VIII). The gestural sheaf notion (especially the global section one) is related to that of global gesture in [16, Section 66.5], which was formulated in the category of topological categories and resembles the global composition concept. However, sheaves are simpler and still useful. We provide the *gestural sheaf* notion for more general categories in Section VII.

By the way, the following results emerged in the present research. 1. Determination of left adjoints to gestural presheaves (Section II.i), which helps to characterize the (generalized) elements of objects of gestures. 2. Determination of left adjoints to the covariant gesture functors (Section VI, based on Section IV). 3. Characterization of sheaves in terms of cotensor products, suggesting a close relation between gestures and sheaves (Section VII).

We include a Glossary of specialized terms that the reader can access by clicking on a red term like sheaf. Also, these hyperlinks direct to some term in boldface if it was defined in the article. However, it is desirable that the reader has a minimum acquaintance with category theory [11, pp. 10-23]. On the other hand, we end each example with the symbol ♣ for organization.

II. Abstract gestures

Throughout this paper all examples are focused on topological gestures whose skeleta are digraphs, for simplicity. However, we will use the general language of *abstract gestures*, which includes topological gestures as a particular case, because it include other instances[1] of gesture theory and allows the relation with several valuable categorical concepts, like cotensor products, presheaves, and sheaves.

Consider the following data:

- A small category $\mathcal{D}$. In gesture theory, we often explore the case when $\mathcal{D}$ is the category pictured in the following diagram.

$$id \circlearrowright [0] \underset{\epsilon_0}{\overset{\epsilon_1}{\rightrightarrows}} [1] \circlearrowleft id \; .$$

  We use this instance for all examples, although *all theoretical results in this paper are valid for an arbitrary $\mathcal{D}$*.
- A category $\mathcal{E}$ with small hom-sets and all small limits. Usually, we work with the case when $\mathcal{E}$ is a category of (generalized) spaces like[2] **Top** (topological spaces), **Loc** (locales), **Cat**(**Top**) (topological categories), and **Cat** (small categories).

[1] See [5, 4] for discussions on these variations.
[2] A discussion of gestures in these categories, for the case of digraphs, is [1].

- A functor[3] $S : \mathcal{D}^{op} \longrightarrow \mathcal{E}$.

Let $\widehat{\mathcal{D}}$ be the category of presheaves on $\mathcal{D}$. Given a presheaf P of $\widehat{\mathcal{D}}$, **the object of P-gestures with respect to S** is the **cotensor product** $P \pitchfork S$, defined as the limit of the composite functor

$$\left(\int P\right)^{op} \xrightarrow{\pi^{op}} \mathcal{D}^{op} \xrightarrow{S} \mathcal{E}, \tag{1}$$

where $\int P$ denotes the category of elements of P.

Example II.1 (Topological gestures for digraphs)**.** Let us suppose that $\mathcal{E} = \mathbf{Top}$. A functor $S : \mathcal{D}^{op} \longrightarrow \mathbf{Top}$ can be identified with a cuadruple (S_1, S_0, d_0, d_1), where S_1 and S_0 are topological spaces and d_0 and d_1 are continuous maps, by defining $S_1 = S([1])$, $S_0 = S([0])$, $d_0 = S(\epsilon_0)$, and $d_1 = S(\epsilon_1)$. We say that (S_1, S_0, d_0, d_1) is a **topological digraph**. Similarly, a presheaf Γ on $\mathcal{D}$ is just a *digraph* (A, V, t, h).

The cotensor product $\Gamma \pitchfork S$ is the limit in **Top** of the diagram consisting of for each $a \in A$ a copy of S_1, for each $z \in V$ a copy of S_0, a copy of d_0 whenever $z = t(a)$, and a copy of d_1 whenever $z = h(a)$. We compute this limit explicitly as the subspace of the product (with the Tychonoff topology)

$$\left(\prod_{a\in A} S_1\right) \times \left(\prod_{z\in V} S_0\right)$$

of all sequences $((c_a)_{a\in A}, (x_z)_{z\in V})$ satisfying $d_0(c_a) = x_{t(a)}$ and $d_1(c_a) = x_{h(a)}$ for each $a \in A$. We call these sequences *gestures*. We discuss the particular case of Mazzola's gestures and more examples in Section III. ♣

II.i. The cotensor adjunction and gestural presheaves

By dualizing[4] [11, Theorem I.5.2], the cotensor construction gives rise to an adjunction (left adjoint on the left)

$$\mathcal{E}(-, S) : \mathcal{E} \rightleftarrows \left(\widehat{\mathcal{D}}\right)^{op} : - \pitchfork S\ ,$$

with associated bijection

$$\mathcal{E}(E, P \pitchfork S) \cong \widehat{\mathcal{D}}(P, \mathcal{E}(E, S(-))), \tag{2}$$

natural in P and E. This bijection sends a natural transformation on the right-hand term, which can be identified with a cone on the functor in Equation (1) with vertex E, to the morphism $E \longrightarrow P \pitchfork S$ given by the universal property of the limit $P \pitchfork S$ of this functor.

We call a functor of the form $- \pitchfork S : \left(\widehat{\mathcal{D}}\right)^{op} \longrightarrow \mathcal{E}$ **gestural presheaf**.

Example II.2. Let us compute the action of a gestural presheaf on a morphism in the case of Example II.1. Consider two digraphs (presheaves) Γ and Γ' with $\Gamma = (A, V, t, h)$ and $\Gamma' = (A', V', t', h')$. A morphism of presheaves $\tau : \Gamma \longrightarrow \Gamma'$ can be identified with the *digraph morphism* (u, v), where $u = \tau_{[1]}$ and $v = \tau_{[0]}$. In particular, note that the category $\widehat{\mathcal{D}}$ is just the category of digraphs. Following the explicit presentation of the space of gestures in Example II.1, the continuous map $\tau \pitchfork S : \Gamma' \pitchfork S \longrightarrow \Gamma \pitchfork S$ sends the sequence $((c_a)_{a\in A'}, (x_z)_{z\in V'})$ to $((c_{u(a)})_{a\in A}, (x_{v(z)})_{z\in V})$. ♣

[3] We can also call it *presheaf on $\mathcal{D}$ with values in $\mathcal{E}$*.

[4] See [2, Section 3.4.1] and [2, Section 3.4.4] for more details.

Example II.3 (Individual gestures)**.** The adjunction is useful to compute the elements of the object of gestures. Suppose we are in the situation of Example II.1. If we take E to be the point space $\{*\}$ and P as a digraph Γ, then the bijection in Equation (2) just says that there is a correspondence between the set $\Gamma \pitchfork S$ and the set of digraph morphisms from Γ to S regarded as a digraph (forget the topological structure). By definition, such a morphism (c, x) sends $a \in A$ to $c_a \in S_1$ and $z \in V$ to $x_z \in S_0$, and satisfies $d_0(c_a) = x_{t(a)}$ and $d_1(c_a) = x_{h(a)}$. This means that it is just a gesture sequence as in Example II.1. ♣

II.ii. Kan extension characterization

Let $\mathbf{y} : \mathcal{D} \longrightarrow \widehat{\mathcal{D}}$ be the Yoneda embedding. The restriction functor between functor categories

$$- \circ \mathbf{y}^{op} : \mathcal{E}^{(\widehat{\mathcal{D}})^{op}} \longrightarrow \mathcal{E}^{\mathcal{D}^{op}}$$

has a right adjoint R, with $R(S) = - \pitchfork S$, since the limit in Equation (1) exists for each S and P [10, Corollary X.3.2]. This means, by definition, that *$- \pitchfork S$ is the right Kan extension of S along $\mathbf{y}^{op}$*. Consequently, we have a bijection

$$\mathcal{E}^{(\widehat{\mathcal{D}})^{op}}(F, - \pitchfork S) \cong \mathcal{E}^{\mathcal{D}^{op}}(F \circ \mathbf{y}^{op}, S) \tag{3}$$

natural in F and S. The functor $\mathbf{y}^{op}_{(-)} \pitchfork S$ can be assumed to be S, so the counit component $\epsilon : \mathbf{y}^{op}_{(-)} \pitchfork S \longrightarrow S$ is the identity [10, Corollary X.3.3]. In this way, the bijection (3) sends a natural transformation $\sigma : F \longrightarrow - \pitchfork S$ to $\sigma_{\mathbf{y}^{op}} : F\mathbf{y}^{op} \longrightarrow \mathbf{y}^{op}_{(-)} \pitchfork S = S$. Conversely, given $\alpha : F \circ \mathbf{y}^{op} \longrightarrow S$, for each presheaf P in $\widehat{\mathcal{D}}$ it induces a cone[5] $\{\alpha_D F(p) : F(P) \longrightarrow S(D) \mid (D, p) \in \int P\}$ on the functor in Equation (1) and hence the component $\sigma_P : F(P) \longrightarrow P \pitchfork S$ of a natural transformation $\sigma : F \longrightarrow - \pitchfork S$.

We show in Example V.1 that Equation (3) generalizes Mazzola's gestural Yoneda lemma [16, Theorem 39, p. 962]. Before, we need to define Mazzola's gestures on topological categories.

III. Generalized Mazzola's gestures

III.i. The functor of an object

Given an object C of $\mathcal{E}$ and a functor $T : \mathcal{D} \longrightarrow \mathcal{E}$ with all its images exponentiable in $\mathcal{E}$, we can construct the functor

$$C^T : \mathcal{D}^{op} \longrightarrow \mathcal{E} : D \xrightarrow{g} D' \mapsto C^{T(D')} \xrightarrow{C^{T(g)}} C^{T(D)},$$

which we denote by S_C. This construction generalizes Mazzola's topological digraph of a topological space and the categorical digraph of a topological category, as shown in the following examples.

Example III.1. Take $\mathcal{E} = \mathbf{Top}$. Consider the continuous endpoint inclusions $i_0, i_1 : \{*\} \longrightarrow I$ of the unit interval I in $\mathbb{R}$. They correspond to the functor $T : \mathcal{D} \longrightarrow \mathbf{Top}$ defined by $T([0]) = \{*\}$, $T([1]) = I$, $T(\epsilon_0) = i_0$, and $T(\epsilon_1) = i_1$. The images of T are (locally) compact Hausdorff and hence exponentiable in **Top**. Given a topological space X, $S_X : \mathcal{D}^{op} \longrightarrow \mathbf{Top}$ corresponds to the *topological digraph* (X^I, X, e_0, e_1) *of* X, where the exponential X^I is the *function space* $\mathbf{Top}(I, X)$ of continuous paths in X, equipped with the compact-open topology [7, p. 558], X is isomorphic to the exponential $X^{\{*\}}$, and e_0 and e_1 are the continuous evaluations at 0 and 1. ♣

[5] Here we identify $p \in P(D)$ with its natural transformation $p : \mathbf{y}_D = \mathcal{D}(-, D) \longrightarrow P$ given by the Yoneda lemma.

Example III.2. Take $\mathcal{E}$ as the category of topological categories **Cat**(**Top**). There are topological functors $\mathbf{i}, \mathbf{j} : \mathbf{1} \longrightarrow \mathbb{I}$ from the final category to the topological category $\mathbb{I}$ of I, defined by $\mathbf{i}_0(*) = 0$, $\mathbf{j}_0(*) = 1$, $\mathbf{i}_1(*) = (0,0)$, and $\mathbf{j}_1(*) = (1,1)$. Once again, they amount to the functor $T : \mathcal{D} \longrightarrow \mathbf{Top}$ defined by $T([0]) = \mathbf{1}$, $T([1]) = \mathbb{I}$, $T(\epsilon_0) = \mathbf{i}$, and $T(\epsilon_1) = \mathbf{j}$. In this case, $\mathbb{I}$ is exponentiable in **Cat**(**Top**) because its spaces of objects I, morphisms ∇, and composable morphisms are locally compact Hausdorff, that is, exponentiable in **Top**; see [2, Theorem 5.3.2].

If $\mathbb{K}$ is a topological category with spaces of objects and morphisms C_1 and C_0 respectively, then $S_{\mathbb{K}} : \mathcal{D}^{op} \longrightarrow \mathbf{Cat}(\mathbf{Top})$ corresponds to the *categorical digraph* $(\mathbb{K}^{\mathbb{I}}, \mathbb{K}, \mathbf{e_0}, \mathbf{e_1})$ *of* $\mathbb{K}$, where $\mathbb{K}^{\mathbb{I}}$ is the category of all topological functors from $\mathbb{I}$ to $\mathbb{K}$ with its set of objects P_0 (that is, of topological functors) topologized as a subspace of $C_1^{\nabla} \times C_0^{I}$ and its set of morphisms P_1 (that is, of natural transformations) topologized as a subspace of $P_0 \times P_0 \times C_1^{I}$, and

$$\mathbf{e}_i : \mathbb{K}^{\mathbb{I}} \longrightarrow \mathbb{K} : F \xrightarrow{\tau} G \mapsto F(i) \xrightarrow{\tau_i} G(i)$$

for $i = 0, 1$. ♣

Example III.3 (General addresses)**.** Take $\mathcal{E}$ as the category of contravariant functors from **Cat**(**Top**) to itself. Given a functor $S : \mathcal{D}^{op} \longrightarrow \mathbf{Cat}(\mathbf{Top})$ and a topological category $\mathbb{A}$, called *address*, we have a functor $\mathbf{Cat}(\mathbf{Top})(\mathbb{A}, S(-)) : \mathcal{D}^{op} \longrightarrow \mathbf{Cat}(\mathbf{Top})$. In fact, each image, which is of the form $\mathbf{Cat}(\mathbf{Top})(\mathbb{A}, \mathbb{D})$, can be enriched with a topological category structure similar to that of the exponential $\mathbb{K}^{\mathbb{I}}$ in Example III.2, although it need not be an exponential in **Cat**(**Top**). By ranging $\mathbb{A}$ over all topological categories we obtain a functor

$$\mathbf{Cat}(\mathbf{Top})(-, S(-)) : \mathbf{Cat}(\mathbf{Top})^{op} \times \mathcal{D}^{op} \longrightarrow \mathbf{Cat}(\mathbf{Top})$$

or equivalently $\mathbf{Cat}(\mathbf{Top})(-, S(-)) : \mathcal{D}^{op} \longrightarrow \mathcal{E}$. In the case when S is a categorical digraph $S_{\mathbb{K}}$ we denote this functor by $@S_{\mathbb{K}}$ and call $\mathbb{A}@S_{\mathbb{K}}$ *the* $\mathbb{A}$*-addressed categorical digraph of* $\mathbb{K}$, according to [16, p. 961]. ♣

III.ii. Gestures

Under the hypotheses of Section III.i, given an object C of $\mathcal{E}$ and a presheaf P on $\mathcal{D}$, we define the *object of* P*-gestures with body in* C as the cotensor product $P \pitchfork S_C$ in Section II. Though this limit can be hard to compute, Equation (2) gives us a simple characterization of its generalized elements, and in particular its points. In fact, for each object E of $\mathcal{E}$ we have the natural bijections (the second one given by the exponential adjunction)

$$\mathcal{E}(E, P \pitchfork S_C) \cong \widehat{\mathcal{D}}(P, \mathcal{E}(E, C^{T(-)})) \cong \widehat{\mathcal{D}}(P, \mathcal{E}(E \times T(-), C)),$$

which mean that E-addressed elements of $P \pitchfork S_C$ correspond to natural transformations from P to $\mathcal{E}(E \times T(-), C)$. In particular, if E is the final object of $\mathcal{E}$, then we obtain that the *points* of $P \pitchfork S_C$ correspond to natural transformations from P to $\mathcal{E}(T(-), C)$, the latter being the *underlying presheaf of* S_C. This leads us to define *a* P*-gesture with body in* C as a natural transformation $P \longrightarrow \mathcal{E}(T(-), C)$.

Example III.4. Let X be a topological space, T as in Example III.1, and Γ a digraph (A, V, t, h). The topological space of Γ-gestures with body in X coincides with the original definition $\Gamma@X$ in [15]. In fact, the limit defining the cotensor $\Gamma \pitchfork S_X$ in Example II.1 for the case of the topological digraph S_X, which is identified with the tuple (X^I, X, e_0, e_1) in Example III.1, is just that in [15, p. 31].

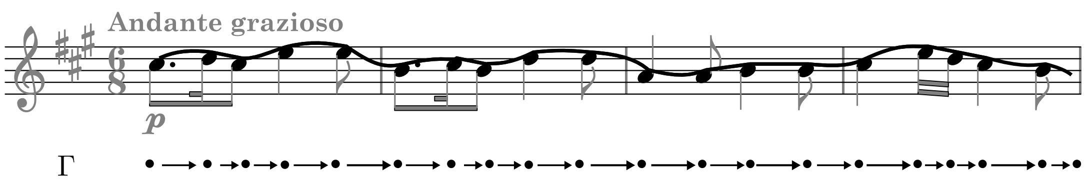

Figure 1: *The first phrase in Mozart's K. 331 as a gesture.*

On the other hand, the definition of an individual gesture as a natural transformation from Γ to $\mathbf{Top}(T(-), X)$ coincides with a digraph morphism from Γ to the underlying digraph of (X^I, X, e_0, e_1) in Example II.3 and [15, p. 29], and a sequence $((c_a)_{a\in A}, (x_z)_{z\in V})$ satisfying $c_a(0) = x_{t(a)}$ and $c_a(1) = x_{h(a)}$, as in Example II.1, where $c_a : I \longrightarrow X$ is a continuous map and $x_z \in X$. ♣

Example III.5 (The score space)**.** We can define gestures on musically meaningful spaces. Let us interpret the Euclidean space $\mathbb{R}^2$ as the *score space*, where a pair (t, p) denotes a sound event with pitch p that occurs at the time t. We choose the unities according to the situation. Here we use the quarter duration as time unity and identify the subset $\mathbb{Z}$ of $\mathbb{R}$ with the diatonic scale indicated by the key involved. For instance, the pair $(1/2, 0)$ denotes the pitch A4 occurring after an eight duration.

Consider the first phrase of Mozart's Piano Sonata[6] K. 331. The melodic contour can be regarded as a gesture in $\mathbb{R}^2$, see Figure 1. As we explain in Section 3, the melodic contour plays an important role in Mozart's variations of this phrase since they can be regarded as the result of transforming the original gesture with homotopies and a sheaf. ♣

Example III.6. Akin to Example III.4, the topological category of Γ-gestures with body in a topological category $\mathbb{K}$, based on the categorical digraph from Example III.2, coincides with $\Gamma @ \mathbb{K}$, as defined in [14, Section 2.2].

In this case, an individual gesture is a digraph morphism from Γ to the underlying digraph of $(\mathbb{K}^{\mathbb{I}}, \mathbb{K}, \mathbf{e_0}, \mathbf{e_1})$, which only takes into account the objects[7] of $\mathbb{K}^{\mathbb{I}}$ (functors) and $\mathbb{K}$. They can be identified with sequences

$$((F_a)_{a\in A}, (C_z)_{z\in V})$$

satisfying $F_a(0) = C_{t(a)}$ and $F_a(1) = C_{h(a)}$, where $F_a : \mathbb{I} \longrightarrow \mathbb{K}$ is a topological functor and C_z is an object of $\mathbb{K}$. ♣

Example III.7. However, there are more abstract gestural notions for topological spaces with musical meaning, other than Mazzola's one. Consider the functor $S : \mathcal{D}^{op} \longrightarrow \mathbf{Top}$ corresponding to the *topological digraph of intervals of the score space* $\mathbb{R}^2$. The space of vertices is the score space $\mathbb{R}^2$, see Example III.5. The space of arrows is the subspace of $\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$ consisting of triples of the form $(x, y, y - x)$, where $y - x$ is the vector difference, that is, *the interval between x and y*. The tail and head are just the first and second product projections. In this case, a gesture is a sequence

$$\left((x_{t(a)}, x_{h(a)}, x_{h(a)} - x_{t(a)})_{a\in A}, (x_z)_{z\in V}\right),$$

where x_z is a sound event. A gesture of intervals between the sound events of the first phrase in Mozart's K. 331 corresponds to Figure 2. ♣

[6]The authors prepared the excerpts from [17] (theme) and the original manuscript (variation 4) at https://mozart.oszk.hu/.

[7]Note that, in this case, points are functors with domain the final category **1** and correspond to objects of the codomains.

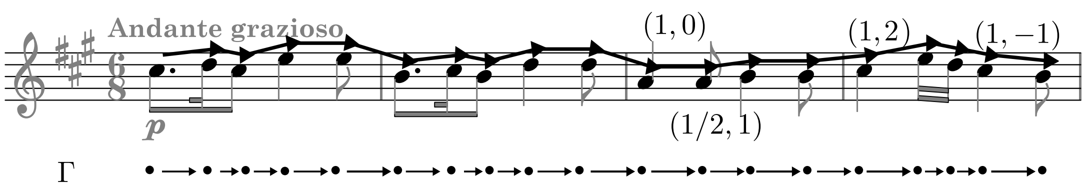

Figure 2: *The first phrase in Mozart's K. 331 as a gesture of intervals. We only consider the third component (interval) of some arrows for simplicity.*

IV. The functor of functors of an object and its left adjoint

The construction in Section III.i induces a functor

$$S_{(-)} : \mathcal{E} \longrightarrow \mathcal{E}^{\mathcal{D}^{op}} : C \xrightarrow{f} C' \mapsto \left(f^{T(D)} : C^{T(D)} \longrightarrow C'^{T(D)} \right)_{D \in \mathcal{D}}. \tag{4}$$

We claim that this functor has a left adjoint. To prove this, first consider a natural transformation $\tau : S \longrightarrow S_C$, where $S, S_C : \mathcal{D}^{op} \longrightarrow \mathcal{E}$ are functors. This natural transformation τ is a family of morphisms $\left(\tau_D : S(D) \longrightarrow C^{T(D)} \right)_{D \in \mathcal{D}}$ making the following diagram commute for each morphism $g : D \longrightarrow D'$ of $\mathcal{D}$.

$$\begin{array}{ccc} S(D) & \xrightarrow{\tau_D} & C^{T(D)} \\ {\scriptstyle S(g)} \uparrow & & \uparrow {\scriptstyle C^{T(g)}} \\ S(D') & \xrightarrow{\tau_{D'}} & C^{T(D')} \end{array}$$

By transposing both composites across the exponential adjunction, the previous commutative diagram is equivalent to the following one, where $\widetilde{\tau_D}$ denotes the transpose of τ_D.

$$\begin{array}{ccc} S(D) \times T(D) & \xrightarrow{\widetilde{\tau_D}} & C \\ {\scriptstyle S(g) \times T(D)} \uparrow & & \uparrow {\scriptstyle \widetilde{\tau_{D'}}} \\ S(D') \times T(D) & \xrightarrow{S(D') \times T(g)} & S(D') \times T(D') \end{array}$$

But this is just a *wedge* from $S \times T$ to C. Thus, we have a bijection

$$\mathcal{E}^{\mathcal{D}^{op}}(S, S_C) \cong \mathrm{Wedge}(S \times T, C). \tag{5}$$

In the case when $\mathcal{E}$ is small-cocomplete all coends exist [10, p. 224] and the definition of coend establishes a bijection

$$\mathrm{Wedge}(S \times T, C) \cong \mathcal{E}\left(\int^{D} S(D) \times T(D), C \right). \tag{6}$$

To sum up, the bijections in Equations (5) and (6) amount to

$$\mathcal{E}^{\mathcal{D}^{op}}(S, S_C) \cong \mathcal{E}\left(\int^{D} S(D) \times T(D), C \right),$$

which can be shown to be natural in C and hence it determines an adjunction [10, Corollary IV.1.2]. We thus have the following theorem.

Theorem IV.1. *If $\mathcal{E}$ is small-cocomplete, then $S_{(-)}$ has as left adjoint the functor*

$$\int^{D} (-)(D) \times T(D) : \mathcal{E}^{\mathcal{D}^{op}} \longrightarrow \mathcal{E},$$

sending S to the coend $\int^{D} S(D) \times T(D)$.

We apply this theorem to gestures on the score space in Example VI.1.

V. A gestural embedding

The Kan adjunction $- \circ \mathbf{y}^{op} \dashv R$ from Section II.ii restricts to an equivalence between $\mathcal{E}^{\mathcal{D}^{op}}$ and the full subcategory of $\mathcal{E}^{\widehat{\mathcal{D}}^{op}}$ of all functors $F : \left(\widehat{\mathcal{D}}\right)^{op} \longrightarrow \mathcal{E}$ that are naturally isomorphic to $- \pitchfork S$ for some S. This follows from [11, II.6.4], since the counit $\epsilon : R(-) \circ \mathbf{y}^{op} \longrightarrow id$ is the identity (Section II.ii), and for each S, $\eta_{-\pitchfork S} : - \pitchfork S \longrightarrow - \pitchfork (\mathbf{y}^{op}_{(-)} \pitchfork S)$ is the identity (check).

In particular, *the functor R is an embedding*. This means that for each pair $S, S' : \mathcal{E}^{\mathcal{D}^{op}} \longrightarrow \mathcal{E}$, R restricts to a bijection

$$\mathcal{E}^{\mathcal{D}^{op}}(S', S) \cong \mathcal{E}^{\widehat{\mathcal{D}}^{op}}(- \pitchfork S', - \pitchfork S). \tag{7}$$

Example V.1 (Gestural Yoneda for topological categories)**.** Let $\mathcal{E}$, $\mathbb{K}$, and $@S_{\mathbb{K}}$ be as in Example III.3, where $S_{\mathbb{K}}$ is as in Example III.2. Let us denote by $@\mathbb{K}$ the gestural presheaf $- \pitchfork @S_{\mathbb{K}}$. By taking F as a functor naturally isomorphic to $- \pitchfork S$ in Equation (7), we obtain

$$\mathcal{E}^{\widehat{\mathcal{D}}^{op}}(@\mathbb{K}, F) \cong \mathcal{E}^{\widehat{\mathcal{D}}^{op}}(@\mathbb{K}, - \pitchfork S) \cong \mathcal{E}^{\mathcal{D}^{op}}(@S_{\mathbb{K}}, S).$$

Following Mazzola's terminology, $@S_{\mathbb{K}} = \overrightarrow{\mathbb{K}}$, F is a *limiting functor*, and $S = \overrightarrow{F}$. This is just the Yoneda lemma for gestures on topological categories [16, Theorem 39, p. 962]. ♣

Example V.2. Suppose that $\mathcal{E} = \mathbf{Top}$. Given a natural transformation $\tau : S' \longrightarrow S$ of topological digraphs, let us compute the associated natural transformation $R(\tau) : - \pitchfork S' \longrightarrow - \pitchfork S$. If Γ is a digraph (A, V, t, h), then the continuous map $R(\tau)_{\Gamma} : \Gamma \pitchfork S' \longrightarrow \Gamma \pitchfork S$ sends a gesture sequence (Example II.1)

$$((c_a)_{a \in A}, (x_z)_{z \in V})$$

to

$$((\tau_{[1]}(c_a))_{a \in A}, (\tau_{[0]}(x_z))_{z \in V}).$$

♣

Example V.3 (Interplay between intervals (Lewin) and gestures (Mazzola) in the score)**.** Let us assume the situation of Example V.2. Take S' as the topological digraph of intervals of the score (Example III.7) and S as the topological digraph of $\mathbb{R}^2$ (Examples III.4 and III.5).

First, we define a natural transformation τ from S' to S. The correspondence on arrows $S'([1]) \longrightarrow (\mathbb{R}^2)^I$ is the exponential transpose of the continuous map

$$S'([1]) \times I \longrightarrow \mathbb{R}^2 : ((x, y, y - x), s) \mapsto x + s(y - x)$$

and the correspondence on vertices is the identity. Hence the correspondence on arrows sends $(x, y, y - x)$ to the linear path with parametrization $\alpha(s) = x + s(y - x)$. This defines a natural transformation since $\alpha(0) = x = \pi_1(x, y, y - x)$ and $\alpha(1) = y = \pi_2(x, y, y - x)$. This ensures the existence (Example V.2) of a natural transformation $R(\tau) : - \pitchfork S' \longrightarrow - \pitchfork S$ that *incarnates* every gesture of intervals as a gesture of line segments.

Conversely, we can define a natural transformation μ from S to S'. The correspondence on arrows sends a path c in $(\mathbb{R}^2)^I$ to $(c(0), c(1), c(1) - c(0))$. This is just the pointwise definition of the continuous map $(e_0, e_1, e_1 - e_0) : (\mathbb{R}^2)^I \longrightarrow S'([1])$, where e_0 and e_1 are the continuous evaluation maps (Example III.1). The correspondence on vertices is the identity again. We thus have $R(\mu)$, which transforms gestures into abstract intervals.

Note that $\mu\tau = id$ and hence $R(\mu)R(\tau) = id$, that is, $\Gamma \pitchfork S'$ is a *retract* of $\Gamma \pitchfork \mathbb{R}^2$ for each digraph Γ. ♣

Now, assume the hypotheses of Section III.i. By composing R with the functor $S_{(-)}$ from Equation (4), we obtain the functor

$$\mathcal{E} \xrightarrow{S_{(-)}} \mathcal{E}^{\mathcal{D}^{op}} \xrightarrow{R} \mathcal{E}^{\widehat{\mathcal{D}}^{op}}.$$

In the case when $T([0])$ is the final object,[8] $S_{(-)}$ is faithful because if $S_f = S_g$, where $f, g : C \longrightarrow C'$ are continuous maps, then in particular $f = S_f([0]) = S_g([0]) = g$. However, the functor $S_{(-)}$ need not be full as shown in the following example.

Example V.4. Let us assume the situation of Example III.4 and that $X = \mathbb{R}^2$. The functor $S_{\mathbb{R}^2}$ is just the topological digraph $((\mathbb{R}^2)^I, \mathbb{R}^2, e_0, e_1)$. We have an endomorphism (u, v) of this topological digraph where v is the constant map with value $(0, 0)$ and u is the constant map with value the circular loop with parametrization $(\cos(2\pi t) - 1, \sin(2\pi t))$ for $t \in I$. This endomorphism is not of the form S_f, otherwise $v = f$ and $u = f^I$, that is, u is the constant map with value the constant path on $(0, 0)$; a contradiction. ♣

In the case when $T([0])$ is the final object, we can correct this drawback by just restricting the morphisms of $\mathcal{E}^{\mathcal{D}^{op}}$ to the image of $S_{(-)}$ and then those of $\mathcal{E}^{\widehat{\mathcal{D}}^{op}}$ to the image of $R \circ S_{(-)}$. This implies that *if* $T([0]) = \mathbf{1}$, *then* $R \circ S_{(-)}$ *is an embedding of* $\mathcal{E}$ *into a subcategory of the category of gestural presheaves* $\mathcal{E}^{\widehat{\mathcal{D}}^{op}}$. This result applies to topological and categorical gestures (Examples III.4 and III.6).

VI. The covariant gesture functor and its left adjoint

Let us suppose that $\mathcal{E}$ is small-cocomplete. The *covariant gesture functor* $P \pitchfork -$ is the composite

$$\mathcal{E}^{\mathcal{D}^{op}} \xrightarrow{\mathcal{E}^{\pi_P}} \mathcal{E}^{(\int P)^{op}} \xrightarrow{\text{Lim}} \mathcal{E}.$$

We claim that both functors in this composite have a left adjoint, so the composite of these adjoints is the desired one. Certainly, the diagonal functor from [11, p. 21] is the left adjoint to Lim and, on the other hand, by [10, Corollary X.3.2], $\mathcal{E}^{\pi_P}$ has a left adjoint since $\mathcal{E}$ is cocomplete and both $\mathcal{D}$ and $\int P$ are small categories.

In turn, *Mazzola's gesture functor* $P \pitchfork S_{(-)}$ is the composite

$$\mathcal{E} \xrightarrow{S_{(-)}} \mathcal{E}^{\mathcal{D}^{op}} \xrightarrow{P \pitchfork -} \mathcal{E}$$

and has a (composite) left adjoint since both functors have left adjoints.

In particular, $P \pitchfork S_{(-)}$ preserves limits. Next, we provide an example illustrating the situation.

[8] In a more general language, this means that T preserves the final object.

Example VI.1. Let us assume the situation of Example III.4. By preservation of the product $\mathbb{R} \times \mathbb{R}$, the topological space of gestures $\Gamma \pitchfork S_{\mathbb{R}^2}$ with skeleton Γ and body in $\mathbb{R}^2$ is homeomorphic to the product $(\Gamma \pitchfork S_{\mathbb{R}}) \times (\Gamma \pitchfork S_{\mathbb{R}})$. In Mazzola's notation: $\Gamma@\mathbb{R}^2 \cong (\Gamma@\mathbb{R}) \times (\Gamma@\mathbb{R})$.

In terms of the interpretation of $\mathbb{R}^2$ as the score space (Example III.5), the homeomorphism corresponds to the decomposition of gestures on the score as pairs of gestures where the first component is a time gesture and the second one a pitch gesture. ♣

VII. Gestural sheaves

Now we discuss the relation between gestures and sheaves in Grothendieck's sense. This section intends to show that given a functor $S : \mathcal{D}^{op} \longrightarrow \mathcal{E}$, the gestural presheaf $- \pitchfork S : \left(\widehat{\mathcal{D}}\right)^{op} \longrightarrow \mathcal{E}$ is a sheaf with values in $\mathcal{E}$ with respect to the canonical Grothendieck topology on $\widehat{\mathcal{D}}$.

First, we rewrite the characterization of sheaves in terms of equalizers within our language of cotensor products and gestures. Let $(\mathcal{C}, J)$ be a site. We say that a *presheaf* $F : \mathcal{C}^{op} \longrightarrow \mathcal{E}$ *with values in* $\mathcal{E}$ is a *sheaf* if and only if for each object C of $\mathcal{C}$ and each covering sieve R in $J(C)$ we have the identity

$$R \pitchfork F = F(C),$$

where the *cotensor product* $R \pitchfork F$ is defined as

$$\mathrm{Lim}\left(\left(\int R\right)^{op} \xrightarrow{\pi_R^{op}} \mathcal{C}^{op} \xrightarrow{F} \mathcal{E}\right).$$

By using the presentation of this limit as an equalizer [10, p. 113], the identity $R \pitchfork F = F(D)$ means that the following diagram, with appropriate arrows, is an equalizer, namely *that defining a sheaf with values in* $\mathcal{E}$; compare with [11, p. 122].

$$F(C) \longrightarrow \prod_{(f:D\to C)\in R} F(D) \rightrightarrows \prod_{\substack{(f:D\to C)\in R\\ m:D'\to D}} F(D')$$

Note also that the condition $R \pitchfork F = F(C)$ (for C ranging over $\mathcal{C}$) above just says that, for each object A of $\mathcal{E}$, the presheaf $\mathcal{E}(A, F(-)) : \mathcal{C}^{op} \longrightarrow$ **Set** is a sheaf. In other words, every matching family

$$\{x_f : A \longrightarrow F(D) | \ (f : D \to C) \in R\}$$

for R of generalized elements of F has a unique amalgamation $x : A \longrightarrow F(C)$; compare with [11, p. 121-122].

Second, we are interested in the case when $\mathcal{C}$ is the category of presheaves $\widehat{\mathcal{D}}$, so we need an appropriate Grothendieck topology on it.[9] The topology that we will use is the canonical one. Recall that [11, p. 126] a Grothendieck topology on $\mathcal{C}$ is *subcanonical* if all representable presheaves on $\mathcal{C}$ are sheaves, and that [6, II.2.5] the *canonical topology* on a category $\mathcal{C}$ is the greatest subcanonical topology. We can rewrite this definition by using the category of elements as follows. A topology J on $\mathcal{C}$ is subcanonical if and only if for each object C of $\mathcal{C}$ and each covering sieve R in $J(C)$ the identity

$$\mathrm{Colim}\left(\int R \xrightarrow{\pi_R} \mathcal{C}\right) = C$$

[9] There is a subtlety here: the category $\widehat{\mathcal{D}}$ is not small, so the definition of topology and site given in [11, p. 110] cannot be used. But this is easily corrected by *changing the universe*, as suggested in [6].

holds. In fact, this equality just says that, for each object C of $\mathcal{C}$, every matching family for R of elements of $\mathcal{C}(-, C)$ has a unique amalgamation. In the case when $\mathcal{C}$ is a category of presheaves, the canonical topology on $\mathcal{C}$ is that having as covering sieves all epimorphic ones. To prove this, note that epimorphic sieves define a topology in every category of presheaves (check), that every sieve of the canonical topology is epimorphic, and recall that, conversely, in a category of presheaves every epimorphic sieve is a sieve of the canonical topology by part 2) of [6, Proposition II.4.3]. In this way, in the case when $\mathcal{C} = \widehat{\mathcal{D}}$, the canonical topology is given by all sieves that are epimorphic; concretely, a sieve R on a presheaf P is an epimorphic family if and only if for each $D \in \mathcal{D}$ and each $a \in P(D)$ there is a natural transformation $\tau : P' \longrightarrow P$ in R and an element $x \in P'(D)$ such that $\tau_D(x) = a$.

Now we can prove the main result of this section.

Theorem VII.1. *Each gestural presheaf* $- \pitchfork S : \left(\widehat{\mathcal{D}}\right)^{op} \longrightarrow \mathcal{E}$ *is a sheaf with values in* $\mathcal{E}$ *for any subcanonical topology on* $\widehat{\mathcal{D}}$*, in particular for the canonical one consisting of all epimorphic sieves.*

Proof. Consider a subcanonical topology J on $\widehat{\mathcal{D}}$. According to the characterization of subcanonical topologies above, for each object P of $\widehat{\mathcal{D}}$ and each sieve R in $J(P)$ we have the identity

$$\mathrm{Colim}\left(\int R \xrightarrow{\pi_R} \widehat{\mathcal{D}}\right) = P.$$

Moreover, since the gestural presheaf, as a right adjoint (Section II.i), transforms colimits in $\widehat{\mathcal{D}}$ into limits in $\mathcal{E}$, by applying $- \pitchfork S$ to the identity above, we obtain that

$$\mathrm{Lim}\left(\left(\int R\right)^{op} \xrightarrow{\pi_R^{op}} \left(\widehat{\mathcal{D}}\right)^{op} \xrightarrow{-\pitchfork S} \mathcal{E}\right) = P \pitchfork S.$$

According to the characterization of sheaves above, this means that $- \pitchfork S$ is a sheaf. □

VIII. The sheaf of topological gestures in Mozart and Beethoven

In particular, Theorem VII.1 says that the gestural presheaf

$$- \pitchfork S_X = -@X : \mathcal{D}^{op} \longrightarrow \mathbf{Top}$$

that results from the situation in Example III.4 is a sheaf.

Recall that (Example II.2) given a digraph morphism

$$\tau = (u, v) : \Gamma = (A, V, t, h) \longrightarrow \Gamma' = (A', V', t', h')$$

the sheaf $\tau@X$ sends a gesture sequence $((c_a)_{a\in A'}, (x_z)_{z\in V'})$ to $((c_{u(a)})_{a\in A}, (x_{v(z)})_{z\in V})$. In particular, if τ is a digraph inclusion (both u and v inclusions), then $\tau@X$ sends a Γ'-gesture to its *restriction* to Γ.

Let us consider the canonical topology on digraphs. An important example of epimorphic sieve on a digraph Γ, with $\Gamma = (A, V, t, h)$, is that generated by a *cover* $\{(A_i, V_i, t_i, h_i) \mid i \in \mathcal{I}\}$ of Γ by subdigraphs, which means that the inclusion pair is a digraph morphism from (A_i, V_i, t, h) to Γ for each $i \in \mathcal{I}$, $\bigcup_{i\in\mathcal{I}} A_i = A$, and $\bigcup_{i\in\mathcal{I}} V_i = V$. A coherent family of topological gestures for this sieve consists of a Σ-gesture g_Σ in X for each member Σ of the sieve. The coherence condition can be reduced to saying that g_{Γ_i} and g_{Γ_j} coincide on $(A_i \cap A_j, V_i \cap V_j, t, h)$, whenever $\Gamma_i = (A_i, V_i, t, h)$

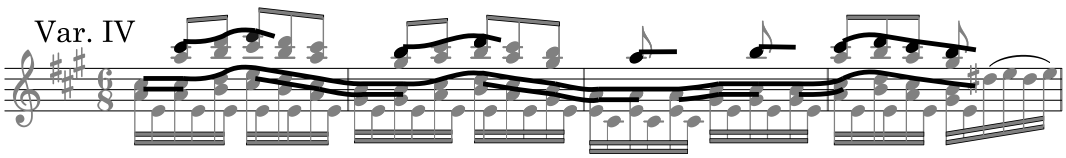

Figure 3: *A global gesture in the fourth variation in Mozart's K. 331 (middle) from local melodic gestures (top and bottom). The harmonic tones are the black dots. We omit the digraphs, which are of the form* $\bullet \to \bullet \cdots \bullet \to \bullet$*, for simplicity.*

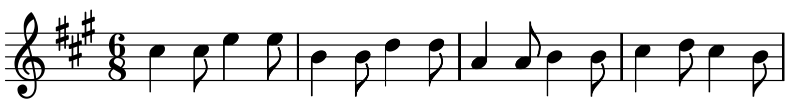

Figure 4: *Harmonic tones in the theme of Mozart's K. 331.*

and $\Gamma_j = (A_j, V_j, t, h)$. The sheaf property ensures that there is a unique *global* Γ-gesture g whose restriction to Γ_i is just the *local* gesture g_{Γ_i}.

In the following two musical examples we consider the case when Γ is a path digraph of the form $\bullet \to \bullet \cdots \bullet \to \bullet$ and each member of the cover is a path subdigraph of the same form. This kind of cover helps to reconstruct melodic contours from smaller fragments. Also, we assume that X is the score space $\mathbb{R}^2$ (Example III.5).

VIII.i. The fourth variation in Mozart's Piano Sonata K. 331

The construction of a global gesture from local ones can be observed in the fourth variation in Mozart's K. 331; see Figure 3. The melody can be reconstructed by following several steps. First, consider the original theme (Example III.5 and Figure 1) and take the harmonic tones as shown in Figure 4. Second, construct melodic fragments in each measure by joining the tones stepwise if necessary, following the contour gestures between harmonic tones, as shown in the top of Figure 3. Finally we coherently paste these fragments (transposed by an octave) by adding new melodic stepwise gestures and following a uniform rhythm (successive eights), as shown in the bottom of Figure 3. We thus obtain a global gesture (middle of Figure 3) that accompanies the melody by *continuation* of the original fragments.

This kind of gestural analysis could be useful to understand underlying topological processes in the theme with variations form, beyond transformational approaches. In this way, we highlight the plasticity of musical thinking, which we access mathematically thanks to the continuity of melodic contours obtained by continuation in the sheaf of gestures in the score. For instance, there is no symmetry (translation or reflection) between the melody in the first bar of the original theme (Figure 1) and the corresponding variation in Figure 3. However, *homotopies*[10] between paths, *preserving the harmonic tones*, seem to provide a valid explanation for the transit from the original fragment to its variation; see Figure 5. The preservation of some musical attribute, such as harmonic tones, seems to be essential for using this kind of explanation since in the score space $\mathbb{R}^2$ any two paths α, β can be transformed into each other by means of the homotopy defined by $(1-t)\alpha + t\beta$ for $t \in [0,1]$.

[10] A homotopy between two paths α, β in X is a path h in $X^{[0,1]}$ such that $h(0) = \alpha$ and $h(1) = \beta$. Similarly we can define homotopies between Γ-gestures in X as paths in the space $\Gamma \pitchfork S_X$.

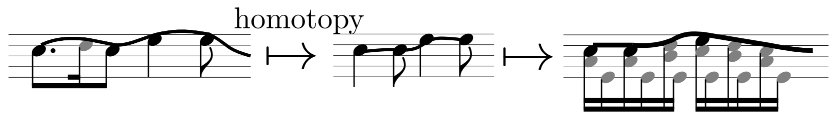

Figure 5: *Transformation of the theme (first bar) in Mozart's K. 331 into the fourth variation (first bar) by means of homotopies between paths that preserve the harmonic tones. The black dots are the harmonic tones.*

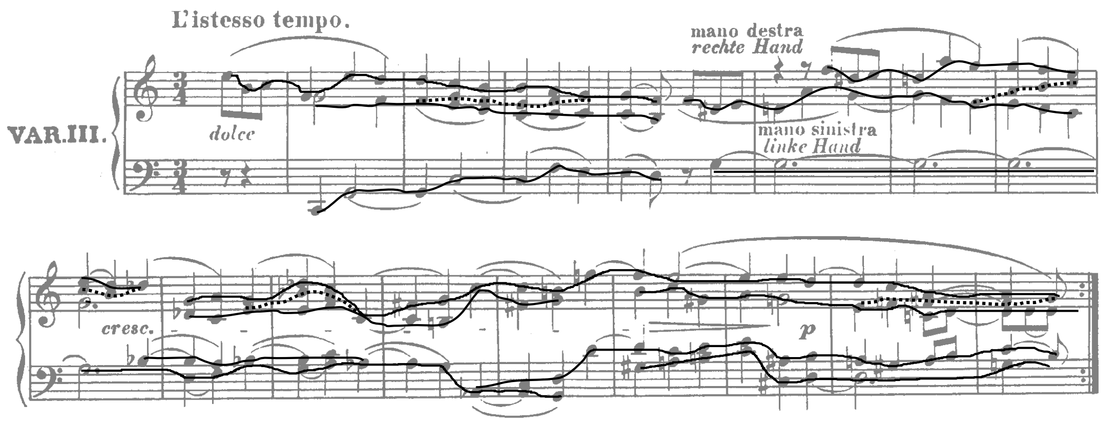

Figure 6: *The third Diabelli variation in Beethoven's Op. 120; taken from [18]. The ramification, fusion, and connection of voices can be explained within the sheaf of topological gestures, contradicting the variety principle in counterpoint. We omit the digraphs again, which are of the form* $\bullet \rightarrow \bullet \cdots \bullet \rightarrow \bullet$*, for simplicity.*

VIII.ii. The third Diabelli variation in Beethoven's Op. 120

The third variation in Beethoven's Op. 120, edition [18], is a very explicit example where the structure of a sheaf seems to be used intuitively. The following discussion refers to Figure 6. We analyze melodic contours by using topological gestures on the score space from Example III.5.

The two upper voices in measures 1-5 are intertwined by means of an intermediate voice, corresponding to the pointed line in measures 3-4. We thus have a global section connecting the two voices, which consists of the contour gesture ranging the middle voice in measure 2, the pointed line in measures 3-4, and the upper voice in measure 5. A similar phenomenon occurs in measures 7-9 and 16-17. In measures 10-11 the alto seems to split into two for a few quarters to fuse again.

The conceptual value of the sheaf involved seems to rely on the fact that it offers an explanation of a phenomenon that does not follow counterpoint rules: the fusion and connection of voices, which directly defies the *variety* principle in counterpoint [8, p. 21].

These contrapuntal ramification and fusion processes are considerably combined with symmetries in this case. The soprano theme in bars 1-5 reappears transposed in the alto in bars 5-9 with some intervallic variations. This imitation can also be regarded as homotopy equivalent to a literal transposition of the soprano theme. Then, in bars 6-9 the soprano forms an octave canon with the melody of the alto, which implies translation symmetries. In this context the voice fusion occurs (bars 7-9).

The topological tools considered (sheaves and homotopies) in the previous examples could be used to compose new music.

Glossary

In what follows $\mathcal{C}$ and $\mathcal{E}$ denote arbitrary categories.

Digraph A *digraph* Γ is a quadruple (A, V, t, h) such that A (*arrows*) and V (*vertices*) are sets and $t, h : A \longrightarrow V$ are functions called *tail* and *head*, respectively. A **digraph morphism** from (A, v, t, h) to (A', V', t', h') is a pair (u, v) of functions $u : A \longrightarrow A'$ and $v : V \longrightarrow V'$ such that $t'u = vt$ and $h'u = vh$.

Presheaf *A presheaf on a category* $\mathcal{C}$ is a contravariant functor from $\mathcal{C}$ to a suitable[11] category of sets. All presheaves on $\mathcal{C}$ form a category $\widehat{\mathcal{C}}$ whose morphisms are natural transformations between presheaves. *Examples*: 1. The **representable functors**, which are of the form $\mathcal{C}(-, C)$ for C object of $\mathcal{C}$, are presheaves on $\mathcal{C}$.

Category of elements Let P be a presheaf on $\mathcal{C}$. We denote by $\int P$ the *category of elements* of P. Its objects are of the form (C, p) where $p \in P(C)$ and C is an object of $\mathcal{C}$. A morphism from (C, p) to (C', p') is a morphism f of $\mathcal{C}$, from C to C', such that $P(f)(p') = p$. We denote by $\pi : \int P \longrightarrow \mathcal{C}$ the natural projection functor. This category is isomorphic to the comma category $\mathbf{y} \downarrow P$. Also, its opposite $(\int P)^{op}$ is isomorphic to $P \downarrow \mathbf{y}^{op}$.

Generalized element Given two objects C and D in a category $\mathcal{C}$, a C**-addressed element** of D is, by definition, a morphism $C \longrightarrow D$ of $\mathcal{C}$. This definition expresses a certain relativity of mathematical objects by means of the introduction of *observers* C for an object D.

Locally compact space A topological space X is *locally compact* if for each point $x \in X$ and each open neighborhood $U \ni x$, there is a compact neighborhood of x contained in U. In the case when X is a Hausdorff space, this definition is equivalent to saying that each point in X has a compact neighborhood. In this way, every compact Hausdorff space is locally compact.

Compact-open topology Let X and Y be topological spaces. The subbasic opens of the *compact-open topology* on the function space X^Y are those of the form $\{c : Y \longrightarrow X \text{ continuous } \mid c(K) \subseteq U\}$, where K is compact in Y and U is open in X. If Y is locally compact Hausdorff, then this makes X^Y an *exponential* in the category of topological spaces [7, p. 558].

Topological category A small category such that its sets of morphisms C_1 and objects C_0 are topological spaces and the identity $e : C_0 \longrightarrow C_1$, domain $d : C_1 \longrightarrow C_0$, codomain $c : C_1 \longrightarrow C_0$, and composition $m : E_2 = E_1 \times_{E_0} E_1 \longrightarrow E_1$ are continuous maps. It is usually written as the tuple (C_1, C_0, e, d, c, m). A **topological functor** $F : \mathbb{K} \longrightarrow \mathbb{D}$ between topological categories is a functor between the underlying categories such that the correspondences on objects and morphisms, namely F_0 and F_1, are continuous. It is usually written as the pair (F_1, F_0).

Topological category of the real unit interval[12] We define it by $\mathbb{I} = (\nabla, I, e', d', c', m')$, where

- $\nabla = \{(x, y) \in I \times I \mid x \leq y \text{ in } I\}$;
- $e' : I \longrightarrow \nabla$ is the diagonal, that is, $e'(x) = (x, x)$;
- $d', c' : \nabla \longrightarrow I$ are the first and second projections respectively;

[11] That is, the sets are the members of a Grothendieck universe [6, Exposé I].
[12] This category was introduced in [14].

- $E_2 = \nabla \times_I \nabla = \{((z,w),(x,y)) \in I^2 \times I^2 \mid x \leq y = z \leq w\}$, and $m' : E_2 \longrightarrow \nabla$ is defined by $m'((z,w),(x,y)) = (x,w)$; and
- we place the usual topology on I, ∇ is a subspace of $I \times I$ (product topology), and E_2 is a subspace of I^4, so e' (diagonal), d', c', and m' (projections) are continuous.

Yoneda lemma Given a presheaf P on $\mathcal{C}$ and an object of $\mathcal{C}$, it establishes a natural bijection between the sets of natural transformations from $\mathcal{C}(-,C)$ to P and $P(C)$. Explicitly, the bijection sends such a natural transformation τ to $\tau_C(id_C)$.

Yoneda embedding The functor $\mathbf{y} : \mathcal{C} \longrightarrow \widehat{\mathcal{C}}$ sending an object C to the representable functor $\mathcal{C}(-,C)$ and a morphism $f : C \longrightarrow D$ to the natural transformation defined by composition with f in each component. This functor is full and faithful by the Yoneda lemma.

Wedge Let $F : \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathcal{E}$ be a functor and E an object of $\mathcal{E}$. A *wedge from F to E* consists of a family of morphisms (α_C) indexed by the objects of $\mathcal{C}$ such that the following diagram commutes for each morphism $f : C \longrightarrow C'$ of $\mathcal{C}$.

$$\begin{array}{ccc} F(C,C) & \xrightarrow{\alpha_C} & E \\ {\scriptstyle F(f,id)} \uparrow & & \uparrow {\scriptstyle \alpha_{C'}} \\ F(C',C) & \xrightarrow[F(id,f)]{} & F(C',C') \end{array}$$

Coend Let $F : \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathcal{E}$ be a functor and E an object of $\mathcal{E}$. A *coend of F* is a pair (E,α), where E is an object of $\mathcal{E}$ and α is a wedge from F to E such that for every wedge β from F to E' there is a unique morphism $h : E \longrightarrow E'$ with $h\alpha_C = \beta_C$ for each object C of $\mathcal{C}$. We denote E by $\int^C F(C,C)$.

Sieve Let C be an object of a category $\mathcal{C}$. A *sieve* on C is a set of morphisms with codomain C that is closed under right composition. *Examples*: 1. The **maximal sieve** $t(C)$ consisting of all morphisms with codomain C. 2. The **sieve generated by a set** X of morphisms with codomain C, defined as the closure of X under right composition. 3. The **restriction sieve** $h^*(S)$ of a sieve S on C along a morphism $h : D \longrightarrow C$, defined as the set of all morphisms f with codomain D such that hf is in S.

Grothendieck topology A *Grothendieck topology* J on a category $\mathcal{C}$ consists of for each object C a set of *covering sieves* $J(C)$ such that i) the maximal sieve $t(C)$ is in $J(C)$, ii) if S is in $J(C)$, then all possible restriction sieves of S are covering sieves, and iii) if all possible restriction sieves of a given one S are covering sieves, then S is a covering sieve. *Examples*: 1. Let T be a topology (of a topological space) regarded as a category (category of a poset). The sieves generated by open coverings of opens in T are the covering sieves of a Grothendieck topology on T. 2. Consider the category $\widehat{\mathcal{C}}$ of presheaves on $\mathcal{C}$. The **epimorphic sieves** of a presheaf P are the sieves S on P such that for each object C of $\mathcal{C}$ the set of images $\{\mathrm{Im}(\tau_C) \mid \tau \in S\}$ covers $P(C)$.

Site Category with a Grothendieck topology $(\mathcal{C}, J)$.

Sheaf A presheaf F on a site $(\mathcal{C}, J)$ is a *sheaf* if for each object C of $\mathcal{C}$ and each covering sieve S in $J(C)$, given a family of *local sections* $\{x_f \mid f \text{ in } S \text{ and } x_f \in F(\mathrm{dom}(f))\}$ such that $F(h)(x_f) = x_{fh}$ whenever the composite fh exists, there is a *unique* $x \in F(C)$ such that $F(f)(x) = x_f$ for each f in S.

In such a case we say that x is a *global section*. In words, we also express the sheaf condition above by saying that every *matching family* (x_f) of elements of F has a unique *amalgamation* x. *Examples*: given the site of the usual topology of $\mathbb{R}$ (respectively $\mathbb{C}$), the presheaf with $P(U)$ defined as the set of all (continuous or differentiable) functions defined on U with values in $\mathbb{R}$ (respectively $\mathbb{C}$) is a sheaf.

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